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## FINITE DIHEDRAL GROUPS AND D.G. NEAR RINGS II

by

J. J. Malone and C. G. Lyons

In this paper the distributively generated near rings generated, respectively, by the inner automorphisms, automorphisms, and endomorphisms of a dihedral group of order  $2n$ ,  $n$  even, are described. For near rings, a theory of successive decompositions generated by idempotents is developed. This theory is used in determining the form of the endomorphism near ring of the dihedral group. The order of the inner automorphism near ring is found to be  $n^3/4$  and that of the automorphism near ring to be  $n^3/2$ . The order of the endomorphism near ring is  $n^7/64$  if  $4|n$  but  $8 \nmid n$ . Otherwise its order is  $2n^3/(n/\gcd(n, 4))^4$ . The radicals of these morphism near rings are also given and the near ring modulo its radical is described.

### 1. Successive decompositions

In this section we extend Theorems 2.1, 2.2, and 2.3 of [6] to cover successive decompositions generated by idempotents. Since the application of this material is to endomorphism near rings, we assume that the near ring  $R$  satisfies the condition that  $0r = 0 = r0$ , for each  $r \in R$ .

**THEOREM 1.1.** *Let  $e$  and  $f$  be idempotents in the near ring  $R$  such that  $ef = 0$ . Then each  $r \in R$  has a unique decomposition*

$$r = [(r - er) - f(r - er)] + f(r - er) + er.$$

Thus  $R = D + N + M$ , where

$$D = \{(r - er) - f(r - er) | r \in R\} = \{x \in R | ex = fx = 0\},$$

$$N = \{f(r - er) | r \in R\} = \{y \in R | ey = 0 \text{ and } fy = y\},$$

$$M = \{er | r \in R\} = \{z \in R | ez = z\},$$

$$\begin{aligned} \text{and } D \cap N = D \cap M = N \cap M &= (D + N) \cap M = D \cap (N + M) \\ &= N \cap (D + M) = \{0\}. \end{aligned}$$

**THEOREM 1.2.**  *$D$  is a right ideal in  $R$ . Each of  $N$  and  $M$  is an  $R$ -subgroup (and subnear ring) of  $R$ .*

The two theorems just stated are proved in a manner similar to that used in proving Theorems 2.1 and 2.2 of [6]. The following theorem is an extension of Theorem 2.3 of [6].

**THEOREM 1.3.** *Let  $R$  be a near ring such that  $(R, +)$  is generated by  $\{r_\gamma | \gamma \in \Gamma, \text{ an index set}\}$ . Let  $e$  and  $f$  be idempotents in  $R$  such that  $ef = 0$ . Then  $M$  is the subgroup generated by  $\{er_\gamma | \gamma \in \Gamma\}$  and  $N = fA_e$  ( $A_e$  is the normal subgroup generated by  $\{r_\gamma - er_\gamma | \gamma \in \Gamma\}$ , see [6]). Also,  $D$  is generated by elements of the form  $fa_2 + a_1 - fa_1 - fa_2$ , with  $a_1$  and  $a_2$  in  $A_e$ .*

**PROOF.** Let  $r \in R$  and let  $r'$  denote  $r - er$  and  $r''$  denote  $r' - fr'$ . Note that  $eg + r' - eg = (eg + r - eg)'$ ,  $g \in R$ . Now  $r$  can be written as  $r = c_1 + c_2 + \cdots + c_k$ , with either  $c_i$  or  $-c_i$  in  $\{r_\gamma | \gamma \in \Gamma\}$ ,  $i = 1, \cdots, k$ . Since the application of this theorem in the following sections is to finite groups, the proof is presented for the case in which each  $c_i$  is in  $\{r_\gamma | \gamma \in \Gamma\}$ . The proof can be adapted to the other cases by noting that

$$(-r_\gamma) - e(-r_\gamma) = -er_\gamma + (er_\gamma - r_\gamma) + er_\gamma.$$

By Theorem 2.3 of [6], we may write

$$\begin{aligned} r &= c'_1 + (ec_1 + c'_2 - ec_1) + \cdots \\ &\quad + (ec_1 + ec_2 + \cdots + ec_{k-1} + c'_k - ec_{k-1} - \cdots - ec_2 - ec_1) \\ &\quad + (ec_1 + ec_2 + \cdots + ec_k). \end{aligned}$$

$$\begin{aligned} \text{Then } r &= c'_1 + (ec_1 + c_2 - ec_1)' + \cdots \\ &\quad + (ec_1 + ec_2 + \cdots + ec_{k-1} + c_k - ec_{k-1} - \cdots - ec_2 - ec_1)' \\ &\quad + (ec_1 + ec_2 + \cdots + ec_k). \end{aligned}$$

