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NUCLEAR SPACES OF MAXIMAL DIAMETRAL DIMENSION

by

Christian Fenske and Eberhard Schock

The diametral dimension $\Delta(E)$ of a locally convex vector space E is known to be a measure for the nuclearity of E . Therefore it is of interest to characterize the class Ω of those locally convex vector spaces, the diametral dimension of which is maximal. We show that the class Ω has the same stability properties as the class \mathcal{N} of all nuclear spaces, and characterize the members of Ω , that are contained in the smallest stability class, by a property of their bornology. At first let us define what we mean by a stability class:

DEFINITION. (a) A *stability class* is a class of locally convex vector spaces, which is closed under the operations of forming

- (S₁) completions
- (S₂) subspaces
- (S₃) quotients by closed subspaces
- (S₄) arbitrary products
- (S₅) countable direct sums
- (S₆) tensor products
- (S₇) isomorphic images.

(b) If E is a locally convex vector space, we denote by $\sigma(E)$, the *stability class of E* , the smallest stability class containing E .

Let us remark, that in (S₆) we choose the projective (π -) topology on the tensor product; but as we shall be solely concerned with nuclear spaces, we could equally well have chosen the (ε -) topology of bi-equi-continuous convergence. Note further, that because of (S₄) a stability class (if not empty) will always be a proper class.

Examples of stability classes are

the class \mathcal{S} of all Schwartz spaces (cf. e.g. [4]),
the class \mathcal{N} of nuclear spaces, or more generally,
the class \mathcal{N}_ϕ of ϕ -nuclear spaces, which are defined as follows:

DEFINITION. Let Φ denote the set of all continuous, subadditive, strictly increasing functions $\phi : [0, \infty) \rightarrow [0, \infty)$ vanishing at 0. Let $\phi \in \Phi$; a locally convex vector space E is a member of \mathcal{N}_ϕ , the class of

ϕ -nuclear spaces, if for every neighbourhood U of 0 in E there is a neighbourhood V of 0 contained in U , such that $\sum \phi(\delta_n(V, U)) < \infty$, where $\delta_n(V, U)$ denotes the n -th Kolmogorov diameter of V with respect to U [1].

By a theorem of Rosenberger [6] for $\phi \in \Phi$ \mathcal{N}_ϕ is a stability class, and so is

$$\mathcal{N}_\Phi := \bigcap_{\phi \in \Phi} \mathcal{N}_\phi.$$

A further example of a stability class is given by the spaces of maximal diametral dimension:

DEFINITION. If E is a locally convex space, we denote by $\Delta(E)$, the diametral dimension of E , the set of all nonnegative sequences δ , such that for each neighbourhood U of 0 in E there is a neighbourhood V of 0 contained in U , such that

$$\lim_{n \rightarrow \infty} \delta_n(V, U) \cdot \delta_n^{-1} = 0$$

(cf. [1]).

Call ω the set of all strictly positive non-increasing sequences of real numbers, and Ω the class of all locally convex vector spaces, such that $\omega \subset \Delta(E)$. The following proposition will show, that Ω is a stability class of nuclear spaces.

PROPOSITION. $\mathcal{N}_\Phi = \Omega$.

PROOF. (a) $\mathcal{N}_\Phi \subset \Omega$: Let $E \in \mathcal{N}_\Phi$, $\delta \in \omega$, U a neighbourhood of 0 in E . Choose $\phi \in \Phi$, such that for all $n \in \mathbb{N}$ $\phi(\delta_n) > 1/(n+1)$: such a function may be obtained by considering the ‘upper boundary’ of the closed convex hull of $\{(0, 0)\} \cup \{(\delta_n, 1/(n+1)); n \in \mathbb{N}\}$. As ϕ is subadditive, $\phi(\delta_n/(n+1)) > \phi(\delta_n)/(n+1) > 1/(n+1)^2$. If $\phi \in \Phi$, $\sqrt{\phi}$ will also be in Φ , so there is a neighbourhood W of 0 such that $\sum \sqrt{\phi(\delta_n(W, U))} < \infty$, hence

$$\lim_{n \rightarrow \infty} (n+1)^2 \phi(\delta_n(W, U)) = 0,$$

and so we may find a neighbourhood V of 0, such that for all $n \in \mathbb{N}$ $\phi(\delta_n(V, U)) < (n+1)^{-2} < \phi(\delta_n/(n+1))$, which means

$$\lim_{n \rightarrow \infty} \delta_n(V, U) \delta_n^{-1} = 0$$

and consequently $\delta \in \Delta(E)$.

