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ON THE FORM OF FUNCTIONS WHICH PRESERVE REGRESSIVE ISOLS[1]

by

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0. Introduction

In [1] J. Barback gives a characterization of those unary recursive functions which map the regressive isols into the regressive isols. In [6] M. Hassett does the same for the binary recursive functions. Following their lead we characterize the n-ary functions which map the cosimple regressive isols into the cosimple regressive isols. Thus our work extends that of [1] and [6] in that (i) the function may have any number of arguments, (ii) the function is not necessarily recursive, and (iii) the arguments of the function are cosimple. Our results however do include the non-cosimple case. The reader may find a precise statement of our results in the last paragraph of this paper.

1. Closure properties

Let ω be the non-negative integers. We refer to elements of ω as *numbers*. If $\alpha \subseteq \omega$ is a set and $n \in \omega$ let $\times^n\alpha$ be the n-fold direct power of α, and if f is a function let δf, ρf denote its domain, range respectively. If $n \in \omega$ and $f : \times^n\omega \to \omega$ we define a function $\hat{f} : \times^n\omega \to \omega$. If $x = (x_0, \cdots, x_{n-1})$ let

$$(1) \qquad \begin{aligned} \hat{f}(x) &= 0 \text{ if } x_i = 0 \text{ for some } i < n, \\ \hat{f}(x) &= f(y) \text{ if } x_i = y_i + 1 \text{ for } i < n. \end{aligned}$$

Note that if $x \in \times^n\omega$ then we write x_i for the i-th component of x. For $j < n$ we define n-ary functions E_jf, Δ_jf by

$$(2) \qquad \begin{aligned} E_jf(x) &= f(y) \text{ if } y_j = x_j + 1 \text{ and } y_i = x_i \text{ for } i \neq j, \\ \Delta_jf(x) &= (E_jf(x)) - f(x). \end{aligned}$$

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Finally let $\Delta = \Delta_0 \cdots \Delta_{n-1}$ be the composition of the Δ_j's. λ is Church's functional operator, i.e., $f = \lambda x f(x)$, and if $x, y \in \times^n\omega$ write $x \leqq y$ if $x_i \leqq y_i$ for $i < n$. In general when notions usually reserved for the one variable case are applied to n-tuples then we are to understand these notions as holding for each component separately.

A function $f \colon \times^n\omega \to \omega$ is called *recursive increasing* if f is recursive and $\Delta \hat{f}(x) \geqq 0$ for every $x \in \times^n\omega$, f is *eventually recursive increasing* if $\lambda x f(x+m)$ is recursive increasing for some $m \in \times^n\omega$ (here $+$ is componentwise addition). If α is a finite set let $|\alpha|$ denote the number of elements in α, and if $n \in \omega$ let $\nu(n) = \{x \in \omega | x < n\}$. Consider h, σ, d, and j which together satisfy

$$\begin{array}{ll} (3) & \begin{array}{l} \sigma \subsetneqq \nu(n),\ h : \sigma \to \omega,\ |\nu(n)-\sigma| = d > 0, \text{ and} \\ j : \nu(d) \to \nu(n)-\sigma \text{ is strictly increasing.} \end{array} \end{array}$$

With h we associate a function $h_n^* : \times^d\omega \to \times^n\omega$ as follows. If $x \in \times^d\omega$ let $h_n^*(x) = y \in \times^n\omega$ where

$$(4) \qquad y_i = h(i) \text{ for } i \in \sigma \text{ and } y_{j(i)} = x_i \text{ for } i < d.$$

f is called *almost recursive increasing* if $f \circ h_n^*$ is eventually recursive increasing for every h that satisfies (3) ($\circ$ denotes composition of functions). Note that every almost recursive increasing function is necessarily recursive.

Let Λ be the isols, Λ_R the regressive isols, $\Lambda^\infty = \Lambda - \omega$ and $\Lambda_R^\infty = \Lambda_R - \omega$.

THEOREM 1: *Let $n > 0$ and $R \subseteq \times^{n+1}\omega$ the graph of a function r.*

(i) *If r is eventually recursive increasing then for each $x \in \times^n\Lambda_R^\infty$ there is exactly one $y \in \Lambda$ such that $(x, y) \in R_\Lambda$.*

(ii) *If r is almost recursive increasing then for each $x \in \times^n\Lambda_R$ there is exactly one $y \in \Lambda$ such that $(x, y) \in R_\Lambda$.*

We start with a lemma and then discuss the notions that are necessary to complete a proof of Theorem 1.

LEMMA 1. *Let $n > 0$ and $f \colon \times^n\omega \to \omega$.*

(i) *f can be represented as $f(x) = \sum_{i \leqq x} \Delta \hat{f}(i)$ where* i is *understood as ranging over $\times^n\omega$.*

(ii) *If $f(x) = \sum_{i \leqq x} b(i)$ is any other representation of f as a sum then $b(i) = \Delta \hat{f}(i)$ for all $i \in \times^n\omega$.*

PROOF. For $n = 1$ the result is due to Barback [1] and for $n = 2$ it is due to Hassett [6]. For $n = 1$

$$f(0) = \hat{f}(1) - \hat{f}(0) = \Delta \hat{f}(0) = \sum_{i \leqq 0} \Delta \hat{f}(i),$$
$$f(x+1) = f(x) + \Delta f(x) = f(x) + \Delta \hat{f}(x+1)$$

because $\hat{f}(x+1) = f(x)$. (i) for $n = 1$ will then follow by induction. Now assume as hypothesis A that (i) holds in the n-ary case and that $f: \times^{n+1} \omega \to \omega$. Let x, i range over ω and y, j over $\times^n\omega$. If $g(y) = f(0, y)$ then by hypothesis A

$$f(0, y) = g(y) = \sum_{j \leqq y} \Delta_1 \cdots \Delta_n \hat{g}(j).$$

But $\hat{g}(j) = \hat{f}(1, j) = \Delta_0 \hat{f}(0, j)$ since $\hat{f}(0, j) = 0$ and hence

$$f(0, y) = \sum_{i \leqq 0,\, j \leqq y} \Delta_0 \cdots \Delta_n \hat{f}(i, j)$$

since the various Δ_i operators commute. Now assume as hypothesis B that (i) holds for a given $x \in \omega$ and all $y \in \times^n\omega$.

$$f(x+1, y) = f(x, y) + \Delta_0 f(x, y).$$

Now by hypothesis B

$$f(x, y) = \sum_{i \leqq x,\, j \leqq y} \Delta_0 \cdots \Delta_n \hat{f}(i, j)$$

and by hypothesis A if $g(y) = f(x, y)$ then

$$f(x, y) = g(y) = \sum_{j \leqq y} \Delta_1 \cdots \Delta_n \hat{g}(j).$$

But $\hat{g}(j) = \hat{f}(x+1, j)$ and hence

$$\Delta_0 f(x, y) = \sum_{j \leqq y} \Delta_0 \cdots \Delta_n \hat{f}(x+1, j).$$

Summing now shows that (i) holds for $x+1$ and all $y \in \times^n\omega$. Thus by induction (i) holds in the $n+1$-ary case and hence by another induction, for all n.

Uniqueness in the one variable case is trivial because if

$$f(x) = \sum_{i \leqq x} b(i) = \sum_{i \leqq x} c(i)$$

then

$$b(0) = f(0) = c(0), b(i+1) = \Delta f(i) = c(i+1).$$

Now assume as hypothesis C hat (ii) holds in the n-ary case and that $f: \times^{n+1}\omega \to \omega$ is represented as

$$f(x) = \sum_{i \leqq x} b(i) = \sum_{i \leqq x} c(i).$$

Let x, i range over ω and y, j over $\times^n\omega$. If $g(y) = f(0, y)$ then

$$g(y) = f(0, y) = \sum_{j \leqq y} b(0, y) = \sum_{j \leqq y} c(0, j)$$

and hence by hypothesis C we have $b(0,j) = c(0, j)$ for all $j \in \times^n\omega$. If $x+1 \in \omega$ then

$$f(x+1, y) = f(x, y)+ \sum_{j \leqq y} b(x+1, j)$$
$$= f(x, y)+ \sum_{j \leqq y} c(x+1, j).$$

Thus if $g(y) = \Delta_0 f(x, y)$ then

$$g(y) = \sum_{j \leqq y} b(x+1, j) = \sum_{j \leqq y} c(x+1, j)$$

and hence by applying hypothesis C to $g(y)$ we have $b(x+1,j) = c(x+1, j)$ for all $j \in \times^n\omega$. Thus (ii) holds in the $n+1$-ary case and hence by induction for all n. q.e.d.