$$\begin{aligned} \text{Set } d_i &= (ec_1 + ec_2 + \cdots + ec_{i-1} + c_i - ec_{i-1} - \cdots - ec_2 - ec_1)', \\ i &= 2, \cdots, k \text{ and } d_1 = c'_1. \end{aligned}$$

$$\begin{aligned} \text{Thus } r &= (d'_1 + fd_1) + (d'_2 + fd_2) + \cdots + (d'_k + fd_k) \\ &\quad + [ec_1 + ec_2 + \cdots + ec_k] \\ &= [d'_1 + (fd_1 + d'_2 - fd_1) + (fd_1 + fd_2 + d'_3 - fd_2 - fd_1) + \cdots \\ &\quad + (fd_1 + fd_2 + \cdots + fd_{k-1} + d'_k - fd_{k-1} - \cdots - fd_2 - fd_1)] \\ &\quad + [fd_1 + fd_2 + \cdots + fd_k] + [ec_1 + ec_2 + \cdots + ec_k]. \end{aligned}$$

Specifically, Theorem 1.3 justifies the following procedure for working successive decompositions. One decomposes the generators of  $R$  by  $e$  to obtain elements of the form  $r_\gamma - er_\gamma$  and of the form  $er_\gamma$ . The elements of the first form are conjugated by the elements of the group generated by the elements of the second form. Then a second idempotent  $f$  is selected from the elements which can be generated by the conjugates. Then

each element in the set of conjugates (call a typical such conjugate  $er + r'y - er$ ) is decomposed as

$$[(er + r'_y - er) - f(er + r'_y - er)] + [f(er + r'_y - er)].$$

Next the elements of the form given in the first term are conjugated by the elements of the group generated by the forms given in the second term. Then these second conjugates generate  $D$ , the  $f(er + r'_y - er)$  elements generate  $N$ , and then  $er_y$  elements generate  $M$ . This procedure may be iterated and extended to apply to a decomposition involving any finite number of idempotents. The technique is applied to discover the properties of the endomorphism near ring discussed in Section 4.

### 2. Endomorphisms of $D_{2n}$

The finite dihedral group of order  $2n$  will be designated by  $D_{2n}$  and will be presented as  $(a, b|a^n, b^2, abab)$ . Elements of  $D_{2n}$  will be given in the form  $a^x b^s, 0 \leq x \leq n-1, 0 \leq s \leq 1$ . For the remainder of this paper it is assumed that  $n$  is even.

LEMMA 2.1. *The proper normal subgroups of  $D_{2n}$  are the subgroups of (a), the normal subgroup  $S$  generated by  $b$ , and the normal subgroup  $T$  generated by  $ab$ .*

PROOF. It is immediate that the subgroups of (a) are normal. Since  $aba^{-1} = a^2b \in S$ , it follows that  $(a^2b)b = a^2 \in S$  and that

$$S = \{a^{2^x} b^s | 0 \leq s \leq 1, 0 \leq x \leq n/2 - 1\} \text{ with } |S| = n.$$

It is readily shown that the normal subgroup generated by any element of the form  $a^{2^x} b$  is  $S$ . Since  $a(ab)a^{-1} = a^3b \in T$ , it follows that  $(a^3b)(ab) = a^2 \in T$  and that  $T = \{a^{2^x} | 0 \leq x \leq n/2 - 1\} \cup \{a^k b | k \text{ is odd}\}$  with  $|T| = n$ .

LEMMA 2.2. *For  $n \geq 4$ ,  $D_{2n}$  has  $n \cdot (n)\phi$  automorphisms and  $n$  inner auto-morphisms.*

THEOREM 2.3.  *$D_{2n}$  has  $(n+2)^2$  endomorphisms.*

PROOF. We allow  $n = 2$  so that  $D_4$  is the Klein group. Note that the Klein group has 6 automorphisms. For  $n = 2$ , the result is well-known (see [2]).

Assume  $n \geq 4$ . As was shown in [7], if  $k > 1, k|n$ , then  $D_{2n}$  contains  $t = n/k$  copies of  $D_{2k}$ . The number of ways  $D_{2n}/(a^k)$  may be mapped onto any one of these is the same as the number of automorphisms of  $D_{2k}$ . For  $k > 1$  and  $k \neq 2$  this number is  $k \cdot (k)\phi$  and for  $k = 2$  the number is 6. So there are exactly  $n \cdot (k)\phi$  endomorphisms with kernel

