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ON INTEGERS WITH MANY SMALL PRIME FACTORS

by

R. Tijdeman

Let p be a prime, $p \geqq 3$, and let $n_1 = 1 < n_2 < \ldots$ be the sequence of all positive integers composed of primes $\leqq p$. It follows from a theorem of Størmer [22], that

$$\liminf_{i \to \infty} (n_{i+1} - n_i) > 2.$$

This result was improved upon by Thue [23]. He derived from his famous result on the approximation of algebraic numbers by rationals that

$$\lim_{i \to \infty} (n_{i+1} - n_i) = \infty. \tag{1}$$

See also Pólya [15]. Siegel's well known improvements [18, 19] of Thue's result made it possible to improve upon (1). Such an improvement can be obtained immediately from a result of Mahler [13, I, Satz 2]. But, as far as I know, it was not before 1965 that such a result was stated explicitly in literature. In a survey paper [7, p. 218] Erdös gave the following improvement of (1). Let $0 < \vartheta < 1$. Then

$$n_{i+1} - n_i > n_i^{1-\vartheta} \quad \text{for } n_i > N_\vartheta. \tag{2}$$

In this paper we give several improvements and generalizations of inequality (2). In our Theorems 1 and 2 the problem of filling the gap between (2) and the opposite result $\lim_{i \to \infty} n_{i+1}/n_i = 1$ (cf. [15]) is solved almost completely. We shall prove that *there exists an effectively computable constant* $\mathrm{C} = \mathrm{C}(p)$ *such that*

$$n_{i+1} - n_i > \frac{n_i}{(\log n_i)^{\mathrm{C}}} \quad \text{for } n_i \geqq 3, \tag{3}$$

while we shall show that this inequality is false for $\mathrm{C} < \pi(p) - 1$. (Here $\pi(x)$ denotes the number of primes $\leqq x$.)

It follows from (1) that for every integer $k \geqq 1$ there exists a number A_{kp} such that $n_{i+1} - n_i > k$ for $n_i > A_{kp}$. Because of its ineffective nature it is impossible to obtain estimates for A_{kp} from the methods of Thue and Siegel. Using a result of Gel'fond, Cassels [5] gave the first "effective proof" of (1). In 1964/65 D.H. Lehmer [11, 12] gave explicit upper bounds for A_{1p}, A_{2p} and A_{4p} for all p. In the Theorems 3 and 4 upper bounds are given for A_{kp} for all k and all p and also for N_ϑ in (2). A consequence of this result is that if two integers a and b with $0 < a < b$ have the same greatest prime factor, then the inequality $b-a \geqq 10^{-6} \log\log a$ holds.

Professor Straus suggested to me a generalization of (2) in another direction, namely for integers n_i which have many prime factors $\leqq p$, but are not necessarily composed of primes $\leqq p$. Such a result is given by Theorem 6.

Finally we solve a conjecture of Wintner [7, p. 218] in the affirmative. We prove even much more in Theorem 7:

Let $0 < \vartheta < 1$. *There exists an infinite sequence of primes* $p_1 < p_2 < \cdots$ *such that, if* $n_1 < n_2 < \cdots$ *is the set of all integers composed of these primes, then*

$$n_{i+1} - n_i > n_i^{1-\vartheta} \quad \text{for } i = 1, 2, \cdots.$$

Recent results proved by the beautiful Gel'fond-Baker method form the basic tools for nearly all proofs in this paper. The results used for the proofs of Theorems 1–5 have all been published. Since it is our main interest to indicate the possibility of the applications, we did not bother too much to get as good results as possible, but it should be noted that it is possible to improve upon our results by using not yet published results of Stark and others, or by improving upon the auxiliary results. This is possible because of additional information available here. For the proofs of Theorems 6 and 7 a not yet published result of Baker [3] has been used. This result is cited in section **7**. I wish to express my thanks to Dr. A Baker for communicating the exact formulation of his result to me.

