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EX-HOMOTOPY THEORY

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The ex-homotopy category has been investigated in recent years by I. M. James, J. Becker, L. Smith, J. F. McClendon, C. A. Robinson and others. In [5] I. M. James develops homotopy theory for ex-spaces as far as Puppe sequences, and in [6] he examines the Puppe sequence in a special case in order to calculate ex-homotopy groups.

Further developments are hampered by the fact that no sufficiently helpful homology theory for ex-spaces is known. In this paper we obviate the need for one and use results from [2] and [3] to derive an EHP-sequence. The technique is to mimic globally the *constructions* of ordinary homotopy theory and then to apply comparison theorems to deduce *theorems* in exhomotopy theory from the corresponding theorems in ordinary homotopy theory. As an example of our main result we calculate in § 6 some ex-homotopy groups involving the Hopf bundle which, to my knowledge, were not previously obtainable.

I am most grateful to Professor James for his help and encouragement during the preparation of this work.

Throughout the paper we consider only Hausdorff spaces and adopt the terminology of [2] and [3]. Let B be a connected, locally finite CW-complex (although our arguments in Sections 1–3 pertain more generally for B any locally contractible, para-compact, locally compact and path connected space). Recall that an ex-space (E, ρ, σ) (over B) consists of a space E and maps $\rho : E \to B$, $\sigma : B \to E$ such that $\rho \cdot \sigma = 1_B$ and $\sigma(B)$ is closed in E.

For ex-spaces E, X the ex-spaces $E_V X$ (wedge sum), $E \times X$ (direct product), ΣE (reduced suspension), and $E \# X$ (smash product) are defined in [5]. The loop ex-space $(\Omega E, \rho', \sigma')$ of the ex-space (E, ρ, σ) has total space the subspace

$$\{\omega \in E^I : \omega(0) = \omega(1) = \sigma\rho\omega(0), \rho(\omega(t)) = \rho(\omega(0)) \text{ for all } t \in I\}$$

of E^I where E^I has the compact open topology. The ex-structure is defined by

$$\rho'(\omega) = \rho(\omega(0)), (\sigma'(b))(t) = \sigma(b) \quad (t \in I, b \in B).$$

Recall from [2] that an ex-space (E, ρ, σ) is said to be *docile* if for each point $b \in B$ there is a closed neighbourhood $W(b)$ such that the restriction ex-space $(\rho^{-1}W(b), \rho|\rho^{-1}W(b), \sigma|W(b))$ over $W(b)$ is ex-homotopically equivalent to the product ex-space $W(b) \times F$, where F is a well-pointed space.

DEFINITION: An ex-space (E, ρ, σ) over B is a *placid ex-space* if σ is a cofibration, ρ is a Hurewicz fibration and the fibre of ρ has the pointed homotopy type of a locally finite CW-complex.

By [2] Theorem 3.6 a placid ex-space is docile. The class of placid ex-spaces over B is closed under product and smash product ([2] Corollary 3.4 and [4] Lemma 8.1).

1. Reduced product ex-spaces

DEFINITION (1.1): An ex-space (E, ρ, σ) is *distance-based* if there exists a map $\psi : E \to [0, 1]$ such that $\psi^{-1}(0) = \sigma(B)$. An ex-space (E, ρ, σ) where the total space E is normal and $\sigma(B)$ is a closed (G, δ)- set of E is distance-based ([10] p. 134). Thus any ex-space with a metrizable total space is distance-based. If E is a distance-based ex-space then so are the ex-spaces ΣE and ΩE.

Let E_n $(n \geqq 2)$ denote the ex-space obtained from the direct product of n copies of the ex-space (E, ρ, σ) by making the identifications, for each $1 \leqq i \leqq n-1$,

$$(e_1, e_2, \cdots, e_{n-1}, \sigma\rho(e_1)) \sim (e_1, \cdots, e_{i-1}, \sigma\rho(e_1), e_i, \cdots, e_{n-1})(e_j \in E).$$

Since $E \backslash \sigma(B)$ is open in E the inclusion ex-map $i_n : E_n \to E_{n+1}$, $i_n(e_1, \cdots, e_n) = (e_1, \cdots, e_n, \sigma\rho(e_1))$, is open and embeds E_n naturally in E_{n+1}. Define the reduced product ex-space to be $E_\infty = \varinjlim E_n$.