$(a^k)$  for  $k > 1$  and  $k \neq 2$ . Also, there are  $3n$  endomorphisms with kernel  $(a^2)$ . For  $k = 1$ ,  $D_{2n}/(a^k) \cong C_2$  and it follows that there are  $n+1$  endomorphisms with kernel  $(a)$ . There are also  $n+1$  endomorphisms with kernel  $S$ ,  $n+1$  with kernel  $T$ , and the 0 map. Write  $3n$  as  $2n+n \cdot (2)\varphi$  and  $n+1$  as  $1+n \cdot (1)\varphi$ . Then the number of endomorphisms of  $D_{2n}$  is

$$n \cdot \sum_{k|n} (k)\varphi + 4n + 4 = n^2 + 4n + 4 = (n+2)^2.$$

### 3. $I(D_{2n})$ and $A(D_{2n})$

As in [7] and [6],  $I(D_{2n})(A(D_{2n}), E(D_{2n}))$  designates the distributively generated near ring generated additively by the inner automorphisms (automorphisms, endomorphisms) of  $D_{2n}$ .  $I$  or  $A$  or  $E$  will be used in cases where no confusion would arise. The conventions used in [7] for designating functions are followed here: the inner automorphism generated by  $a^x b^s$  is denoted by  $[a^x b^s]$ ; the endomorphism  $\alpha$  such that  $(a^x b^s)\alpha = a^{xy+sz} b^s$ ,  $0 \leq y, z \leq n-1$ , is presented in terms of the images of  $a$  and  $b$  as  $[a^y, a^z b]$ ; and the function on  $D_{2n}$  which maps each power of  $a$  to  $e$  and each element outside  $(a)$  to some fixed  $d \in D_{2n}$  is given as  $(e, d)$ .

Because the form of the needed idempotent function varies as  $n$  varies, it is more convenient to present  $I$  and  $A$  as is done below rather than to use the procedure of Section 1.

**THEOREM 3.1.**  $|I(D_{2n})| = n^3/4$ .

**PROOF.** The theorem is true for  $n = 2$ . Assume  $n \geq 4$ . Note that  $[a^k] = [a, a^{-2k}b]$  and  $[a^k b] = [a^{-1}, s^{2k}b]$ . Set  $\delta = (ee \cdots e|a^2 a^2 \cdots a^2)$ , where the  $2n$ -tuple is used to indicate, in order, the images of

$$e, a, a^2, \dots, a^{n-1}, b, ab, a^2 b, \dots, a^{n-1} b.$$

The bar  $|$  is inserted as a matter of convenience between the images of  $a^{n-1}$  and  $b$ . Since  $\delta = -[e] + [a]$ ,  $\delta \in I$ . Moreover,  $[a^k] = k\delta + [e]$ ,  $k = 0, 1, \dots, n/2 - 1$ . Now set

$$\alpha = (ee \cdots e|ea^2 a^4 \cdots a^{-2}||ea^2 a^4 \cdots a^{-2}),$$

where the symbol  $||$  separates the image of  $a^{(n/2)-1}b$  and the image of  $a^{n/2}b$ . Since  $\alpha = [e] + [b]$ ,  $\alpha \in I$ . Also,

$$[a^{(n/2)-k}b] = -[a^k] + \alpha, k = 0, 1, \dots, (n/2) - 1.$$

Obviously,  $\alpha + \delta = \delta + \alpha$  and  $(\alpha) \cap (\delta) = (e, e)$ . Also,  $-[e] + \delta + [e] = -\delta$  and  $-[e] + \alpha + [e] = -\alpha$ . In total then, for some choice of integers  $u, v$ , and  $w$ , an arbitrary element of  $I$  can be given in the form  $u[e] + v\alpha + w\delta$ . Thus  $I(D_{2n})$  is the semidirect sum  $([e]) + [(\alpha) \oplus (\delta)]$ ,  $(\alpha) \oplus (\delta)$  normal in  $(I, +)$  and  $|I| = n(n/2)(n/2) = n^3/4$ .

**THEOREM 3.2.**  $|A(D_{2n})| = n^3/2, n \geq 4.$

**PROOF.** Let  $\beta = (ee \cdots e|aa \cdots a).$  Since  $\beta = [a, b] + [a, a^{-1}b],$   $\beta \in A.$  Obviously,  $\beta \notin I(D_{2n}).$  An arbitrary automorphism of  $D_{2n}$  has the form  $[a^k, a^x b], (k, n) = 1$  (in particular,  $k$  is an odd integer). If  $x$  is even set  $\psi = [a, a^x b] \in I$  and if  $x$  is odd set  $\psi = \beta + [a, a^{x-1} b],$  where  $[a, a^{x-1} b] \in I.$  Then  $[a^k, a^x b] = ((k-1)/2)\alpha + k\psi,$  i.e. all automorphisms are in the group generated by  $I$  and  $\beta.$  Since  $2\beta = \delta$  we have, in particular, that  $\beta + \delta = \delta + \beta.$  Also,  $\alpha + \delta = \delta + \alpha$  and  $-[e] + \beta + [e] = -\beta.$  Thus  $(\beta)$  is a normal subgroup of  $(A, +)$  and, in fact,  $A = (\beta) + I.$  Thus  $|A(D_{2n})| = (n(n^3/4))/(n/2) = n^3/2.$  Note that if  $4|n,$  we may state that