1. As usual $\pi(x)$ denotes the number of primes $\leqq x$. The number of distinct prime factors and the greatest prime factor of a positive integer a are denoted by $\omega(a)$ and $P(a)$, respectively. For the sake of brevity we write e.c. for effectively computable. In this paper all constants $\mathrm{c}, \mathrm{c}_1, \mathrm{c}_2, \cdots$ and $\mathrm{C}, \mathrm{C}_1, \mathrm{C}_2, \ldots$ are e.c.

2. We use a result of Fel'dman to prove

THEOREM 1. *Let a and b be positive integers*, $3 < a < b$. *Put* $r = \omega(ab)$, $p = P(ab)$. *Then*

$$b-a > \frac{a}{(\log a)^{\mathrm{C}_1}},$$

where $\mathrm{C}_1 = \mathrm{c}_1^{r^4}(\log p)^{14r^2}$ *and* c_1 *is an e.c. absolute constant.*

PROOF. Without loss of generality we may assume $b \leqq 2a$ and $r \geqq 2$. Let $a = \Pi_{j=1}^r p_j^{\alpha_j}$ and $b = \Pi_{j=1}^r p_j^{\beta_j}$ be prime factorizations of a and b.

Hence

$$\log \frac{b}{a} = \sum_{j=1}^{r} (\beta_j - \alpha_j) \log p_j .$$

According to [9, Theorem 1] the following estimate holds:

$$|\sum_{j=1}^{r} (\beta_j - \alpha_j) \log p_j| > \exp\{-(1+\log H_0)(4^{6r^2}(c+\log(1+p)))^{14r^2}\},$$

where

$$H_0 = \max_j (1+|\beta_j - \alpha_j|) \leqq 1 + \frac{\log b}{\log 2} \leqq 2 + \frac{\log a}{\log 2},$$

and c is some e.c. absolute constant. Hence

$$\frac{b}{a} - 1 > \log \frac{b}{a} > \exp\{-c_1^{r^4} (\log p)^{14r^2} \log\log a\} = (\log a)^{-C_1 r^4 (\log p)^{14r^2}}.$$

This proves the theorem.

COROLLARY. *Let $n_1 < n_2 < \cdots$ be the sequence of integers composed of primes not greater than p. Then there exists an e.c. constant* $C = C(p)$ *such that*

$$n_{i+1} - n_i > \frac{n_i}{(\log n_i)^C} \text{ for } n_i \geqq 3.$$

PROOF. We have $r \leqq 4p/(3 \log p)$. (See e.g. [17, formula (3.6)] or [10])

3. The following theorem shows that the constants C_1 and C in the preceding section cannot be replaced by constants smaller than $r-1$ and $\pi(p)-1$, respectively. The gap meant by Erdös at [7,p. 128] is therefore filled up almost completely.

THEOREM 2. *Let $P = \{p_1, \cdots, p_r\}$ be a given set of primes, $r > 1$, and put $p = \max_j p_j$. Then there exist infinitely many pairs of integers a, b such that*

$$0 < b - a < \frac{(r \log p)^r a}{(\log a)^{r-1}}. \tag{4}$$

PROOF. Let T be a positive integer and consider all numbers of the form $t_1 \log p_1 + t_2 \log p_2 + \cdots + t_r \log p_r$ with $0 \leqq t_j \leqq T$ for $j = 1, \cdots, r$. We have $(T+1)^r$ non-negative numbers of absolute value $\leqq rT \log p$. It follows that there are two among them, $\alpha_1 \log p_1 + \cdots + \alpha_r \log p_r$ and $\beta_1 \log p_1 + \cdots + \beta_r \log p_r$ say, such that

$$0 < (\beta_1 \log p_1 + \cdots + \beta_r \log p_r) - (\alpha_1 \log p_1 + \cdots + \alpha_r \log p_r) \leqq \frac{r \log p}{(T+1)^r - 1}.$$

Hence,

$$(5) \qquad 0 < (\beta_1-\alpha_1)\log p_1 + \cdots + (\beta_r-\alpha_r)\log p_r < \frac{r\log p}{T^{r-1}}.$$