Next we introduce the various concepts that will appear in the proof of Theorem 1. We start with the notion of a frame. Let P be the set of all subsets of ω and let Q be the set of all finite subsets of ω. F is a *frame* if $F \subseteq \times^n Q$ and F is closed under componentwise intersection (also denoted by $\cap$; in general when discussing frames we extend set theoretic operations and relations componentwise and use the usual symbols to denote these extensions). Let $F^* = \{\alpha \in \times^n Q | (\exists\beta)(\alpha \subseteq \beta \in F)\}$ and let $C_F(\alpha)$ be the minimal with respect to $\subseteq$ element $\beta \in F$ such that $\alpha \subseteq \beta$, provided $\alpha \in F^*$ (such an element exists because F is closed under intersections). $C_F(\alpha)$ is undefined for $\alpha \notin F^*$. F is a *recursive frame* if C_F is a partial recursive function. If $R \subseteq \times^n\omega$ then F is called an *R-frame* if $|\alpha| \in R$ for every $\alpha \in F$. An element $\xi \in \times^n P$ is *attainable* from F (in symbols $\xi \in A(F)$) if for every $\alpha \in \times^n Q$, $\alpha \subseteq \xi$ there is a $\beta \in F$ such that $\alpha \subseteq \beta \subseteq \xi$. For any $\alpha \in \times^n P$ let $\langle\alpha\rangle$ be the recursive equivalence type of α (the componentwise convention is also used here). Then if $R \subseteq \times^n\omega$, the extension R_Λ used in Theorem 1, i.e., the Nerode extension (cf. [7]) is

(5) $$R_\Lambda = \{x \in \times^n\Lambda | (\exists\xi, F)(F \text{ is a recursive } R\text{-frame for which } \xi \in A(F) \text{ and } x = \langle\xi\rangle)\}.$$

The next set of notions that enter into Theorem 1 center on the definition of a regressive function. A function t is called *regressive* if it is a one-one mapping of ω into ω for which there exists a partial recursive function p such that

(6) $$\rho t \subseteq \delta p,\ pt(0) = t(0), \text{ and } pt(n+1) = t(n)$$

for all $n \in \omega$. If t is a regressive function then we can find a partial recursive p which satisfies (6) as well as

(7) $$\rho p \subsetneqq \delta p \text{ and } (\exists n)(p^n(x) = p^{n+1}(x)) \text{ for } x \in \delta p$$

where p^n is the n-th iterate of p. A function p satisfying (6) and (7) is called a *regressing function* of t. Then we define a function p^* by

$$(8) \qquad \delta p^* = \delta p \text{ and } p^*(x) = (\mu n)(p^n(x) = p^{n+1}(x)) \text{ for } x \in \delta p.$$

The function p^* is also partial recursive, $\rho t \subseteq \delta p^*$ and $p^*t(n) = n$ for all $n \in \omega$. We also define a set valued function $\bar{p}$ by

$$(9) \qquad \delta \bar{p} = \delta p \text{ and } \bar{p}(x) = \{p^n(x) | n < p^*(x)\} \text{ for } x \in \delta p.$$

$\bar{p}$ is also partial recursive and in particular $\bar{p}t(0) = \emptyset$ (= the empty set). A function t is called *retraceable* if it is regressive and strictly increasing, and any regressing function p of t which also satisfies

$$(10) \qquad p(x) \leqq x \text{ for all } x \in \delta p$$

is called a *retracing function* of t. A set is called *regressive* if it is finite or the range of a regressive function and is called *retraceable* if it is finite or the range of a retraceable function. An isol is called *regressive* if it contains a regressive set. By [2] every regressive isol contains a retraceable set so there is no need to define retraceable isol. Let Λ_R be the set of all regressive isols.

Let $j : \times^2\omega \to \omega$ be the usual recursive one-one mapping of $\times^2\omega$ onto ω defined by

$$(11) \qquad j(x, y) = x + \tfrac{1}{2}(x+y)(x+y+1)$$

and let k, l be its first, second inverse respectively. If $\alpha, \beta \subseteq \omega$ let $j(\alpha \times \beta) = \{j(x, y) | x \in \alpha \text{ and } y \in \beta\}$. Identify j_2 with j and inductively define $j_n : \times^n\omega \to \omega$, a one-one onto map, by

$$(12) \qquad j_{n+1}(x_0, \cdots, x_n) = j(j_n(x_0, \cdots, x_{n-1}), x_n).$$

For $i < n$ let k_i be the i-th inverse to j_n. We shall also use variations of the obvious notation $j_n(\alpha_0 \times \cdots \times \alpha_{n-1})$, etc.

Proof of Theorem 1: First assume that r is a recursive increasing function so that by Lemma 1 we can represent r as

$$(13) \qquad r(x) = \sum_{i \leqq x} \Delta\hat{r}(i), \ \Delta\hat{r}(i) \geqq 0 \text{ for } i \in \times^n\omega.$$

Let $T = (T_0, \cdots, T_{n-1}) \in \times^n\Lambda_R^\infty$ and choose retraceable functions t_i and retracing functions p_i such that p_i is a retracing function of t_i and $\rho t_i - \{t_i(0)\} = \tau_i \in T_i$ for $i < n$. We use componentwise conventions throughout; in particular we define functions p, p^*, and $\bar{p}$ by $\delta p = \delta p^* = \delta \bar{p} = \delta p_0 \times \cdots \times \delta p_{n-1}$ and if $x \in \delta p$ then

$$(14) \qquad \bar{p}(x) = (\bar{p}_0(x_0), \cdots, \bar{p}_{n-1}(x_{n-1}))$$

and similarly for p and p^*. Also if a, $x \in \times^n\omega$ then define

$$p^a(x) = (p_0^{a_0}(x_0), \cdots, p_{n-1}^{a_{n-1}}(x_{n-1})). \tag{15}$$

Lastly if $x \in \times^n A$ and $y \in A$ write (x, y) for the concatenation of x and y, an element of $\times^{n+1}A$. If $x \in \delta p$ let

$$\xi(x) = \cup \{j(\{j_n(p^a(x))\} \times v(\Delta\hat{r}(b)))|a+b = p^*(x)\} \tag{16}$$

where a, $b \in \times^n\omega$ and $+$ is componentwise addition. $\xi(x)$ makes sense because $\Delta\hat{r}(b) \geqq 0$ for every $b \in \times^n\omega$. Let

$$F = \{(\bar{p}(x), \xi(x))|x \in \delta p \wedge (\exists a)(\forall i < n)p^a(x)_i = t_i(0)\}. \tag{17}$$

We claim that F is a recursive R-frame. If x, $y \in \delta p$ then there exists a, b, $z \in \times^n\omega$ such that $z = p^a(x) = p^b(y)$. Choose such a z so as to be componentwise maximal. First we show that

$$\bar{p}(x) \cap \bar{p}(y) = \bar{p}(z), \tag{18}$$

$$\xi(x) \cap \xi(y) = \xi(z). \tag{19}$$

Re. (18). If $u \in \bar{p}(z)_i$ then $u = p^c(z)_i$ for some $c < p^*(z)$. But $p^*(x) = a+p^*(z)$,

$$p^*(y) = b+p^*(z) \text{ so } u = p^c(z)_i = p^{c+a}(x)_i = p^{c+b}(y)_i$$

and $c+a < p^*(x)$, $c+b < p^*(y)$. Consequently $u \in \bar{p}(x)_i \cap \bar{p}(y)_i$ i.e., $\bar{p}(z) \subseteq \bar{p}(x) \cap \bar{p}(y)$. Conversely if $u \in \bar{p}(x)_i \cap \bar{p}(y)_i$ then by the maximality of z we have $u = p^c(x)_i$ for some $a \leqq c < p^*(x)$. But then

$$u = p^{(c-a)+a}(x)_i = p^{c-a}(z)_i$$

and $c-a < p^*(z)$ because $p^*(x)-a = p^*(z)$. Thus $u \in \bar{p}(z)_i$, i.e., $\bar{p}(x) \cap \bar{p}(y) \subseteq \bar{p}(z)$.