$$A = ((ee \cdots e|a^{n/2} a^{n/2} \cdots a^{n/2})) \oplus I.$$

Also note that for some choice of integers  $u, v,$  and  $w,$  an arbitrary element of  $A$  can be written in the form  $u[e] + v\alpha + w\beta.$

**THEOREM 3.3.** *Let  $J(R)$  designate the radical of a near ring  $R.$  Then  $J(I(D_{2n})) = \Phi_n + [(\alpha) \oplus (\delta)]$  and  $J(A(D_{2n})) = \Phi_n + [(\alpha) \oplus (\beta)],$  where  $\Phi_n$  is the subring whose additive group is the Frattini subgroup of the group  $([e]).$*

**PROOF.** Since  $(I, +)$  is solvable, it follows from Lemma 2.1 of [1] that each maximal right ideal of  $I$  contains the commutator subgroup of  $(I, +),$  namely  $(2\alpha) \oplus (2\delta).$

Note that, if  $u$  is odd  $(u[e] + v\alpha + w\delta)\delta = \delta$  and

$$(u[e] + v\alpha + w\delta)\alpha = -2w\delta + (1 - 2v)\alpha.$$

Thus, if  $K$  is a maximal right ideal which contains  $u[e] + v\alpha + w\delta,$  then  $\delta \in K$  and  $\alpha \in K.$  It follows that  $K = (t[e]) + [(\alpha) \oplus (\delta)],$  where  $t$  is an odd integer chosen so that  $(t[e])$  is a maximal subgroup of  $([e]).$  Lemma 2.1 of [1] assures us that a maximal right ideal of  $I$  is also a maximal  $I$ -subgroup. If  $L$  is a right ideal which contains no element for which  $u$  is odd, then  $L$  is contained in the maximal right ideal  $(2[e]) + [(\alpha) \oplus (\beta)].$

If  $I$  is replaced by  $A$  and  $\delta$  is replaced by  $\beta$  in the two paragraphs above, the statements which result are correct. Since, in our case, the radical is the intersection of the maximal right ideals, the theorem follows for both  $I$  and  $A.$

By direct observation or by Theorem 2.2 of [1] it can be seen that  $J(I)$  and  $J(A)$  are nilpotent.

**THEOREM 3.4.**  $I/J(I) \cong A/J(I) \cong Z/(q),$  where  $q$  is the product of the distinct prime factors of  $n.$

**PROOF.** Note that  $\Phi_n$  is generated additively by  $q[e]$  (see 8.1 of p. 134 of [4]). From this and the theorems of this section it follows that  $|I/J(I)| =$

$|A/J(A)| = q$  and that the additive groups of the quotient near rings are cyclic. Since each of the two quotient near rings has a multiplicative identity, each is isomorphic to the ring of integers modulo  $q$  (see the Corollary of p. 147 of [3]).

#### 4. $E(D_{2n})$

In this section the form of  $E(D_{2n})$  is displayed. This is done thru the use of the technique of successive decompositions which was developed in the first section. We start by restating the results of Theorem 2.3 in tabular form.

Endomorphisms of $D_{2n}$				
	Form	Kernel	Number	Comment
(1)	$[e, e]$	$D_{2n}$	1	
(2)	$[a^y, a^x b]$	$(e)$	$n \cdot (n)\varphi$	$(y, n) = 1, 0 \leq x \leq n-1$
(3)	$[d, e]$	$S$	$n+1$	$ d  = 2$
(4)	$[d, d]$	$T$	$n+1$	$ d  = 2$
(5)	$[e, d]$	$(a)$	$n+1$	$ d  = 2$
(6)	$[a^{n/2}, a^x b]$	$(a^2)$	$n$	$0 \leq x \leq n-1$
(7)	$[a^x b, a^{x+n/2} b]$	$(a^2)$	$n$	$0 \leq x \leq n-1$
(8)	$[a^x b, a^{n/2}]$	$(a^2)$	$n$	$0 \leq x \leq n-1$
(9)	$[a^{st}, a^x b]$	$(a^k)$	$\sum_{k n} n \cdot (k)\varphi$	$k \neq 1, 2, n; t n;$ $(s, k) = 1; 0 \leq x \leq n-1$