If E is distance-based by the function ψ an ex-map $f : E_\infty \to \Omega\Sigma E$ may be defined as follows. For $(e_1, \cdots, e_n) \in E_n \backslash E_{n-1}$ $(n \geqq 1)$ set $a_i = \psi(e_i)/\sum_{j=1}^n \psi(e_j)$ and define

$$(1.2) \qquad (f(e_1, \cdots, e_n))(t) = \begin{cases} [e_1, t/a_1] & 0 \leqq t \leqq a_1 \\ [e_2, (t-a_1)/a_2] & a_1 \leqq t \leqq a_1 + a_2 \\ \cdot & \\ \cdot & \\ \cdot & \\ [e_n, (t - \sum_{i=1}^{n-1} a_i)/a_n] & \sum_{i=1}^{n-1} a_i \leqq t \leqq \sum_{i=1}^{n} a_i. \end{cases}$$

Take $E_0 = \sigma(B)$, $E_1 = E$. Then (1.2) defines f on $E_\infty \backslash E_0$. The map f is extended to an ex-map $f : E_\infty \to \Omega\Sigma E$ by defining $(f(\sigma(b)))(t) = \sigma_{\Sigma E}(b)$, $0 \leqq t \leqq 1$.

Let F be the fibre $\rho^{-1}(b)$ of E at the point $b \in B$. We then have $E_n \cap \rho_n^{-1}(b) = F_n$, $E_\infty \cap \rho_\infty^{-1}(b) = F_\infty$ where the right-hand sides are obtained by the reduced product construction for the pointed space $(F, \sigma(b))$. By D. Puppe's refinement [11] (p. 234 Theorem 17.3) of the theorem of I. M. James [8] we know that $f|_b$ is a homotopy equivalence if $(F, \sigma(b))$ is an h-well-pointed, path-connected space which admits a numerable null-homotopic covering. The same method of proof as in [2] Proposition 3.8 establishes that E_n, E_∞ and ΩE are docile ex-spaces if E is a docile ex-space. By [2] Theorem 3.9 we then obtain

PROPOSITION (1.3): *Let (E, ρ, σ) be a docile distance-based ex-space with fibre having the pointed homotopy type of a connected locally finite CW-complex. Then the ex-map $f: E_\infty \to \Omega\Sigma E$ in (1.2) is an ex-homotopy equivalence.*

2. The reduced join of ex-spaces

Let (E, ρ, σ) and (X, ρ', σ') be ex-spaces. The total space of the reduced join $E * X$ is obtained from $E \times X \times I$ by making the identifications

$$\begin{array}{ll} (e, x, 0) \sim (\sigma\rho(e), x, 0) & e \in E,\ x \in X \\ (e, x, 1) \sim (e, \sigma'\rho'(x), 1) & e \in E,\ x \in X \\ (\sigma(b), \sigma'(b), t) \sim (\sigma(b), \sigma'(b), 0) & b \in B,\ t \in I. \end{array}$$

The projection $E * X \to B$ takes $(e, x, t)/\sim$ to $\rho(e)$, and the section $B \to E * X$ takes b to $(\sigma(b), \sigma'(b), 0)/\sim$. There is a natural collapsing ex-map $E * X \to \Sigma(E \# X)$, which, by [15] p. 239, induces a homotopy equivalence between the fibres of $E * X$ and $\Sigma(E \# X)$ over any point $b \in B$ if the fibres of E and X over b are polyhedra.

By [2] Proposition 3.8 and Theorem 3.9 one has

PROPOSITION (2.1): *Let E, X be docile ex-spaces with fibres having the pointed homotopy type of locally finite CW-complexes. Then the collapsing ex-map $E * X \to \Sigma(E \# X)$ is an ex-homotopy equivalence.*

I remark in passing that by an application of [2] Theorem 3.9 similar to that in (1.3) or (2.1) a Hilton-Milnor theorem for ex-spaces may be obtained (see [4]).

3. Ex-homotopy exact sequences

The material in this section is a straightforward generalization of the corresponding results in homotopy theory. The reader is referred to [4] for more detailed proofs.

Let $(Z, Z'$, (X, X'), (W, W') be ex-space pairs ([3] Part 1 Section 4).