Re. (19). If $u \in \xi(z)$ then $u = j(j_n(p^c(z)), v)$ for some $c \leqq p^*(z)$ and $v < \Delta\hat{r}(p^*(z)-c)$. Then

$$p^c(z) = p^{c+a}(x) = p^{c+b}(y)$$

and $v < \Delta\hat{r}(p^*(x)-(c+a))$, $v < \Delta\hat{r}(p^*(y)-(c+b))$ because $p^*(z)-c = p^*(x)-(c+a) = p^*(y)-(c+b)$. Thus $u \in \xi(x) \cap \xi(y)$, i.e., $\xi(z) \subseteq \xi(x) \cap \xi(y)$. Conversely if $u \in \xi(x) \cap \xi(y)$ then $u = j(j_n(p^c(x)), v)$ for some $c \leqq p^*(x)$ and $v < \Delta\hat{r}(p^*(x)-c)$. But $u \in \xi(y)$ and hence by the maximality of z we have $a \leqq c$. Then $p^c(x) = p^{c-a}(z)$ and $v < \Delta\hat{r}(p^*(z)-(c-a))$ because $p^*(x)-c = p^*(z)-(c-a)$. Thus $u \in \xi(z)$, i.e., $\xi(x) \cap \xi(y) \subseteq \xi(z)$.

By (18) and (19) F is closed under intersection and hence is a frame. Since r is a recursive function, so is $\Delta\hat{r}$, and since p is partial recursive, F

and consequently F^* are r.e. (recursively enumerable). Given $\alpha \in F^*$ there is a unique $x \in \delta p$ which is the componentwise minimal element satisfying $\alpha \subseteq (\bar{p}(x), \xi(x))$. By using (9), (14), and (16) we can effectively analyze the elements of α and find this x. Then the partial recursiveness of $\Delta \hat{r}$ and p allows us to compute $(\bar{p}(x), \xi(x))$. Thus C_F is partial recursive and F is a recursive frame. If $x \in \delta p$ and $p^*(x) = a$ then by (9) and (14) $|\bar{p}(x)| = a$ componentwise, and by computing the cardinality of (16)

$$|\xi(x)| = \sum_{i \leqq a} \Delta \hat{r}(i).$$

Thus by (13) and (17) F is a recursive R-frame.

Recall $\tau_i = \rho t_i - \{t_i(0)\}$ for $i < n$, and let $\tau = (\tau_0, \cdots, \tau_{n-1})$,

$$t(i) = (t_0(i_0), \cdots, t_{n-1}(i_{n-1})) \text{ for } i \in \times^n \omega, \tag{20}$$

$$\xi = \cup \{j(\{j_n(t(i))\} \times v(\Delta \hat{r}(i))) | i \in \times_n \omega\}. \tag{21}$$

Now observe by a standard argument that if ξ contained an infinite r.e. subset then so would $j_n(\tau_0 \times \cdots \times \tau_{n-1})$ which we know is impossible. Thus ξ is an isolated set. If we evaluate $\xi(x)$ for $x = t(a)$, $a \in \times^n \omega$, we obtain

$$\xi(t(a)) = \cup \{j(\{j_n(t(i))\} \times v(\Delta \hat{r}(i))) | i \leqq a\} \tag{22}$$

since $\rho t \subseteq \delta p$ and $p^i(t(a)) = t(a-i)$. By computing $\bar{p}(t(a))$ and using (17) we easily get $(\tau, \xi) \in A(F)$. But $\langle \tau \rangle = T$ so by (5), $(T, \langle \xi \rangle) \in R_\Lambda$. The uniqueness of $\langle \xi \rangle$ follows because $R(x, y) \wedge R(x, y') \rightarrow y = y'$ is a Horn formula valid in ω and hence by [7] is valid in Λ. Now we proceed to consider the case where r is only eventually recursive increasing. Then there is a function $s = \lambda x r(x+m)$ which is recursive increasing for some $m \in \times^n \omega$. Let $S \subseteq \times^{n+1} \omega$ be the graph of s and note that the Horn formula

$$R(v_0 + m_0, \cdots, v_{n-1} + m_{n-1}, \; v_n) \text{ iff } S(v_0, \cdots, v_n)$$

is valid in ω and hence in Λ. Then if $T \in \times^n \Lambda_R^\infty$ and $T - m = (T_0 - m_0, \cdots, T_{n-1} - m_{n-1})$ then $T - m \in \times^n \Lambda_R^\infty$ and we may apply what we have already proved to S. Uniqueness follows as above. This completes our proof of (i). Finally if r is almost recursive increasing and $T \in \times^n \Lambda_R$ let $\sigma = \{i < n | T_i \text{ is finite}\}$ and define h on σ by $h(i) = T_i$. If $\sigma = v(n)$ there is nothing to prove because r is a function. Otherwise let d be as in (3) and note that $r \circ h_n^*$ (cf. (4)) is a d-ary eventually recursive increasing function whose graph is

$$S_h R = \{x \in \times^{d+1} \omega | h_{n+1}^*(x) \in R\}.$$

By [7]

$$S_h(R_\Lambda) = (S_h R)_\Lambda$$

and hence (ii) follows by applying (i) to $S_h R$ and that $x \in \times^d \Lambda_R^\infty$ for which $h_n^*(x) = T$. Uniqueness follows as above. This completes our proof of (ii). q.e.d.

Let Λ_Z be the cosimple isols, i.e., those isols which contain at least one set with r.e. complement,

$$\Lambda_{RZ} = \Lambda_R \cap \Lambda_Z, \Lambda_Z^\infty = \Lambda_Z - \omega,$$

and

$$\Lambda_{RZ}^\infty = \Lambda_{RZ} - \omega.$$

COROLLARY 1. *Let $n > 0$ and $R \subseteq \times^{n+1}\omega$ the graph of a function r.*

(i) *If r is eventually recursive increasing then for each $x \in \times^n \Lambda_{RZ}^\infty$ there is exactly one $y \in \Lambda$ such that $(x, y) \in R_\Lambda$ and moreover this $y \in \Lambda_Z$.*

(ii) *If r is almost recursive increasing then for each $x \in \times^n \Lambda_{RZ}$ there is exactly one $y \in \Lambda$ such that $(x, y) \in R_\Lambda$ and moreover this $y \in \Lambda_Z$.*

PROOF. We show that if r is recursive increasing and $T \in \times^n \Lambda_{RZ}^\infty$ then there is a $y \in \Lambda_Z$ such that $(T, y) \in R_\Lambda$. (i) and (ii) then follow by metatheorems just as in the proof of Theorem 1. By a result of [2] we can find retraceable functions t_i such that ρt_i is cosimple and $\rho t_i - \{t_i(0)\} = \tau_i \in T_i$ for $i < n$. To complete our proof it will suffice to show that ξ as defined in (21) has a r.e. complement. Let k_i for $i < n$ be defined as in (12). Then

$$j(x, y) \in \omega - \xi \text{ iff } (\exists i < n) k_i(x) \in \omega - \rho t_i \quad \text{or } y \geqq \Delta \hat{r}(p^*(x)). \tag{23}$$

Since each $\omega - \rho t_i$ is r.e. (23) defines an r.e. condition and hence $\omega - \xi$ is r.e. q.e.d.