Let  $m$  be an integer such that  $0 \leq m \leq n-1$ . If  $(m, n) = 1$ , then  $[a^m, a^x b]$  is an automorphism included in form (2). If  $(m, n) = t$ , then  $m = (m/t) \cdot t$  with  $(m/t, n/t) = 1$  and  $[a^m, a^x b]$  is an endomorphism included in (9). If  $m = 0$ , then  $[a^m, a^x b]$  is a form which includes all endomorphisms of (5) except  $[e, a^{n/2}]$ . Thus  $[a^y, a^x b], 0 \leq y, x \leq n-1$ , gives  $n^2$  endomorphisms and includes all endomorphisms of (2), (5), (6), and (9) with the exception of  $[e, a^{n/2}]$ . So there are six different forms for the endomorphisms of  $D_{2n}$ , i.e. for the generators of  $E$ . The idempotent functions of  $E$  which will be used to generate the decompositions will be designated by  $\gamma$ 's. The generators themselves will be designated by  $\alpha$ 's. The procedure followed is that given in the last paragraph of Section 1. If  $\gamma_1$  is chosen to be  $[e, b]$ , the following table is obtained.

$\alpha$	$\alpha - \gamma_1 \alpha$	$\gamma_1 \alpha$
$[a^y, a^x b]$	$(ea^y a^{2y} a^{3y} \dots   ea^y a^{2y} a^{3y} \dots)$	$[e, a^x b]$
$[d, e]$	$(eded \dots   eded \dots)$	$[e, e]$
$[d, d]$	$(eded \dots   eded \dots)$	$[e, d]$
$[e, a^{n/2}]$	$(eeee \dots   eeee \dots)$	$[e, a^{n/2}]$
$[a^x b, a^{x+n/2} b]$	$(ea^x bea^x b \dots   ea^x bea^x b \dots)$	$[e, a^{x+n/2} b]$
$[a^x b, a^{n/2}]$	$(ea^x bea^x b \dots   ea^x bea^x b \dots)$	$[e, a^{n/2}]$

Now it is seen that  $M(\gamma_1) = \{(e, g) | g \in D_{2n}\}$  and that as a group  $M(\gamma_1)$  is isomorphic to  $D_{2n}$ . Among the  $\alpha - \gamma_1 \alpha$  there are just two forms:

$$(ea^y a^{2y} a^{3y} \dots | ea^y a^{2y} a^{3y} \dots)$$

and

$$(eded \dots | eded \dots).$$

Since the case  $d = a^{n/2}$  is included in the first of these forms we may assume that the  $d$  of the second form can be given as  $a^x b$ .

Now these two forms must be conjugated by the  $\gamma_1 \alpha$ 's. This conjugation yields the generators of  $A(\gamma_1)$ :

- (1)  $= (ea^y a^{2y} a^{3y} \dots | ea^y a^{2y} a^{3y} \dots),$
- (2)  $= (ea^x bea^x b \dots | ea^x bea^x b \dots),$
- (3)  $= (ea^y a^{2y} a^{3y} \dots | ea^{-y} a^{-2y} a^{-3y} \dots),$
- (4)  $= (ea^x bea^x b \dots | ea^{2y-x} bea^{2y-x} b \dots),$
- (5)  $= (ea^x bea^x b \dots | ea^{2y+x} bea^{2y+x} b \dots).$

For the second decomposition, set  $\gamma_2 = [ab, e] \in A(\gamma_1)$ . It follows that:

- $\gamma_2(1) = (ea^y ea^y \dots | ea^y ea^y \dots),$
- $\gamma_2(2) = (ea^x bea^x b \dots | ea^x bea^x b \dots),$
- $\gamma_2(3) = (ea^{-y} ea^{-y} \dots | ea^{-y} ea^{-y} \dots),$
- $\gamma_2(4) = (ea^{2y-x} bea^{2y-x} b \dots | ea^{2y-x} bea^{2y-x} b \dots),$
- $\gamma_2(5) = (ea^{2y+x} bea^{2y+x} b \dots | ea^{2y+x} bea^{2y+x} b \dots).$

Thus,  $M(\gamma_2) = \{(egeg \dots | egeg \dots) | g \in D_{2n}\}$  and as a group is isomorphic to  $D_{2n}$ . In  $A(\gamma_2)$  we have:

- (1)  $-\gamma_2(1) = (eea^{2y} a^{2y} \dots a^{(n-2)y} | eea^{2y} a^{2y} \dots a^{(n-2)y}),$
- (2)  $-\gamma_2(2) = (eeee \dots | eeee \dots),$
- (3)  $-\gamma_2(3) = (ea^{2y} a^{2y} a^{4y} a^{4y} \dots e | eea^{-2y} a^{-2y} \dots a^{-(n-2)y}),$
- (4)  $-\gamma_2(4) = (ea^{2x-2y} ea^{2x-2y} \dots | eeee \dots),$
- (5)  $-\gamma_2(5) = (ea^{-2y} ea^{-2y} \dots | eeee \dots).$