Without loss of generality we may assume that $\alpha_j\beta_j = 0$ for $j = 1, \ldots, r$. Putting $a = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ and $b = p_1^{\beta_1}\cdots p_r^{\beta_r}$, we obtain $a^2 \leqq ab \leqq p^{rT}$. Hence, $T \geqq 2\log a/(r\log p)$. Substituting this estimate in (5) we see that

$$(6) \qquad \log\frac{b}{a} < \frac{(r\log p)^r}{(2\log a)^{r-1}}.$$

We know from (5) that $\log(b/a) \to 0$, if $T \to \infty$. Hence,

$$(7) \qquad \log\frac{b}{a} \geqq \frac{1}{2}\left(\frac{b}{a}-1\right) \quad \text{for } T \geqq T_0.$$

The inequality (4) follows from (6) and (7). By letting $T \to \infty$ we obtain infinitely many pairs a, b subject to (4).

4. One cannot deduce an explicit bound for the N_ϑ in (2) from the proof of Siegel. The following proof based on the Gel'fond-Baker theory shows that N_ϑ can be chosen such that

$$N_\vartheta < \exp\left\{\left(\frac{e^{8(p/\log p)^2}}{\vartheta}\right)^{13p(/\log p)^2}\right\}.$$

THEOREM 3. *Let* $0 < \vartheta < 1$. *Let a and b be positive integers such that* $a < b < a + a^{1-\vartheta}$. *Put* $r = \omega(ab)$ *and* $p = P(ab)$. *Then*

$$a < \exp\left\{\left(\frac{e^{4r^2}\log p}{\vartheta}\right)^{7r^2}\right\}.$$

PROOF. We have $a > 1$ and $r \geqq 2$. Let $a = \Pi_{j=1}^r p_j^{\alpha_j}$ and $b = \Pi_{j=1}^r p_j^{\beta_j}$ be prime factorizations of a and b respectively. Hence,

$$|\alpha_j|, |\beta_j| \leqq \frac{\log b}{\log 2} \leqq 1 + \frac{\log a}{\log 2} \leqq 3\log a \text{ for } j = 1, \cdots, r.$$

We have

$$0 < \sum_{j=1}^r (\beta_j-\alpha_j)\log p_j = \log\frac{b}{a} < \frac{b}{a} - 1 < a^{-\vartheta},$$

and therefore

$$|\sum_{j=1}^r (\beta_j-\alpha_j)\log p_j| < e^{-\delta H}$$

where $\delta = \vartheta/3$ and $H = \max_j(\beta_j-\alpha_j) \leqq 3\log a$. It follows from Baker's well known result [1] that

$$\max_j |\beta_j - \alpha_j| < \left(\frac{4^{(r+1)^2} \text{ og } p}{\vartheta/3}\right)^{(2r+1)^2} < \left(\frac{e^{4r^2} \log p}{\vartheta}\right)^{7r^2-1}.$$

Hence,

$$a \leqq \exp\left\{r \log p \left(\frac{e^{4r^2 \log p}}{\vartheta}\right)^{7r^2-1}\right\} \leqq \exp\left\{\left(\frac{e^{4r^2} \log p}{\vartheta}\right)^{7r^2}\right\}.$$

This proves the theorem. We recall that $r \leqq 4p/(3 \log p)$.

5. The next theorem gives an upper bound for a similar to the one in Theorem 3. The present estimate is better if $b-a$ is rather small and r is large. A similar method of proof was given by Ramachandra [16, Theorem 3]. Here we use Baker's result on the integer solutions of the diophantine equation $y^2 = x^3 + k$.