Next we introduce the n-ary minimum function. If $x \in \times^n \omega$ then min (x) is simply the least number in the set $\{x_0, \cdots, x_{n-1}\}$. A function $f : \times^n \omega \to \omega$ is called *recursive regular* if there exist unary recursive increasing functions $g_0, \cdots, g_{n-1}$ such that

$$f(x_0, \cdots, x_{n-1}) = \min (g_0(x_0), \cdots, g_{n-1}(x_{n-1})). \tag{24}$$

We use the convention $g(x) = (g_0(x_0), \cdots, g_{n-1}(x_{n-1}))$ so (24) may be written in the shorter form $f(x) = \min g(x)$. f is called an *eventually recursive regular* function if $\lambda x f(x+m)$ is recursive regular for some $m \in \times^n \omega$. The *essential domain* of a recursive regular function as given in (24) is $\{i < n | \rho g_i \text{ is infinite}\}$, and it is said to be of *essentially n variables* if its essential domain is $\nu(n)$. It is easy to see that the notion of essential domain can be unambiguously extended to eventually recursive regular functions. Finally f is called *almost recursive regular* if $f \circ h_n^*$ is eventually recursive regular for every h that satisfies (3).

THEOREM 2. *Let* $n > 0$ *and* $R \subseteq \times^{n+1}\omega$ *the graph of a function* r.

(i) *If* r *is eventually recursive regular then for each* $x \in \times^n \Lambda_R^\infty$ *there is exactly one* $y \in \Lambda$ *such that* $(x, y) \in R_\Lambda$ *and moreover this* $y \in \Lambda_R$.

(ii) *If* r *is almost recursive regular then for each* $x \in \times^n \Lambda_R$ *there is exactly one* $y \in \Lambda$ *such that* $(x, y) \in R_\Lambda$ *and moreover this* $y \in \Lambda_R$.

PROOF. We start with a detailed analysis of the n-ary min function. First we show that min is recursive increasing. Since the computation is rather nasty we proceed in a roundabout way. Let $b : \times^n\omega \to \omega$ be defined by $b(i) = 1$ if all the components of $i \in \times^n\omega$ are equal to one another but different from 0, and $b(i) = 0$ in all other cases. Let

$$r(x) = \sum_{i \leqq x} b(i) \text{ for } i \in \times^n\omega \tag{25}$$

and observe that

$$r(x) = \sum_{0 < i \leqq \min(x)} 1 = \min(x).$$

Lemma 1 and (25) then give $\Delta\hat{r}(i) = b(i)$ for all $i \in \times^n\omega$, i.e., that min is a recursive increasing function. Let $R \subseteq \times^{n+1}\omega$ be the graph of r. By Theorem 1 for all $x \in \times^n\Lambda_R$ there is exactly one $y \in \Lambda$ such that $(x, y) \in R_\Lambda$. In particular we can apply this result to the T used in the proof of Theorem 1 (we use the same notation as in that proof). We claim that the y such that $(T, y) \in R_\Lambda$ belongs to Λ_R. It suffices to show that ξ as computed in (21) is regressive. For $i \in \omega$ let

$$t(i) = (t_0(i), \cdots, t_{n-1}(i)) \tag{26}$$

and observe that ξ is recursively equivalent to

$$\xi' = \cup \{j_n(t(i)) | 0 < i \in \omega\}. \tag{27}$$

Then

$$q(x) = j_n(p_0k_0(x), \cdots, p_{n-1}k_{n-1}(x))$$

if $x \in \delta p$ and $x \neq j_n(t(1))$ and $qj_n(t(1)) = j_n(t(1))$ is a partial recursive function that retraces ξ'. This takes care of the case when all components of T are infinite. Now assume that at least one component of T is finite and that $T_0 = m$ is the smallest such component. Then the Horn formula

$$R(m, m+v_1, \cdots, m+v_{n-1}, v_n) \to m = v_n$$

is valid in ω and hence in Λ so that the value of y is $m \in \omega \subseteq \Lambda_R$. This proves that Theorem 2 holds for the min function. Now let f be recursive regular as given in (24). Let F be the graph of f, G_i the graph of g_i and R the graph of $r = \min$. By [1] for all $x \in \Lambda_R$ there is exactly one $y \in \Lambda$ such that $(x, y) \in G_{i\Lambda}$ and moreover this $y \in \Lambda_R$. Then the Horn formula

$$G_0(u_0, v_0) \wedge \cdots \wedge G_{n-1}(u_{n-1}, v_{n-1}) \wedge R(v_0, \cdots, v_{n-1}, v_n) \to F(u_0, \cdots, u_{n-1}, v_n)$$

is valid in ω and hence in Λ. This, the same result for r and the g_i, and the fact that f is a function guarantee that for all $x \in \times^n \Lambda_R$ there is a $y \in \Lambda_R$ such that $(x, y) \in F_\Lambda$. (i) and (ii) follow from this result by using metatheorems just as the corresponding parts of Theorem 1 follow from the result for recursive increasing functions. We do not stop to prove this here. q.e.d.

Corollary 1 and Theorem 2 can be combined to give

COROLLARY 2. *Let $n > 0$ and $R \subseteq \times^{n+1}\omega$ the graph of a function r.*

(i) *If r is eventually recursive regular then for each $x \in \times^n \Lambda_{RZ}^\infty$ there is exactly one $y \in \Lambda$ such that $(x, y) \in R_\Lambda$ and moreover this $y \in \Lambda_{RZ}$.*

(ii) *If r is almost recursive regular then for each $x \in \times^n \Lambda_{RZ}$ there is exactly one $y \in \Lambda$ such that $(x, y) \in R_\Lambda$ and moreover this $y \in \Lambda_{RZ}$.*

2. Converse properties

THEOREM 3. *Let $n > 0$ and $R \subseteq \times^{n+1}\omega$ the graph of a function r.*

(i) *If for each $x \in \times^n \Lambda_{RZ}^\infty$ there is a $y \in \Lambda$ such that $(x, y) \in R_\Lambda$ then r is an eventually recursive increasing function.*

(ii) *If for each $x \in \times^n \Lambda_{RZ}$ there is a $y \in \Lambda$ such that $(x, y) \in R_\Lambda$ then r is an almost recursive increasing function.*

The following machinery is necessary in order to complete a proof of Theorem 3.

LEMMA 2. *For any $d > 0$ there exist retraceable functions $t_0, \cdots, t_{d-1}$ such that $\tau_i = \rho t_i$ is cosimple for each $i < d$, and if $\theta = \tau_0 \times \cdots \times \tau_{d-1}$ and p is any partial recursive function whose domain is a subset of $\times^d\omega$ and which assumes values in $\times^d\omega$ such that $\theta \subseteq \delta p$ and $p(\theta) \subseteq \theta$ then there is an $m \in \times^d\omega$ such that $pt(n) \leqq t(n)$ for every $n \in \times^d\omega$ with $m \leqq n$* (cf. (20) *for a definition of $t(n)$, $\leqq$ is used componentwise).*

PROOF. Our proof consists of a priority argument involving finite injuries. Let $q_n(x)$, $n \in \omega$, $x \in \times^d\omega$ be a partial recursive function of $d+1$ variables which with index n enumerates the partial recursive functions of d variables x, which assume values in $\times^d\omega$. Let $q_n^s(x) = y$ if $q_n(x) = y$ is computed in s or fewer steps; otherwise we say that $q_n^s(x)$ is undefined. Throughout this proof the variable i will be restricted to $\nu(d)$ and whenever we assert a statement S_i we shall understand it to mean the assertion of S_0 and $\cdots$ and S_{d-1}. For each $n \in \omega$ let $\mu_i(n)$ be a movable marker. Our proof will consist of a stage by stage construction of the functions $t_i^s(n)$.

Stage 0: Let $t_i^0(0) = 1$ and at the end of this stage no markers are attached to numbers. Then go on to stage 1.