These last five types are of three forms:

$$\begin{aligned} (1) &= (eea^{2y}a^{2y} \cdots a^{(n-2)y} \mid eea^{2y}a^{2y} \cdots a^{(n-2)y}), \\ (2) &= (ea^{2y}a^{2y}a^{4y} \cdots e \mid eea^{-2y}a^{-2y} \cdots a^{-(n-2)y}), \\ (3) &= (ea^{2y}ea^{2y} \cdots \mid eeee \cdots). \end{aligned}$$

If  $n = 4$ , then  $(1) = (eea^2a^2 \mid eea^2a^2)$ ,  $(2) = (ea^2a^2e \mid eea^2a^2)$ , and  $(3) = (ea^2ea^2 \mid eeee)$ . Since  $(3) = (1) + (2)$  and each of  $(1)$  and  $(2)$  is in the additive center of  $E(D_8)$ , it follows that  $A(\gamma_2) = ((1)) \oplus ((2))$  and that  $E(D_8) = A(\gamma_2) + M(\gamma_2) + M(\gamma_1)$  has order 256. Until the statement of Theorem 4.1 it is assumed that  $n \neq 4$ .

For  $n \neq 4$ , we conjugate forms  $(1)$ ,  $(2)$ , and  $(3)$  by the elements of  $M(\gamma_2)$  and obtain two additional forms:

$$\begin{aligned} (4) &= (eea^{2y}a^{-2y} \cdots a^{-(n-2)y} \mid eea^{2y}a^{-2y} \cdots a^{-(n-2)y}), \\ (5) &= (ea^{-2y}a^{2y}a^{-4y} \cdots e \mid eea^{-2y}a^{2y} \cdots a^{(n-2)y}). \end{aligned}$$

These last five forms furnish the generators for  $A(\gamma_2)$ .

To begin the third decomposition let  $\gamma_3$  be the element of form  $(1)$  for which  $y = 1$ . Then:

$$\begin{aligned} \gamma_3(1) &= (eea^{2y}a^{2y} \cdots a^{(n-2)y} \mid eea^{2y}a^{2y} \cdots a^{(n-2)y}) \\ &= \gamma_3(2) = \gamma_3(4) = \gamma_3(5) \text{ and } \gamma_3(3) = [e, e]. \end{aligned}$$

So  $M(\gamma_3)$  is generated additively by  $\gamma_3$  and is a cyclic group of order  $n/2$ .

In  $A(\gamma_3)$  we have:

$$\begin{aligned} (1) - \gamma_3(1) &= (eeee \cdots \mid eeee \cdots), \\ (2) - \gamma_3(2) &= (ea^{2y}ea^{2y} \cdots \mid eea^{-4y}a^{-4y} \cdots a^{-2(n-2)y}), \\ (3) - \gamma_3(3) &= (ea^{2y}ea^{2y} \cdots \mid eeee \cdots), \\ (4) - \gamma_3(4) &= (eeea^{-4y}ea^{-8y} \cdots a^{-(n-2)y} \mid eeea^{-4y}ea^{-8y} \cdots a^{-2(n-2)y}), \end{aligned}$$

and

$$(5) - \gamma_3(5) = (ea^{-2y}ea^{-6y}ea^{-10y} \cdots a^{-(n-2)y} \mid eea^{-4y}ea^{-8y}e \cdots e).$$

The conjugation of these five functions by the elements of  $M(\gamma_3)$  produces no new forms so, omitting the identity map, we take the remaining forms as giving the generators of  $A(\gamma_3)$ . However, the study of  $A(\gamma_3)$  is facilitated by introducing a new set of generating forms for  $A(\gamma_3)$ . These are defined by:

$$\begin{aligned} \lambda_1 &= (3) - \gamma_3(3), \\ \lambda_2 &= (3) - \gamma_3(3) - ((2) - \gamma_3(2)) \\ &= (eeee \cdots \mid eea^{4y}a^{4y}a^{8y}a^{8y} \cdots a^{-4y}), \end{aligned}$$

$$\begin{aligned} \lambda_3 &= -((5) - \gamma_3(5)) + \lambda_1 - \lambda_2 - ((4) - \gamma_3(4)) \\ &= (ea^{4y}ea^{12y}ea^{20y} \cdots a^{-4y} | eeee \cdots), \\ \lambda_4 &= (4) - \gamma_3(4). \end{aligned}$$

Since  $(3) - \gamma_3(3) = \lambda_1$ ,  $(2) - \gamma_3(2) = \lambda_1 - \lambda_2$ ,  $(4) - \gamma_3(4) = \lambda_4$ , and  $(5) - \gamma_3(5) = \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4$ , the  $\lambda$ 's do provide a set of generators for  $A(\gamma_3)$ . For each  $i$  the set of functions of the form  $\lambda_i$  constitute a cyclic subgroup. Henceforth we identify  $\lambda_i$  with the element of that form for which  $y = 1$ , i.e.  $\lambda_i$  generates that cyclic subgroup. Note that  $|\lambda_1| = n/2$  and that  $|\lambda_2| = |\lambda_3| = |\lambda_4| = n/\gcd(n, 4)$ .