(b) $\Omega \subset \mathcal{N}_\Phi$: Let $E \in \Omega$, $\phi \in \Phi$, U a neighbourhood of 0 in E . Choose a neighbourhood V of 0, such that $\delta_n(V, U) < \phi^{-1}(1/(n+1)^2)$.

COROLLARY. Ω is a stability class.

We shall now be investigating the smallest nontrivial stability class, i.e. $\sigma(\mathbf{R})$.

THEOREM. *A locally convex vector space is in $\sigma(\mathbf{R})$, if and only if E is isomorphic to a subspace of a product of $\bigoplus_{\mathbf{N}} \mathbf{R}$.*

PROOF¹. It suffices to prove the ‘only if’ part. We introduce an auxiliary class Σ as follows: A locally convex vector space E belongs to Σ , if E possesses a basis $\mathcal{U}(E)$ of neighbourhoods of 0, such that for all $U \in \mathcal{U}(E)$ $E/\ker p_U$ (with the quotient topology) is isomorphic to a subspace of $\bigoplus_{\mathbf{N}} \mathbf{R}$, where p_U denotes the seminorm associated with U . Note that a subspace of $\bigoplus_{\mathbf{N}} \mathbf{R}$ is again an at most countable sum of real lines. Note further, that E is a subspace of a product of $\bigoplus_{\mathbf{N}} \mathbf{R}$ if and only if $E \in \Sigma$. For, suppose E is subspace of

$$X := \left(\bigoplus_{\mathbf{N}} \mathbf{R} \right)^A$$

and choose a neighbourhood U_0 of 0 in X , such that

$$X/\ker p_{U_0} = \bigoplus_{\mathbf{N}} \mathbf{R}.$$

Let $U = U_0 \cap E$, then $\ker p_U = \ker p_{U_0} \cap E$, and we have a continuous injection $i : E/\ker p_U \rightarrow X/\ker p_{U_0}$. But $X/\ker p_{U_0}$ carries the finest locally convex topology, so $E/\ker p_U$ is itself an at most countable sum of real lines. The theorem will be proven, if we show, that the class of locally convex spaces which are subspaces of a product of $\bigoplus_{\mathbf{N}} \mathbf{R}$ is a stability class. So let us check conditions (S₁) to (S₆):

(S₁): If E is a subspace of $(\bigoplus_{\mathbf{N}} \mathbf{R})^A$, then \tilde{E} is just the closure of E in $(\bigoplus_{\mathbf{N}} \mathbf{R})^A$, since $(\bigoplus_{\mathbf{N}} \mathbf{R})^A$ is complete.

(S₂): If E is a subspace of $(\bigoplus_{\mathbf{N}} \mathbf{R})^A$, and F is a subspace of E , then clearly F is a subspace of $(\bigoplus_{\mathbf{N}} \mathbf{R})^A$.

(S₃): Let F be a closed subspace of $E \in \Sigma, U \in \mathcal{U}(E), \pi_F : E \rightarrow E/F, \pi_V : E/F \rightarrow (E/F)/\ker p_{\pi_F(U)}, \pi_U : E \rightarrow E/\ker p_U$ canonical projections, $V := \pi_F(U)$. We shall show, that any seminorm $q : (E/F)/\ker p_V \rightarrow \mathbf{R}$ is continuous. $W := q^{-1}([0, 1])$ is absorbing and absolutely convex, and so is $\pi_U \pi_F^{-1} \pi_V^{-1}(W)$. But since $E/\ker p_U$ is isomorphic to a subspace of $\bigoplus_{\mathbf{N}} \mathbf{R}, \pi_U \pi_F^{-1} \pi_V^{-1}(W)$ contains an open neighbourhood \mathcal{O} of 0. Then the open neighbourhood $\pi_V \pi_F \pi_U^{-1}(\mathcal{O})$ will be contained in $W + \pi_V \pi_F(\ker p_U)$; and since $F + \ker p_U \subset \ker p_V, \pi_V \pi_F(\ker p_U) = 0$.