Stage $s+1$: We first state our inductive hypothesis in (28)–(31) below. Assume at the end of stage s that we have defined numbers $t_i^s(n)$ for $n \leqq s$ such that

$$t_i^s(0) = 1,\ kt_i^s(n+1) = t_i^s(n) \text{ for all } n < s \tag{28}$$

(cf. (11) for a definition of k) and that $\mu_i(n)$ is attached to $t_i^s(m_i)$, $m \in \times^d v(s)$ only if (29) and (30) are true. We also assume (31).

$$n \leqq m_i, \tag{29}$$

$$q_n^s t^s(m) = y \text{ is defined and } t_j^s(m_j) < y_j < t_j^s(m_j+1) \text{ for at least one } j < d, \tag{30}$$

$$\text{no other markers are attached to numbers.} \tag{31}$$

Construction: If there is an $n \in \omega$ such that the $\mu_i(n)$ are not attached to numbers at the end of stage s and there is an $m \in \times^d v(s)$ such that

$$n \leqq m_i, \tag{32}$$

$$q_n^{s+1}(t^s(m)) = y \text{ is defined and } t_j^s(m_j+1) \leqq y_j \text{ for at least one } j < d, \tag{33}$$

$$\text{for no } n' < n \text{ and } m_i' \geqq m_i \text{ is there a } \mu_i(n') \text{ attached to a } t_i^s(m_i') \tag{34}$$

then go to step B below; otherwise go to step A.

Step A: Let

$$t_i^{s+1}(x) = t_i^s(x) \text{ for } x \leqq s, \text{ and}$$

$$t_i^{s+1}(s+1) = j(t_i^{s+1}(s), 0). \tag{35}$$

Then go on to stage $s+1$.

Step B: Find the least n and the lexicographically least m which together satisfy (32)–(34). For each $n' > n$ detach the $\mu_i(n')$ from numbers. Attach $\mu_i(n)$ to $t_i^s(m_i)$. Find the least number z_i such that

$$\max(y_i, t_i^s(s)) < j(t_i^s(m_i), z_i). \tag{36}$$

Let

$$t_i^{s+1}(x) = t_i^s(x) \text{ for } x \leqq m_i,$$

$$t_i^{s+1}(m_i+1) = j(t_i^{s+1}(m_i), z_i), \text{ and} \tag{37}$$

$$t_i^{s+1}(x+1) = j(t_i^{s+1}(x), 0) \text{ for } m_i < x \leqq s. \tag{38}$$

Then go on to stage $s+1$.

This completes the construction. Our first claim is that each marker moves finitely often. Assume that for some stage u each $\mu_i(n')$ is never moved after stage u for each $n' < n$ (after stage u the $\mu_i(n')$ may or may not be attached to any numbers). If at any stage $s > u$ the $\mu_i(n)$ are attached to the $t_i^s(m_i)$ then by (34) $\mu_i(n)$ will never move again. This proves our claim. Our second claim is that $\lim_s t_i^s(n)$ exists for every n. If there is a stage u such that markers are not attached to numbers at or after stage u then $t_i^r(n) = t_i^s(n)$ for every $n \leqq r$, $n \leqq s$, $u \leqq r$, $u \leqq s$ by (35). If there is no such u then for infinitely many n the $\mu_i(n)$ reach final positions attached to numbers and if n is such a number and $\mu_i(n)$ is never moved at or after stage u then $t_i^r(x) = t_i^s(x)$ for every $x \leqq n$, $n \leqq r$, $n \leqq s$, $u \leqq r$, $u \leqq s$ by (32). This completes the argument. Now define

(39) $$t_i(n) = \lim_s t_i^s(n), \tau_i = \rho t_i, \theta = \tau_0 \times \cdots \times \tau_{d-1}.$$

We must show that

(40) t_i as a retraceable function,

(41) τ_i has a r.e. complement,

(42) if p is partial recursive and $\theta \subseteq \delta p$, $p(\theta) \subseteq \theta$ then there is an $m \in \times^d\omega$ such that $pt(n) \leqq t(n)$ for every $n \in \times^d\omega$ with $m \leqq n$.

(43) τ_i is immune.

Re. (40). We can show by using our inductive hypothesis and (35), (37) and (38) that there are functions $z_i : \omega \to \omega$ such that

(44) $$t_i(0) = 1, \ t_i(n+1) = j(t_i(n), \ z_i(n)).$$

By the basic properties of the j function t_i is a strictly increasing function which is retraced by $p(x) = k(x)$, $x \neq 1$, $p(1) = 1$.

Re. (41). $u \in \omega - \tau_i$ if and only if

(45) $$(\exists s > u) u \notin \rho t_i^s$$

by (37). But (45) is clearly a r.e. predicate.

Re. (42). If p is a partial recursive function mapping a subset of $\times^d\omega$ into $\times^d\omega$ then $p = q_n$ for some n. Assume that $\theta \subseteq \delta p$ and $p(\theta) \subseteq \theta$. Say that a set $R \subseteq \times^d\omega$ is *totally unbounded* if for every $x \in \times^d\omega$ there is a $y \in R$ with $x \leqq y$. We will show that $R = \{x \in \times^d\omega | \text{not } pt(x) \leqq t(x)\}$ is not totally unbounded. Suppose that it is. Choose $u \in \omega$ such that $\mu_i(n')$, $n' \leqq n$ do not move at or after stage u. Say that $\mu_i(n')$ is attached to numbers *by* stage s if $\mu_i(n')$ was attached to numbers at some stage prior to s and subsequently was not moved. The $\mu_i(n)$ are not attached to numbers by any stage $s \geqq u$ for otherwise (30) would imply that $p(\theta) \nsubseteq \theta$. Choose $m \in \times^d\omega$ such that if $n' < n$ and $\mu_i(n')$ are attached to

$t_i^s(m_i')$ by any stage $s \geqq u$ then $m' \leqq m$. Since R is totally unbounded there is an $m' \in R$ such that $m \leqq m'$ and $n \leqq m_i'$. Choose the lexicographically least such m'. Finally there is a $u' \geqq u$ such that $q_n^s t^s(m') = y$ is defined and $t_i^s(x) = t_i(x)$ for $s \geqq u'$, $x \leqq m_i'+1$. Now the $\mu_i(n)$ are not attached to numbers by stage u' and not $q_n^s t^s(m') \leqq t^s(m')$ for $s \geqq u'$ by our choice of m' and u'. But $p(\theta) \subseteq \theta$ so that $t_j^s(m_j'+1) \leqq y_j$ for $s \geqq u'$ and some $j < d$. Then the $\mu_i(n)$ are attached to numbers at stage $u'+1$ by (32)–(34) which contradicts our choice of u. So R is not totally unbounded which proves (42).

Re. (43). Suppose some τ_j were not immune. Take $j = 0$ for notational convenience. Then by [4] t_0 is a strictly increasing recursive function. Define a partial recursive function p by $p(t_0(n), y) = (t_0(n+1), y)$ for $y \in \times^{d-1}\omega$. But then p violates (42) so τ_0 must be immune. q.e.d.

We need one more bit of notation. Let F be a frame and let $\alpha = (\alpha_0, \cdots, \alpha_{n-1}) \in F^*$ where the α_i need not be the individual components of α (we allow the α_i to be blocks). We define

$$C_F^{\surd}(a_0, \cdots, a_i^{\surd}, \cdots, a_{n-1})$$

(where $\surd$ occurs as a superscript to exactly one α_i) to be that value β_i such that for some $\beta_j, j \neq i$ we have $C_F(\alpha) = (\beta_0, \cdots, \beta_{n-1})$ and $\alpha_j \subseteq \beta_j$ for $j < n$. We say that $(\alpha_0, \cdots, \alpha_{n-1})$ is *unenlarged* by F at α_i if and only if

$$C_F^{\surd}(a_0, \cdots, a_i^{\surd}, \cdots, a_{n-1}) = a_i.$$

Since the α_i generally are blocks this notation allows us to make statements about α without spelling out its individual components.