Obviously,  $(\lambda_1) \cap (\lambda_2) = [e, e]$  so that the sum of  $(\lambda_1)$  and  $(\lambda_2)$  is direct. Consider  $((\lambda_1) \oplus (\lambda_2)) \cap (\lambda_3)$ . This intersection is the same as  $(\lambda_1) \cap (\lambda_3)$ . If  $x\lambda_1 = y\lambda_3$ , then in particular  $n|2x - 4y$  and  $n|2x - 12y$  so that  $n|8y$ . Thus an element in the intersection is the identity map or has order 2. In the latter case  $a^{2x}$  must be  $a^{n/2}$ . If  $4 \nmid n$ , this is impossible and the intersection is trivial so that the sum  $(\lambda_1) + (\lambda_2) + (\lambda_3)$  is direct and has order  $(n/2)^3 = (n/\gcd(n, 4))^3$ . If  $4|n$  but  $8 \nmid n$ , the intersection is trivial since  $n \nmid (4y - (n/2))$ , i.e.  $y\lambda_3$  cannot be of order 2. Thus the sum is direct and has order  $n^3/32$ . If  $8|n$ , the intersection has order 2 and the sum is a subgroup of order  $((n/2) \cdot (n/4)^2)/2 = (n/\gcd(n, 4))^3$ . Since  $(\lambda_2) \cap (\lambda_4) = [e, e]$ , it follows that  $((\lambda_1) + (\lambda_2) + (\lambda_3)) \cap (\lambda_4) = [e, e]$ . Hence  $A(\gamma_3) = ((\lambda_1) + (\lambda_2) + (\lambda_3)) \oplus (\lambda_4)$  has order  $(n/\gcd(n, 4))^4$  unless  $4|n$  but  $8 \nmid n$ . In that case,  $|A(\gamma_3)| = n^4/128$ .

Since  $E = [ [[A(\gamma_3)] + M(\gamma_3)] + M(\gamma_2) ] + M(\gamma_1)$  and  $|E| = |A(\gamma_3)| \cdot (n/2) \cdot (2n)^2$  we have the following theorem. Note that the case  $n = 4$  is included in the statement of the theorem.

**THEOREM 4.1.** *If  $4|n$  but  $8 \nmid n$ , the order of  $E(D_{2n})$  is  $n^7/64$ . Otherwise the order of  $E(D_{2n})$  is  $2n^3(n/\gcd(n, 4))^4$ .*

### 5. The Radical of $E(D_{2n})$

**LEMMA 5.1.** *Let  $\sigma \in J(E)$ . Then  $(D_{2n})\sigma \leq (a^2)$ .*

**PROOF.** If, for  $c \in D_{2n}$ ,  $c\sigma \notin (a^2)$ , then there is a function  $\pi$  of  $M(\gamma_2)$  or  $M(\gamma_1)$  for which  $g = c$  so that  $c\sigma\pi = c$ . But then  $\sigma\pi \in J(E)$  and  $\sigma\pi$  fixes  $c$ . This is a contradiction since  $J(E)$  must be nilpotent (see Theorem 2.2 of [1]).

Note that  $(a^2)$  is a fully invariant abelian subgroup of  $D_{2n}$  and that each element of  $E$  when restricted to  $(a^2)$  is an endomorphism of  $(a^2)$ . Thus the restriction is completely determined by its action on  $a^2$ .

**LEMMA 5.2.** *Let  $\sigma \in J(E)$ . Then  $a^2\sigma = a^{2r}$ , where  $r$  is divisible by each*

prime factor of  $n$  if  $4|n$  and is divisible by each odd prime factor of  $n$  if  $4 \nmid n$ .

PROOF. If  $a^2\sigma = a^{2r}$ , then  $a^2\sigma^2 = a^{2r \cdot r}$ , etc. In order for  $\sigma$  to be nilpotent, there must exist a  $k$  such that  $n|2r^k$ . Thus the condition on  $x$  which was stated in the Lemma is a necessary and sufficient condition for  $\sigma$  to be nilpotent.