¹ We are very grateful to the referee for drawing our attention to a slip in the first version of this proof.

So W is indeed a neighbourhood of 0 . As $(E/F)/\ker p_V$ is a nuclear space carrying the finest locally convex topology, it must be isomorphic to a subspace of $\bigoplus_N \mathbf{R}$.

(S₄) and (S₅) are clear.

(S₆): Let

$$E, F \in \Sigma, U \in \mathcal{U}(E), V \in \mathcal{U}(F), \pi_U : E \rightarrow E/\ker p_U,$$

$$\pi_V : E \rightarrow E/\ker p_V, \rho : E \otimes_\pi F \rightarrow (E \otimes_\pi F)/\ker (p_U \otimes p_V)$$

canonical projections. Consider a seminorm q on $E \otimes_\pi F/\ker (p_U \otimes p_V)$. Then $W := q^{-1}([0, 1])$ is absorbing and absolutely convex; hence $(\pi_U \otimes \pi_V)\rho^{-1}(W)$ is an absorbing and absolutely convex set in

$$E/\ker p_U \otimes_\pi F/\ker p_V \cong \bigoplus_{N'} \mathbf{R} \otimes_\pi \bigoplus_{N''} \mathbf{R} = \bigoplus_{N' \times N''} \mathbf{R},$$

where N' and N'' are at most countable. So $(\pi_U \otimes \pi_V)\rho^{-1}(W)$ contains an open neighbourhood \mathcal{O} of 0 , and so does W , since the open neighbourhood $\rho(\pi_U \otimes \pi_V)^{-1}(\mathcal{O})$ is contained in $W + \rho(\ker \pi_U \otimes \pi_V)$ and $\rho(\ker \pi_U \otimes \pi_V) = 0$. That means, that $E \otimes_\pi F/\ker p_U \otimes p_V$ carries the finest locally convex topology, so it is again a subspace of $\bigoplus_N \mathbf{R}$.

REMARK. Diestel, Morris and Saxon [2] define a ‘variety’ of locally convex spaces as a class, which is closed under the operations (S₂), (S₃), (S₄), and (S₇). They show, that $\sigma(\mathbf{R})$ is the second smallest variety.

At this stage the question naturally arises, whether Ω actually equals $\sigma(\mathbf{R})$. One feels that this should be true, if the diametral dimension of a space is indeed a measure for its ‘nuclearity’, for this would mean, that maximal diametral dimension should determine the smallest stability class. On the other hand, the following proposition may perhaps provide a method to refute the equality $\sigma(\mathbf{R}) = \Omega$.

PROPOSITION. *Let $E \in \Omega$. Then $E \in \sigma(\mathbf{R})$, if and only if E has the following property*

(PB) *there is a basis \mathcal{U} of neighbourhoods of 0 in E , such that for all $U \in \mathcal{U}$ $E/\ker p_U$ is bornological.*

PROOF. If $E \in \sigma(\mathbf{R}) = \Sigma$, E clearly has property (PB), since an at most countable sum of real lines is bornological. Conversely, let $E \in \Omega$ and $U \in \mathcal{U}$. Now note, that bounded sets in $E/\ker p_U$ are finite-dimensional. For, if B is bounded in $E/\ker p_U$, for each non-increasing sequence δ of positive reals

$$\lim_{n \rightarrow \infty} \delta_n^{-1} \delta_n(B, U_0) = 0$$

(where U_0 is the image of U in $E/\ker p_U$), since $E/\ker p_U$ is in Ω , too. This

means, that $\delta_n(B, U_0) = 0$ for $n > n_0$, which implies, that B is finite-dimensional, since p_{U_0} is a norm. Now choose an algebraic basis

$$(e_i)_{i \in I} \text{ for } E/\ker p_U.$$

The identity map

$$\text{id} : \bigoplus_I \mathbf{R} \cdot e_i \rightarrow E/\ker p_U$$

is a continuous bijection; and if $E/\ker p_U$ is bornological, id will be open, too, hence an isomorphism. But then I must be at most countable, since $E/\ker p_U$ is nuclear.