Proof of Theorem 3: (i) Assume that $R \subseteq \times^{n+1}\omega$ is the graph of a function r and that for all $x \in \times^n \Lambda_{RZ}^{\infty}$ there is a $y \in \Lambda$ such that $(x, y) \in R_\Lambda$. Let $t_0, \cdots, t_{n-1}$ be the functions that were constructed in Lemma 2, $\rho t_i - \{t_i(0)\} = \tau_i \in T_i$, and $T = (T_0, \cdots, T_{n-1})$. In particular there must be a $y \in \Lambda$ such that $(T, y) \in R_\Lambda$. Let $\tau = (\tau_0, \cdots, \tau_{n-1})$ and choose a $\eta \subseteq \omega$ and recursive R-frame F such that $(\tau, \eta) \in A(F)$. We also let p_i be a retracing function of t_i and adhere to the componentwise conventions used for the proof of Theorem 1, in particular to (9), (14), (15), and (20). Let

$$\delta = \{x \in \times^n\omega | x \in \delta p \text{ and } (\bar{p}(x), \emptyset) \in F^*\}. \tag{46}$$

δ is r.e. and hence we may define a partial recursive function q whose domain is δ and which assumes values in $\times^n\omega$ by

$$q(x) = \max C_F^{\surd}(\bar{p}(x)^{\surd}, \emptyset) \tag{47}$$

where the max is taken in the componentwise sense. Also let

$$\theta = \rho t_0 \times \cdots \times \rho t_{n-1}. \tag{48}$$

Then $\theta \subseteq \delta q$ and $q(\theta) \subseteq \theta$ because $(\tau, \eta) \in A(F)$. Hence by Lemma 2 there is an $m \in \times^n\omega$ such that $q(t(i)) \leqq t(i)$ for all $i \geqq m$. This implies that

$$C_F^{\surd}((\bar{p}t(i))^{\surd}, \emptyset) = \bar{p}t(i),$$

i.e., $(\bar{p}t(i), \emptyset)$ is unenlarged by F at $\bar{p}t(i)$ for all $i \geqq m$. Now let

$$A = \{a | (a, \emptyset) \in F^* \text{ and } C_F^{\surd}(a^{\surd}, \emptyset) = a\} \tag{49}$$

and define a function φ with domain A and range $\subseteq Q$ by

$$\varphi(a) = C_F^{\surd}(a, \emptyset^{\surd}). \tag{50}$$

Notice that $\bar{p}t(i) \in A$ for $i \geqq m$, that A is a r.e. family of finite sets and that φ is a partial recursive function. For the moment let α, $\beta \in A$. We claim that

$$\alpha \cap \beta \in A, \tag{51}$$

$$|\alpha| = |\beta| \text{ implies } |\varphi(\alpha)| = |\varphi(\beta)|, \tag{52}$$

$$\varphi(\alpha \cap \beta) = \varphi(\alpha) \cap \varphi(\beta). \tag{53}$$

Re. (51) $(\alpha, \emptyset) \in F^*$, $(\beta, \emptyset) \in F^*$ and hence $(\alpha \cap \beta, \emptyset) \in F^*$.

$$a \cap \beta \subseteqq C_F^{\surd}((a \cap \beta)^{\surd}, \emptyset) \subseteqq C_F^{\surd}(a^{\surd}, \emptyset) \cap C_F^{\surd}(\beta^{\surd}, \emptyset) = a \cap \beta.$$

Re. (52). $(\alpha, \varphi(\alpha)) \in F$, $(\beta, \varphi(\beta)) \in F$ and since $|\alpha| = |\beta|$ and R is the graph of a function we have $|\varphi(\alpha)| = |\varphi(\beta)|$.

Re. (53).

$$\varphi(a \cap \beta) = C_F^{\surd}(a \cap \beta, \emptyset^{\surd}) \subseteqq C_F^{\surd}(a, \emptyset^{\surd}) \cap C_F^{\surd}(\beta, \emptyset^{\surd}) = \varphi(a) \cap \varphi(\beta)$$

so $\varphi(\alpha \cap \beta) \subseteq \varphi(\alpha) \cap \varphi(\beta)$. Now $(\alpha, \varphi(\alpha)) \in F$ and $(\beta, \varphi(\beta)) \in F$ so $(\alpha \cap \beta, \varphi(\alpha) \cap \varphi(\beta)) \in F$. But R is the graph of a function and consequently $\varphi(\alpha \cap \beta) = \varphi(\alpha) \cap \varphi(\beta)$.

Consider any a, $b \in \times^n\omega$ with $a \geqq m$ and $a_i + 1 = b_i$ for $i < n$. Let $\alpha = \bar{p}t(a)$, $\beta = \bar{p}t(b)$ and note that $\alpha \subseteq \beta$, $|\alpha| = a$, $|\beta| = b$, and $\alpha, \beta \in A$. Define G and ψ on G by

$$G = \{\gamma - \alpha | \alpha \subseteq \gamma \subseteq \beta\}, \ \psi(\gamma) = \varphi(\alpha \cup \gamma). \tag{54}$$

It follows immediately from (51)–(54) that

(55) G is closed under intersection, $\psi(\gamma \cap \gamma') = \psi(\gamma) \cap \psi(\gamma')$, and if $|\gamma| = |\gamma'|$ then $|\psi(\gamma)| = |\psi(\gamma')|$.

Thus G is a frame and ψ is a frame map which induces a numerical function. Next define the *essential* ψ^* on G by

$$\psi^*(\gamma) = \psi(\gamma) - \cup \{\psi(\gamma') | \gamma' \in G \text{ and } \gamma' \subseteq \gamma, \gamma' \neq \gamma\}. \tag{56}$$

By well-known methods (cf. 5.8 of [7]) (55) implies that

(57) if $\gamma \neq \gamma'$ then $\psi^*(\gamma) \cap \psi^*(\gamma') = \emptyset$,

(58) if $|\gamma| = |\gamma'|$ then $|\psi^*(\gamma)| = |\psi^*(\gamma')|$,

(59) $\psi(\gamma) = \cup \{\psi^*(\gamma') | \gamma' \in G \text{ and } \gamma' \subseteq \gamma\}$.

Now define $c : \times^n\omega \to \omega$ by $c(x) = |\psi^*(\gamma)|$ if $x = |\gamma|$ for some $\gamma \in G$ and $c(x) = 0$ otherwise. $c(x)$ is well defined by (58). Let

$$f(x) = \sum_{i \leqq x} c(i)\binom{x}{i}$$

for $x \in \times^n\omega$ and summation on a variable $i \in \times^n\omega$. Since $(\alpha \cup \gamma, \psi(\gamma)) \in F$ for $\gamma \in G$, (57) and (59) give $r(a+x) = f(x)$ for $x \in \times^n\{0, 1\}$. If $\sigma \subseteq v(n)$ let $\Delta_{i\in\sigma}$ be the composition of the Δ_i for $i \in \sigma$. Then by well-known properties of combinatorial functions (cf. [7])

(60) $$\Delta_{i\in\sigma} r(a) = \Delta_{i\in\sigma} f(0) = c(x) \text{ where } x \in \times^n\omega \text{ and } x_i = 1 \text{ for } i \in \sigma,\ x_i = 0 \text{ for } i \notin \sigma.$$

Thus $\Delta_{i\in\sigma} r(m+x) \geqq 0$ for every $x \in \times^n\omega$. Let $s = \lambda x r(x+m)$. We show that s is recursive increasing. Consider any $x \in \times^n\omega$. Let $\sigma = \{i | x_i \neq 0\}$ and $y \in \times^n\omega$, $y_i = x_i$ for $i \in \sigma$ and $y_i = 1$ for $i \notin \sigma$. Thus

$$\Delta_{i\notin\sigma} \hat{s}(x) = \hat{s}(y)$$

by (1). Let $z \in \times^n\omega$, $y_i = z_i + 1$ for $i < n$. Then

$$\Delta\hat{s}(x) = \Delta_{i\in\sigma}\hat{s}(y) = \Delta_{i\in\sigma} s(z) = \Delta_{i\in\sigma} r(m+z) \geqq 0$$

and s is increasing. Let $B = \{(a, b) | a \in \times^n\omega,\ b \in \omega \text{ and } (\exists\alpha, \beta)(a = |\alpha|, b = |\beta| \text{ and } (\alpha, \beta) \in F)\}$. Then B is a r.e. set which makes s a recursive function. Thus r is eventually recursive increasing which proves (i). In order to prove (ii) let h, σ, d and j satisfy (3). Then

$$S_h R = \{x \in \times^{d+1}\omega | h^*_{n+1}(x) \in R\}$$

is the graph of a d-ary function $s = r \circ h^*_n$ and since $S_h(R_\Lambda) = (S_h R)_\Lambda$ we easily see that $S_h R$ satisfies the hypothesis of (i). Then s is eventually recursive increasing by (i) and r is almost recursive increasing. q.e.d.