THEOREM 5.3.  $J(E) = \{\sigma \in E | (D_{2n})\sigma \subseteq (a^2) \text{ and } a^2\sigma = a^{2r}, r \text{ as in Lemma 5.2}\}$ .

PROOF. By the previous two lemmas, the given conditions on  $\sigma$  are necessary for  $\sigma$  to be in  $J(E)$ . Conversely, it is easily seen that the set of elements which obey the given conditions is closed under addition. Let  $\pi \in E$ . Recall that  $(a^2)$  is fully invariant so that there exists a  $k$  such that  $a^2\pi = a^{2k}$ . But then,  $c\sigma\pi \subseteq (a^2)$  for any  $c \in D_{2n}$  and  $a^2\sigma\pi = a^{2rk}$ , where  $rk$  satisfies the conditions on  $r$  given in Lemma 5.2. Thus the given set is an  $E$ -subgroup. By Theorem 2.7 of [5] this  $E$ -subgroup is nilpotent (the nilpotence may also be noted by inspection) and by Theorem 2.5 of [5] the set is contained in  $J(E)$ .

We now use Theorem 5.3 to determine the size of  $J(E)$  and the nature of the ring  $E/J(E)$ . Theorem 2.2 of [1] assures us that the quotient near ring is actually a ring. First we determine  $J(E)$  for  $D_8$ . From Theorem 5.3 we have that

$$J(E(D_8)) = ((ea^2ea^2 | eeee)) + ((ea^2ea^2 | ea^2ea^2)) + ((eeee | a^2a^2a^2a^2)).$$

It is interesting to note that  $|J(E(D_8))| = 8$ ,  $|J(A(D_8))| = 16$ , and  $|J(I(D_8))| = 8$ . In this case,  $E/J(E)$  has order 32 and additively is the direct sum of groups of order 2. In the discussion of  $J$  which follows it is assumed that  $n \geq 6$ .

Let  $K = \{\sigma \in J(E) | a^2\sigma = e, \text{ i.e. } r = n/2\}$ . The set  $K$  is easily seen to be a nilpotent  $E$ -subgroup and, in fact,  $K = A(\gamma_3) \oplus M' \oplus M''$ , where

$$M' = \{\sigma \in M(\gamma_1) | g \in (a^2)\} \text{ and } M'' = \{\sigma \in M(\gamma_2) | g \in (a^2)\}.$$

Thus  $K \subseteq J(E)$ . Also,  $|M'| = |M''| = n/2$ . Note that  $K$  has the null multiplication.

Now consider those elements of  $J$  for which  $a^2\sigma \neq e$ . If such an element is presented in the form for the decomposition of  $E$  given in Section 4, then the summand from  $M(\gamma_3)$  cannot be  $[e, e]$ . This summand from  $M(\gamma_3)$  can have any element of  $K$  added to it. Of course, the summand from  $M(\gamma_3)$  must be nilpotent and so we define  $M''' = \{\sigma \in M(\gamma_3) | a^2\sigma = a^{2r}, r \text{ as in Lemma 5.2}\}$ . Note that  $M'''$  is analogous to the  $\Phi_n$  which

appears in the decomposition of  $J(I)$  or  $J(A)$  and that  $|M'''| = n/2p_2 \cdots p_m$  if  $4 \nmid n$  and  $|M'''| = n/4p_2 \cdots p_m$  if  $4|n$ , the  $p_i$  being the distinct prime factors on  $n$ . All told,

$$J(E) = A(\gamma_3) \oplus M' \oplus M'' \oplus M'''.$$

Let  $q = 2p_2 \cdots p_m$ .

The following table sums up the information on the order of  $J(E)$ .

	$ A(\gamma_3) $	$ M' $	$ M'' $	$ M''' $	$ J(E) $	$ E/J(E) $
$2 n, 4 \nmid n$	$n^4/16$	$n/2$	$n/2$	$n/q$	$n^7/2^6q$	$8q$
$4 n, 8 \nmid n$	$n^4/128$	$n/2$	$n/2$	$n/2q$	$n^7/2^{10}q$	$16q$
$8 n$	$n^4/256$	$n/2$	$n/2$	$n/2q$	$n^7/2^{14}q$	$16q$

In the quotient structure  $E/J(E)$ ,  $M(\gamma_1)$  and  $M(\gamma_2)$  become rings of order 4 and characteristic 2 and  $A(\gamma_3)$  is reduced to order 1.  $M(\gamma_3)$  is reduced to a ring defined on a cyclic group of order  $q/2$  if  $4 \nmid n$  and to a ring defined on a cyclic group of order  $q$  if  $4|n$ . Overall,  $E/J(E)$  is a ring with identity with order as given in the table above.

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