Taking a different approach, one could try to prove the equality $\sigma(\mathbf{R}) = \Omega$ by showing, that $\sigma(\mathbf{R})$ and Ω are generated by the same ideal of operators. This is, however, not possible, since the equality $\Omega = \mathcal{N}_\Phi$ implies, that neither $\sigma(\mathbf{R})$ nor Ω is generated by an ideal:

DEFINITION. Let \mathcal{I} be an ideal of operators. *The class of locally convex vector spaces generated by \mathcal{I}* consists of all locally convex vector spaces E with the following property: For each neighbourhood U of 0 in E there is a neighbourhood V of 0 contained in U , such that the canonical map $E(V, U) : E_V \rightarrow E_U$ belongs to \mathcal{I} (E_U denotes the completion of the normed vector space $(E, p_U)/\ker p_U$).

THEOREM. $\sigma(\mathbf{R})$ and Ω are not generated by an ideal.

PROOF. Suppose, there is an ideal \mathcal{I} generating $\sigma(\mathbf{R})$ or Ω . We shall obtain a contradiction by constructing a function $\phi \in \Phi$ and a locally convex vector space E in the class generated by \mathcal{I} , such that E is not ϕ -nuclear. As $\sigma(\mathbf{R})$ and Ω consist of nuclear spaces, we may assume, that \mathcal{I} is an ideal of compact operators between separable Hilbert spaces. As the ideal of operators with finite-dimensional images does not generate a stability class, \mathcal{I} contains an operator S with infinite-dimensional range. By combining S with a partial isometry, we may obtain a compact self-adjoint operator $T \in \mathcal{I}$, such that the eigenvalues (λ_n) of T form a decreasing sequence of positive reals. By construction, the sequence space

$$A := \{ \xi \mid \forall k \in \mathbf{N} \sum |\xi_n| \lambda_n^{-k} < \infty \}$$

will be in the class generated by \mathcal{I} . But clearly, A is not ϕ -nuclear, if we choose a function $\phi \in \Phi$, such that for all $n \in \mathbf{N}$ $\phi(\lambda_n^n) > 1/(n+1)$.

Finally we observe, that \mathcal{N} and $\sigma(\mathbf{R})$ share still another ‘restricted’ stability property, as is shown by the following

PROPOSITION. *A Fréchet space E belongs to $\sigma(\mathbf{R})$, if and only if E'_b belongs to $\sigma(\mathbf{R})$.*

This proposition may equally well be stated as

PROPOSITION. *A Fréchet space E belongs to Ω , if and only if E'_b belongs to Ω .*

PROOF. (a) Let $E \in \Omega$. E has property (PB), so $E \in \sigma(\mathbf{R})$. As we proved already, this implies, that E possesses a basis \mathcal{U} of neighbourhoods of 0, such that for $U \in \mathcal{U}$ $E/\ker p_U$ is a subspace of $\bigoplus_N \mathbf{R}$. So $E/\ker p_U$ being a Fréchet space, too, is finite dimensional, which means, that E is a closed subspace of $\Pi_N \mathbf{R}$. Then E is itself an at most countable product of real lines, so $E' \in \sigma(\mathbf{R})$.

(b) If $E' \in \Omega$, E' is nuclear, so E is nuclear and reflexive. Choose a hilbertian neighbourhood U of 0 in E' and a hilbertian bounded set B in E' . Then we have for all $n \in \mathbf{N}$ $\delta_n(B, U) = \delta_n(B^0, U^0)$. So, if B is bounded in E and U is a neighbourhood of 0 in E , the image of B in $E/\ker p_U$ is finite-dimensional. As E has property (PB), this implies, as we have already seen, that $E \in \Sigma = \sigma(\mathbf{R})$.

We did not include this stability property in our definition of a stability class, since there exist stability classes of nuclear spaces, which do not possess this property, e.g. the class of strongly nuclear spaces (cf. [5]).

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