THEOREM 4. *Let a and b be positive integers, $a < b$. Put $r = \omega(ab)$ and $p = P(ab)$. Then*

$$a \leqq \exp\{(10(b-a)p^r)^{10^5}\}.$$

PROOF. Let $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $b = p_1^{\beta_1} \cdots p_r^{\beta_r}$ be prime factorizations of a and b respectively. Let $a = ve^3$ and $b = wf^2$, where e, f, v, w are positive integers such that v is cubefree and w is squarefree. We have

$$(vw^2f)^2 = (vwe)^3 + (b-a)v^2w^3.$$

Thus $(x, y) = (vwe, vw^2f)$ is an integer solution of the diophantine equation $y^2 = x^3 + k$, where $k = (b-a)v^2w^3$. It follows from Baker's estimate for such solutions [2], that

$$vw^2f = \max(vwe, vw^2f) \leqq \exp\{(10^{10}(b-a)v^2w^3)^{10^4}\}.$$

We have $|v| \leqq p^{2r}$, $|w| \leqq p^r$. Hence,

$$\text{(8)} \quad a < b = wf^2 \leqq \exp\{2(10^{10}(b-a)p^{7r})^{10^4}\}$$
$$\leqq \exp\{(10(b-a)p^{3r/4})^{10^5}\},$$

which implies our assertion.

Using $r \leqq 4p/(3 \log p)$ we see from (8) that

$$\text{(9)} \qquad a \leqq \exp\{(10(b-a)e^p)^{10^5}\}.$$

So we obtain

COROLLARY. *Let k and p be positive integers. Let A_{kp} be the smallest integer a such that if $b-a = k$ and $P(ab) = p$ then $a \leqq A_{kp}$. Then*

$$\log A_{kp} \leqq (10ke^p)^{10^5}.$$

For $k = 1, 2$ and 4 D. H. Lehmer [11, 12] has given better bounds for

A_{kp}. He proved for example $\log A_{1p} \leqq c_2 p^2 e^{p/2}$ for some absolute constant c_2 and all p. He also included tables of numerical results for $p \leqq 41$. It appears that the upper bounds for A_{kp} are rather crude. It would be of great interest to replace the factor e^p by some constant power of p. This would be possible, if Hall's conjecture[4] on the solutions of $y^2 = x^3+k$ were proved. (M. Hall Jr. has conjectured that if x, y are integers with $x^3 \neq y^2$, then $|x^3-y^2| > |x|^{\frac{1}{2}}$ for $x > x_0$.)

6. The following theorem has direct consequences for the number theoretic function $f_3(n)$ introduced by Erdös and Selfridge [8]. In their notation it implies $f_3(n) > 10^{-6} \operatorname{loglog} n$, which is the first non-trivial lower bound for $f_3(n)$. They conjectured $f_3(n) > (\log n)^{c_3}$ for some absolute constant c_3 and all n. (This would also be a consequence of Hall's conjecture.)

THEOREM 5. *Let a and b be positive integers such that $a < b$ and $P(a) = P(b)$. Then*

(i) $\log(b-a)+P(a) \geqq 10^{-6} \log\log a$,

(ii) $\qquad b-a \qquad \geqq 10^{-6} \log\log a$.

PROOF. We know from the above Corollary that

$$\log a \leqq \log A_{b-a,\, P(a)} \leqq (10(b-a)e^{P(a)})^{10^5}.$$

Hence, by $P(a) \geqq 2$,

$$\log\log a \leqq 5.10^5(\log(b-a)+P(a)).$$

This implies (i). Since $P(a) \leqq b-a$, we have

$$\log\log a \leqq 5.10^5(\log(b-a)+b-a) \leqq 10^6(b-a).$$

7. In this section we generalize (2) in a direction which was suggested by Professor E. G. Straus. As Professor Baker pointed out in a discussion with Professor Straus the following recent result of his makes it possible to give an inequality like (2) for integers containing many small prime factors.

If $\alpha_1, \cdots, \alpha_n$ are non-zero algebraic numbers with degrees at most d, and if the heighths of $\alpha_1, \cdots, \alpha_{n-1}$ and α_n are at most A' and $A(\geqq 2)$, respectively, then there is an effectively computable number C, *depending only on n, d and A' such that, for any δ with $0 < \delta < \frac{1}{2}$, the inequalities*

$$0 < |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| < \left(\frac{\delta}{B'}\right)^{C \log A} e^{-\delta B}$$

have no solution in rational integers $b_1, \cdots, b_{n-1}$ and $b_n(\neq 0)$ with absolute values at most B and B', respectively.