There is an easy consequence of this material which should be stated. First we need some definitions. A set S of regressive isols is *strongly recursively independent* if for any $n > 0$, any distinct elements $x_0, \cdots, x_{n-1} \in S$, and for any $R \subseteq \times^{n+1}\omega$ the graph of a function r, if there is a $y \in \Lambda$ such that $(x_0, \cdots, x_{n-1}, y) \in R_\Lambda$ then r is eventually recursive increasing. A set S of cosimple isols is *r.e.* if it can be arranged in a sequence $x_0, x_1, \cdots$ such that for some recursive functions f, for each n,

$f(n)$ is the r.e. index of an r.e. set whose complement is in x_n. We state without proof the following

COROLLARY 3. *There exists an infinite r.e. set* $S \subseteq \Lambda_{RZ}$ *which is strongly recursively independent*

THEOREM 4. *Let* $n > 0$ *and* $R \subseteq \times^{n+1}\omega$ *the graph of a function* r.

(i) *If for each* $x \in \times^n\Lambda_{RZ}^\infty$ *there is a* $y \in \Lambda_R$ *such that* $(x, y) \in R_\Lambda$ *then* r, *is an eventually recursive regular function.*

(ii) *If for each* $x \in \times^n\Lambda_{RZ}$ *there is a* $y \in \Lambda_R$ *such that* $(x, y) \in R_\Lambda$ *then* r *is an almost recursive regular function.*

Several notions and lemmas are necessary in order to complete a proof of Theorem 4. In [3] a Turing degree $D(x)$ is associated with every $x \in \Lambda_R$. It is the (unique) degree $D(\xi)$ of any retraceable set $\xi \in x$. In case $x \in \Lambda_{RZ}$ we know from [4] that x contains a retraceable set with r.e. complement. Thus in this case $D(x)$ is r.e. degree. For us the most important property of this association is that (cf. [3])

$$\text{if } x, y \in \Lambda_R^\infty \text{ and } x \leqq y \text{ then } D(x) = D(y). \tag{61}$$

Let $g_0, \cdots, g_{n-1}$ be a sequence of unary functions. We say that $g = (g_0, \cdots, g_{n-1})$ is *jointly stricly increasing* if each g_i is increasing, has an infinite range, and $g(x) \neq g(x+1)$ for each $x \in \omega$. We state without proof the obvious

LEMMA 3. *Let* $t_0, \cdots, t_{n-1}$ *be retraceable functions and let* $g_0, \cdots, g_{n-1}$ *be jointly strictly increasing recursive functions. Then* $\xi = \{j_n(tg(i)) | i \in \omega\}$ *is retraceable and the degree of* ξ *is the least upper bound of* $\{D(\rho t_i) | i < n\}$.

LEMMA 4. *Let* $n > 0$ *and* $R \subseteq \times^{n+1}\omega$ *the graph of a function* r. *If* r *is recursive increasing, for all* $x \in \times^n\Lambda_{RZ}^\infty$ *there is a* $y \in \Lambda_R$ *such that* $(x,y) \in R_\Lambda$, *and* $B = \{i \in \times^n\omega | \Delta\hat{r}(i) > 0\}$ *is totally unbounded then* (i) there is a finite set $B_0 \subseteq B$ *and jointly strictly increasing recursive functions* $g_0, \cdots, g_{n-1}$ *such that* $B - B_0 = \{g(i) | i \in \omega\}$, *and* (ii) r *is an eventually recursive regular function of essentially* n *variables.*

PROOF. Let $T \in \times^n\Lambda_{RZ}^\infty$ and $t_0, \cdots, t_{n-1}$ retraceable functions with $\rho t_i - \{t_i(0)\} = \tau_i \in T_i$. Since r is recursive increasing we know from Theorem 1 that there is a $y \in \Lambda$ such that $(T, y) \in R_\Lambda$ and moreover that $\xi \in y$ where ξ is given by (21). We also know that ξ is regressive because $y \in \Lambda_R$. Now consider the set

$$\xi_1 = \{j(j_n(t(i)), 0) | i \in B\}. \tag{62}$$

We see that $\xi_1 \subseteq \xi$, ξ_1 is infinite, and ξ_1 is recursively separable from $\xi - \xi_1$ by the recursive set $\{x | l(x) = 0\}$. Recall that l is the second inverse of j. Hence $\langle \xi_1 \rangle \leqq \langle \xi \rangle$ and since Λ_R is closed under predecessors, ξ_1 is an infinite regressive set, regressed say by a partial recursive function q. With q we can associate a partial recursive function q' mapping a subset of $\times^n\omega$ into $\times^n\omega$ by

$$q'(x) = y \text{ iff } q(j(j_n(x), 0)) = j(j_n(y), 0). \tag{63}$$

Let p_i retrace t_i and adhere to the componentwise conventions given in (14). Since B is a recursive set we can extend q' to a partial recursive function q_1 by

$$\begin{aligned} q_1(x) &= q'(x) \text{ if } x \in \delta p \cap \delta q' \text{ and } p^*(x) \in B \\ q_1(x) &= x \qquad \text{if } x \in \delta p \text{ and } p^*(x) \notin B. \end{aligned} \tag{64}$$

Let θ be as in (48). We readily see that q_1 agrees with q' on $\{t(i)|i \in B\}$, $\theta \subseteq \delta q_1$ and $q_1(\theta) \subseteq \theta$. We are now in a position to use Lemma 2 for the t_i constructed in that lemma and conclude that there is an $m \in \times^n\omega$ such that $q_1(t(i)) \leqq t(i)$ for all $i \geqq m$. In particular this implies that $q'(t(i)) \leqq t(i)$ for all $i \in B_1 = \{i \in B | i \geqq m\}$. But then there exist jointly strictly increasing functions $g_0, \cdots, g_{n-1}$ such that $B_1 = \{g(i)|i \in \omega\}$ (it is best to see this by ones self). Since B_1 is recursive so are the g_i. Let $B_0 = B - B_1$. We claim that B_0 is a finite set. Let

$$\xi_1' = \{j(j_n(t(i)), 0)|i \in B_1\}. \tag{65}$$

Now $\xi_1' \subseteq \xi_1$, ξ_1' is infinite, and ξ_1' is readily seen to be recursively separable from $\xi_1 - \xi_1'$. Hence $\langle \xi_1' \rangle \leqq \langle \xi_1 \rangle$, so ξ_1' is regressive, and by (61) has the same degree as ξ_1. By using Lemma 3 and the form of B_1 above, we compute the degree of ξ_1' (and hence of ξ_1) as the least upper bound of $\{D(\tau_i)|i < n\}$. Now assume that B_0 is infinite. Since B_0 is not totally unbounded, by a simple numerical argument (used for example in [7]) there must be a non-empty $\sigma \subsetneqq \nu(n)$, $|\nu(n) - \sigma| = d$ and $h : \sigma \to \omega$ such that $S_h B_0$ is a totally unbounded subset of $\times^d\omega$. For notational ease assume that h specifies the last $n-d$ arguments of B_0. Then we can find an $a \in \times^{n-d}\omega$ such that

$$B_0' = \{i \in \times^d\omega | (i, a) \in B_0\} \tag{66}$$

is totally unbounded. Define

$$\xi_0' = \{j(j_n(t(i, a)), 0)|i \in B_0'\}. \tag{67}$$

Now $\xi_0' \subseteq \xi_1$, ξ_0' is infinite, and we readily see that ξ_0' is recursively separable from $\xi_1 - \xi_0'$. Hence ξ_0' is regressive and has the same degree as ξ_1. Now apply the same line of reasoning that was used to compute the degree of ξ_1 to ξ_0'. Since the situations are essentially the same we can compute the degree of ξ_0' (and hence of ξ_1) as the least upper bound of