Composition on the left by an ex-map $f : (Z, Z') \to (X, X')$ induces a pointed function $f_{\#} : \pi(W, W'; Z, Z') \to \pi(W, W'; X, X')$, and composition on the right induces a pointed function $f^{\#} : \pi(X, X'; W, W') \to \pi(Z, Z'; W, W')$. By restricting the domain and codomain of an ex-map $g : (W, W') \to (Z, Z')$ to W' and Z' respectively one obtains an ex-map $\delta(g) : W' \to Z'$. This boundary operation respects ex-homotopy and defines a pointed function $\delta : \pi(W, W'; Z, Z') \to \pi(W', Z')$.

By a Puppe sequence argument one deduces

PROPOSITION (3.1): (*Exact ex-homotopy sequence of a pair*).
Let W be an ex-space and (X, X') be an ex-space pair. Then the sequence $\cdots \to \pi(\Sigma W, X') \xrightarrow{i\#} \pi(\Sigma W, X) \xrightarrow{j\#} \pi(CW, W; X, X') \xrightarrow{\delta} \pi(W, X') \xrightarrow{i\#} \pi(W, X)$, *where $i : X' \to X$ and $j : (X, \sigma_X(b)) \to (X, X')$ are inclusions, is exact.*

The proof of [11] p. 378 Theorem 15 generalizes to yield

PROPOSITION (3.2): (*Exact ex-homotopy sequence of a triple*).
Let W, X, X' and X'' be ex-spaces such that (X, X') and (X', X'') are ex-space pairs. Then the sequence $\cdots \to \pi(C\Sigma W, \Sigma W; X, X') \xrightarrow{\delta} \pi(CW, W; X', X'') \xrightarrow{i\#} \pi(CW, W; X, X'') \xrightarrow{j\#} \pi(CW, W; X, X')$ *is exact, where $i : (X', X'') \to (X, X'')$ and $j : (X, X'') \to (X, X')$ are the inclusions.*

DEFINITION (3.3): Let E, X and K be ex-spaces. An ex-map $q : E \to X$ has the *ex-homotopy lifting property for K* if, given an ex-map $g : K \to E$ and an ex-homotopy $F : K \times I \to X$ such that $F_0 = q \cdot g$, there exists an ex-homotopy $G : K \times I \to E$ such that $G_0 = g$ and $q \cdot G = F$.

DEFINITION (3.4): The ex-map $q : E \to X$ is an *ex-fibration* if it has the ex-homotopy lifting property for all ex-spaces K. If B is a CW-complex the ex-map $q : E \to X$ is a *Serre ex-fibration* if it has the ex-homotopy lifting property for all ex-complexes K. (Recall from [7] Section 5 that the ex-space (K, ρ, σ) over B is an *ex-complex* if K is a CW-complex with sub-complex $\sigma(B)$. An ex-complex is *proper* if the projection ρ is a cellular map. If K is a proper ex-complex then CK and ΣK are proper ex-complexes.)

Example of an ex-fibration: Let (X, ρ, σ) be an ex-space. Set $P(X) = \{w \in X^I : \rho(w(t)) = \rho(w(0)) \text{ for all } t \in I\}$, and assign to $P(X)$ the subspace topology from X^I, where X^I has the compact open topology. The space $P(X)$ possesses a natural projection onto $B(w \mapsto \rho(w(0)))$ and also a section ($b \mapsto$ constant path at $\sigma(b)$, $(b \in B)$). Hence $P(X)$ is an ex-space. The map $q : P(X) \to X$, $q(w) = w(1)$, is an ex-fibration.

The proof ([4] Proposition 5.4) of the next proposition is lengthy but not difficult.