Using this result one can deduce results related to the p-adic analogues of the theorems of Thue-Siegel-Roth and Baker as developed by Mahler, Coates, Sprindžuk and others. (See for example [13, 14, 6, 20, 21].) We note that one of the related conjectures at the end of Mahler's book [14, Appendix III] was proved by Straus, but that this result has not been published. We prove here

THEOREM 6. *Let* $0 < \vartheta < 1$. *Let* a *and* b *be positive integers such that* $a < b \leqq a + a^{1-\vartheta}$. *Furthermore, let* p, r, a_1, a_2, b_1, b_2 *be positive integers such that* $a = a_1 a_2$, $b = b_1 b_2$, $P(a_1 b_1) \leqq p$, $\omega(a_1 b_1) \leqq r$. *Then there exists an e.c. constant* $\eta = \eta(p, r, \vartheta) > 0$ *such that*

$$a_2 b_2 > a^{\eta} - 1. \tag{10}$$

PROOF. Put $A = \max(a_2, b_2)$. It suffices to prove $A > a^{\eta} - 1$. First we assume $A \geqq 2$. Let $a_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, and $b_1 = p_1^{\beta_1} \cdots p_r^{\beta_r}$ be prime factorizations of a_1 and b_1 respectively. Hence,

$$\log \frac{b}{a} = (\beta_1 - \alpha_1) \log p_1 + \cdots + (\beta_r - \alpha_r) \log p_r + \log \frac{b_2}{a_2}. \tag{11}$$

We have $|\beta_j - \alpha_j| \leqq \log b / \log 2$ for $j = 1, \cdots, r$. We apply the above mentioned result of Baker to the linear form at the right hand side of (11). We take

$$d = 1,\ n = r+1,\ A' = p,\ A = \max\ (a_2,\ b_2),\ \delta = \vartheta/5,$$

$B = \log b / \log 2$ and $B' = 1$. Hence, there exists an e.c. constant $C_2 = C_2(p,r)$ such that

$$\left| (\beta_1 - \alpha_1) \log p_1 + \cdots + (\beta_r - \alpha_r) \log p_r + \log \frac{b_2}{a_2} \right| \geqq \left(\frac{\vartheta}{5}\right)^{C_2 \log A} e^{-(\vartheta \log b)/(5 \log 2)}.$$

Since $b \leqq 2a$, it follows that

$$\log \frac{b}{a} \geqq A^{C_2 \log (\vartheta/5)} (2a)^{-\vartheta/3}.$$

On the other hand,

$$\log \frac{b}{a} < \frac{b}{a} - 1 \leqq a^{-\vartheta}.$$

The combination of these inequalities gives

$$a^{\vartheta} < (2a)^{\vartheta/3} A^{C_2 \log (5/\vartheta)},$$

or, equivalently,

$$a^2 < 2A^{3C_2 \vartheta^{-1} \log (5/\vartheta)}.$$

Let $\eta > 0$ be chosen so small that

$$(12) \qquad 3C_2 \vartheta^{-1} \log \frac{5}{\vartheta} < \frac{1}{\eta}.$$

Then $a^2 \leqq 2A^{1/\eta}$. Hence $A \geqq a^\eta$.

Now we consider the remaining case $A = 1$. We see from Theorem 3 that there exists an e.c. constant $C_3 = C_3(p, r, \vartheta) > 0$ such that $a < C_3$. We choose $\eta > 0$ so small that (12) is fulfilled and, moreover,

$$\eta < \frac{\log 2}{\log C_3}.$$

Hence, $A = 1 > C_3^\eta - 1 > a^\eta - 1$. This completes the proof of the theorem.

8. It is easy to see that Theorem 6 is no longer valid, if we replace (10) by $a_2 b_2 > a^\vartheta + 1$. For example, let $\vartheta = 1 - w^{-1}$, $w \in \mathbb{Z}$, $w > 0$. Take $a = 2^{lw}$, $b = 2^{lw} + 2^l$, $l \in \mathbb{Z}$, $l > 0$ and $p = 2$. Hence, $a < b \leqq a + a^{1-\vartheta}$, while $a_2 b_2 \leqq 2^{l(w-1)} + 1 = a^\vartheta + 1$.