$\{D(\tau_i)|i < d\}$. Also by [4] we know that for each non-recursive r.e. degree there is a cosimple retraceable set of that degree. Choose cosimple retraceable τ_i such that $\{D(\tau_i)|i < d\}$ and $\{D(\tau_i)|i < n\}$ have distinct least upper bounds and evaluate ξ_1 for this choice of τ_i. Then we obtain two contradictory values for the degree of ξ_1 which proves that B_0 must be finite. This completes our proof of (i). For (ii) we may assume without loss of generality that the $m \in \times^n\omega$ and the jointly strictly increasing functions $g_0, \cdots, g_{n-1}$ of (i) have been chosen so that

$$B_0 \subseteq \{i \in \times^n\omega | i \leqq m\}, B_1 \subseteq \{i \in \times^n\omega | m \leqq i, m \neq i\}. \tag{68}$$

Next we define a number $c \in \omega$ and for each $d < n$ unary functions r_d by

$$\begin{aligned} c &= \sum_{i \leqq m} \Delta\hat{r}(i), \text{ and for } x \in \omega, \\ r_d(x) &= c + \sum_{g_d(i) \leqq x + m_d} \Delta\hat{r}(g(i)). \end{aligned} \tag{69}$$

Then if $x \in \times^n\omega$ we use Lemma 1, (68) and (69) to get

$$r(x+m) = c + \sum_{g(i) \leqq x+m} \Delta\hat{r}(g(i)) = \min(r_0(x_0), \cdots, r_{n-1}(x_{n-1})).$$

Since g_d is recursive so is r_d and since g_d has an infinite range so does r_d. Clearly r_d is an increasing function which implies that $s = \lambda x r(x+m)$ is recursive regular, i.e., r is an eventually recursive regular function. q.e.d.

The purpose of Lemma 4 was as much to exposit the method of degrees as it was to obtain a specific result. In the following proof certain degree computations will be omitted when it is obvious from the proof of Lemma 4 how to obtain them.

Proof of Theorem 4. We start with the case where r is a recursive increasing function. Let $B = \{i \in \times^n\omega | \Delta\hat{r}(i) > 0\}$. If B is totally unbounded then r is eventually recursive regular by Lemma 4 so there is nothing to do, and if B is finite the result is trivial. From now on assume that B is infinite but not totally unbounded. Then by induction we can show that there is a number $m \in \omega$ and B_i, B'_i, h_i, σ_i, d_i for $i < m$ such that $\{B_i | i < m\}$ is a partition[1] of B into disjoint subsets, h_i, σ_i, d_i satisfy (3), $B'_i \subseteq \times^{d_i}\omega$ is totally unbounded, and $B_i = \{(h_i)^*_n(x) | x \in B'_i\}$. Let t_i, τ_i, T_i and ξ be as in the proof of Lemma 4. We shall use most of the notation of Lemma 4 except for the new name

$$\bar{\xi} = \{j(j_n(t(i)), 0) | i \in B\}. \tag{70}$$

Then just as before $\bar{\xi}$ is an infinite regressive subset of ξ. For each $x < m$ let

$$\xi_x = \{j(j_n(t(i)), 0) | i \in B_x\}. \tag{71}$$

[1] Except for an unimportant finite set.

Using methods of Lemma 4 we can show that ξ_x is an infinite subset of ξ and that ξ_x, $\xi - \xi_x$ are recursively separable. Thus ξ_x is regressive and has the same degree as ξ. Reproduce enough of the proof of Lemma 4 to compute the degree of ξ_x as the least upper bound of $\{D(\tau_i)|i \in \nu(n) - \sigma_x\}$ and conclude that the σ_x for $x < m$ are the same. Assume now for notational ease that there is a number d such that h_x specifies the last $n-d$ components of B_x. We can then find $a_x \in \times^{n-d}\omega$ such that for each $x < m$

$$B_x = \{(i, a_x)|i \in B'_x\},$$
$$\xi_x = \{j(j_n(t(i, a_x)), 0)|i \in B'_x\}. \tag{72}$$

Now define

$$c : \times^d\omega \to \omega \text{ by } c(i) = \sum_{x<m} \Delta\hat{r}(i, a_x),$$

$$s(x) = \sum_{i \leq x} c(i) \text{ for } x \in \times^d\omega, \text{ and} \tag{73}$$

$$\eta = \cup \{j(\{j_d(t(i))\} \times \nu(c(i)))|i \in \times^d\omega\},$$

where

$$t(i) = (t_0(i_0), \cdots, t_{d-1}(i_{d-1})) \text{ for } i \in \times^d\omega.$$

Now B is recursive, hence the B_x are recursive, hence the B'_x are recursive, hence c is recursive, hence s is recursive, and s is also increasing since $\Delta\hat{s}(i) = c(i) \geqq 0$ for $i \in \times^d\omega$. Let S be the graph of s. Then as in the proof of Theorem 1 we know $(\tau, \eta) \in A(G)$ for some recursive S-frame G. Now $C = \{i \in \times^d\omega|\Delta\hat{s}(i) > 0\} = \{i \in \times^d\omega|(\exists x < m) i \in B'_x\}$ is clearly totally unbounded so we may apply Lemma 4 to s once we know that

$$(\forall x \in \times^d\Lambda^{\infty}_{RZ})(\exists y \in \Lambda_R)(x, y) \in S_\Lambda.$$

But c is a recursive function and consequently ξ and η are recursively equivalent (again it is best to see this by ones self). Hence s is an eventually recursive regular function of essentially d variables. Choose $a \in \times^{n-d}\omega$ so as to satisfy $a \geqq a_x$ for each $x < m$. Then $r(x, y+a) = s(x)$ for $x \in \times^d\omega$, $y \in \times^{n-d}\omega$ which proves that r is an eventually recursive regular function whose essential domain is $\nu(d)$. To obtain (i) of Theorem 4 we note that if r meets the hypothesis of (i) then $s = \lambda x r(x+m)$ for some $m \in \times^n\omega$ also meets these hypotheses and moreover is a recursive increasing function. Apply what we have just proved so as to conclude that s is eventually recursive regular and hence so is r. (ii) is proved in the same way. q.e.d.

We summarize our results below. For each function $r : \times^n\omega \to \omega$ we let the Nerode extension r_Λ be defined by $r_\Lambda(x) = y$ iff $x \in \times^n\Lambda$, $y \in \Lambda$ and $(x, y) \in R_\Lambda$ where R is the graph of r. Thus r_Λ may or may not be defined on all of $\times^n \Lambda$. By Theorems 1 and 3 r_Λ maps $\times^n \Lambda_R$ into Λ iff r_Λ maps $\times^n \Lambda_{RZ}$ into Λ_Z iff r is almost recursive increasing, and by Theorems 2 and 4 r_Λ maps $\times^n \Lambda_R$ into Λ_R iff r_Λ maps $\times^n \Lambda_{RZ}$ into Λ_{RZ} iff r is almost recursive regular. To complete the picture we quote some results of [5] and [7]. r_Λ maps $\times^n \Lambda$ into Λ iff r_Λ maps $\times^n \Lambda_Z$ into Λ_Z iff r is almost recursive combinatorial.

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