9. Finally we give another application of the result of Baker mentioned in **7**. Wintner communicated the following problem to Erdös orally [7, p. 218].

Does there exist an infinite sequence of primes $p_1 < p_2 < \cdots$ such that if $n_1 < n_2 < \cdots$ is the set of all integers composed of the p's then $\lim_{i\to\infty}(n_{i+1} - n_i) = \infty$?

Here we solve the conjecture in the affirmative and prove even much more:

THEOREM 7. *Let $0 < \vartheta < 1$. There exists an infinite sequence of primes $p_1 < p_2 < \cdots$ such that if $n_1 < n_2 < \cdots$ is the set of all integers composed of these primes then*

$$(13) \qquad n_{i+1} - n_i > n_i^{1-\vartheta}.$$

PROOF. We construct a sequence $p_1 < p_2 < \cdots$ with the required property by induction. Let $p_1 = 3$. It is obvious that (13) holds for powers of 3. Now assume that we have a sequence $p_1 < \cdots < p_r$ such that

$$(14) \qquad b - a > a^{1-\vartheta}$$

for all integers a, b with $a < b$ and composed of these primes. Let p be a prime, $p > p_r$. Suppose that there exist integers a and b composed of the primes $p_1, \ldots, p_r, p$ such that $0 < b - a < a^{1-\vartheta}$. Let $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r} p^\alpha$ and $b = p_1^{\beta_1} \cdots p_r^{\beta_r} p^\beta$ be prime factorizations of a and b, respectively. Then $\alpha \neq \beta$ because of our induction hypothesis applied to a/p^α and b/p^β. We have

$$(15) \qquad \left|\sum_{j=1}^{r} (\beta_j - \alpha_j) \log p_j + (\beta - \alpha) \log p\right| = \log \frac{b}{a} < \frac{b}{a} - 1 < a^{-\vartheta}.$$

Furthermore, $|\beta_j - \alpha_j| \leqq \log b \leqq \log 2a$ for $j = 1, \cdots, r$ and $|\beta - \alpha| \leqq \log b/\log p \leqq \log (2a)/\log p$. We apply Baker's result (see **7**) with $d = 1$, $n = r+1$, $A' = p_r$, $A = p$, $\delta = \vartheta/4$, $B = \log (2a)$ and $B' = \log (2a)/\log p$. Hence, there exists an e.c. constant $C_4 = C_4(p, r)$ such that

$$(16) \qquad \left|\sum_{j=1}^{r} (\beta_j - \alpha_j) \log p_j + (\beta - \alpha) \log p\right| \geqq \left(\frac{\vartheta \log p}{4 \log (2a)}\right)^{C_4 \log p} e^{-\vartheta \log (2a/4)}.$$

It follows from (15) and (16) that

$$a^{\vartheta} \leqq (2a)^{\vartheta/4} \left(\frac{4 \log (2a)}{\vartheta \log p}\right)^{C_4 \log p} \leqq a^{\vartheta/2} \left(\frac{8 \log a}{\vartheta \log p}\right)^{C_4 \log p}.$$

This implies

$$\frac{\log a}{\log p} \leqq \frac{2C_4}{\vartheta} \log \left(\frac{8 \log a}{\vartheta \log p}\right).$$

Thus there exist e.c. constants $C_5 = C_5(p, r, \vartheta)$ and $C_6 = C_6(\vartheta)$ such that $x \leqq C_5 \log (C_6 x)$, where $x = \log a/\log p$. Hence, there exists an e.c. constant $C_7 = C_7(p, r, \vartheta)$ such that

$$\frac{\log a}{\log p} = x \leqq \tfrac{1}{2} C_7.$$

So we obtain $a \leqq p^{\frac{1}{2}C_7}$ and $b \leqq 2a \leqq p^{C_7}$, whence $\alpha, \beta \leqq C_7$ and $\alpha_j, \beta_j \leqq C_7 \log p$ for $j = 1, \cdots, r$.

Let T be a large integer. We know from [17, formula (3.8)] that the number of primes in the interval $[T/2, T]$ is larger than $3T/(10 \log T)$ for $T \geqq 41$. For each prime p in the interval $[T/2, T]$ we apply the above argument. Hence,

$$(17) \qquad a = p_1^{\alpha_1} \cdots p_r^{\alpha_r} p^{\alpha},\ b = p_1^{\beta_1} \cdots p_r^{\beta_r} p^{\beta}, \quad 0 < b - a < a^{1-\vartheta}$$

implies that $\alpha, \beta \leqq C_7$ and $\alpha_j, \beta_j \leqq C_7 \log T$ for $j = 1, \cdots, r$. The number of possible choices for $\alpha_1, \cdots, \alpha_r, \alpha, \beta_1, \cdots, \beta_r, \beta$ is at most $(C_7 + 1)^{2r+2} (\log T)^{2r}$. Assume that all these integers are fixed. Then $1 < b/a < 1 + a^{-\vartheta}$ implies that

$$(18) \qquad C_8 < p^{\beta - \alpha} < C_8(1 + a^{-\vartheta}),$$

where $C_8 = p_1^{\alpha_1 - \beta_1} \cdots p_r^{\alpha_r - \beta_r}$. We recall that $\alpha \neq \beta$. First suppose $\beta > \alpha$. Then

$$C_8^{1/(\beta-\alpha)} < p < C_8^{1/(\beta-\alpha)} (1 + a^{-\vartheta})^{1/(\beta-\alpha)} \leqq C_8^{1/(\beta-\alpha)} (1 + a^{-\vartheta}).$$

Hence, p is contained in a fixed interval of length

$$C_8^{1/(\beta-\alpha)} a^{-\vartheta} < p a^{-\vartheta} \leqq T a^{-\vartheta}.$$

Secondly suppose $\beta < \alpha$. Then from (18)

$$(1+a^{-\vartheta})^{1/(\beta-\alpha)}C_8^{1/(\beta-\alpha)} < p < C_8^{1/(\beta-\alpha)}.$$

Hence, p is contained in a fixed interval of length

$$C_8^{1/(\beta-\alpha)}(1-(1+a^{-\vartheta})^{1/(\beta-\alpha)}) \leqq p((1+a^{-\vartheta})^{1/(\alpha-\beta)}-1) \leqq pa^{-\vartheta} \leqq Ta^{-\vartheta}.$$

It turns out that in both cases the number of primes p for which (17) with fixed $\alpha_1, \cdots, \alpha_r, \alpha, \beta_1, \cdots, \beta_r, \beta$ is possible does not exceed $Ta^{-\vartheta}$. Since $2a \geqq b \geqq p \geqq T/2$, we see that $Ta^{-\vartheta} \leqq 4T^{1-\vartheta}$. We conclude that the total number of primes in the interval $[T/2, T]$ for which integers a and b subject to (17) can be found is at most

$$4T^{1-\vartheta}(C_7+1)^{2r+2}(\log T)^{2r}. \tag{19}$$

We have to exclude these primes. However, there are more than $3T/(10 \log T)$ primes in this interval. For sufficiently large T the number of primes in $[T/2, T]$ is greater than (19) and we can take p_{r+1} out of the remaining set. Doing so every pair of integers a and b with $a < b$ and composed of $p_1, \cdots, p_{r+1}$ satisfies $b-a > a^{1-\vartheta}$. Now the proof has been completed by induction.

10. Remarks. (i) It follows from the above proof that for every ϑ with $0 < \vartheta < 1$ it is possible effectively to give a sequence $P_1, P_2, \cdots$ such that there exists a sequence $p_1 < p_2 < \cdots$ with the required property and with $P_j/2 \leqq p_j \leqq P_j$ for all j. (ii) $0 < \vartheta < 1$. It follows from Theorem 2 that there does not exist a constant $C_9 = C_9(\vartheta)$ such that Theorem 7 is valid if (13) is replaced by the inequality

$$n_{i+1}-n_i > \frac{n_i}{(\log n_i)^{C_9}}.$$

It is interesting to compare this with Theorem 1.

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