PROPOSITION (3.5): *Let E be an ex-space and (X, X') an ex-space pair over the [CW-complex] space B. Let $q : E \to X$ be a [Serre] ex-fibration, and write E' for the subex-space $q^{-1}(X')$ of E. Then for any [proper ex-complex] ex-space K the pointed function $q_\# : \pi(CK, K; E, E') \to \pi(CK, K; X, X')$ is bijective.*

PROPOSITION (3.6): (*Exact ex-homotopy sequence of an ex-fibration*).
Let E and X be ex-spaces over the [CW-complex] space B, and let $q : E \to X$ be a [Serre] ex-fibration. For any [proper ex-complex] ex-space K the sequence

$$\cdots \xrightarrow{q_\#} \pi(\Sigma^2 K, X) \xrightarrow{\delta'} \pi(\Sigma K, D) \xrightarrow{i_\#} \pi(\Sigma K, E) \xrightarrow{q_\#} \pi(\Sigma K, X) \xrightarrow{\delta'} \pi(K, D) \xrightarrow{i_\#} \pi(K, E)$$

is exact, where D is the subex-space $q^{-1}(\sigma_X(B))$ of E, i is the inclusion ex-map: $D \subset E$, and $\delta' = \delta \cdot \tilde{q}_\#^{-1}$

$$(\tilde{q}_\# : \pi(CK, K; E, D) \to \pi(CK, K; X, \sigma_X(b)), \delta : \pi(CK, K; E, D) \to \pi(K, D)).$$

Proposition 3.6 may be proved by applying Proposition 3.5 to the exact sequence of the ex-space pair (E, D). As with the analogous exact sequences in homotopy theory except near their tails the exact sequences of (3.1), (3.2) and (3.6) are exact sequences of abelian groups.

4. A relative comparison theorem

Let K be a proper ex-complex over B.

THEOREM (4.1): (*Relative Comparison Theorem*)
Let (E_1, E_2), (X_1, X_2) be ex-space pairs over B where the projections $\rho_{E_1}, \rho_{E_2}, \rho_{X_1}, \rho_{X_2}$ are Serre fibrations.
Suppose that, for some $n \geqq 1$, $f : (E_1, E_2) \to (X_1, X_2)$ is an ex-map whose restriction to a fibre is n-connected.[1] Then the function $f_\# : \pi(CK, K; E_1, E_2) \to \pi(CK, K; X_1, X_2)$ is bijective for $\dim K < n-1$, surjective for $\dim K \leqq n-1$.

PROOF: We construct the ex-space $(P, \bar{\rho}, \bar{\sigma})$ where $P = \{w \in E_1^I | w(1) \in E_2,\ \rho(w(t)) = \rho(w(0)) \text{ for all } t \in [0, 1]\}$, $\bar{\rho}(w) = \rho_{E_1}(w(0))$, and $(\bar{\sigma}(b))(t) = \sigma_{E_1}(b)$ for all $t \in I$. The ex-map $p : P \to E_1$, $p(w) = w(0)$, is an ex-fibration. Set $D = p^{-1}(\sigma_{E_1}(B))$ and regard D as a subex-space of P over B. Since ρ_{E_2} and ρ_{E_1} are Serre fibrations the projection, ρ_D say, of D is a Serre fibration.

[1] i.e. if F_1, F_2, Y_1, Y_2 are the fibres of E_1, E_2, X_1, X_2 respectively over some point of B, then $f|_{pt\#} : \pi_i(F_1, F_2) \to \pi_i(Y_1, Y_2)$ is bijective for $i < n$, surjective for $i \leqq n$.

Define the subex-space $P'E_1$ of $P(E_1)$ over B to be the subex-space with total space $P'E_1 = \{w \in E_1^I | w(0) \in \sigma_{E_1}(B),\ \rho(w(t)) = \rho(w(0))$ for all $t \in I\}$. The map $p' : P'E_1 \to E_1$, $p'(w) = w(1)$ is an ex-fibration, and $D = p'^{-1}(E_2)$. Since $P'E_1$ is ex-contractible, by Proposition 3.1 $\pi(CK, K; P'E_1, D) \xrightarrow{\delta} \pi(K, D)$ is bijective regardless of dim K. Also, by Proposition 3.5, $\pi(CK, K; P'E_1, D) \xrightarrow{p'_\#} \pi(CK, K; E_1, E_2)$ is bijective (regardless of dim K).

The ex-map f induces a commutative diagram

$$\begin{array}{ccccc}
\pi(CK, K; E_1, E_2) & \underset{\approx}{\xleftarrow{p'_\#}} & \pi(CK, K; P'E_1, D) & \underset{\approx}{\xrightarrow{\delta}} & \pi(K, D) \\
\downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} \\
\pi(CK, K; X_1, X_2) & \underset{\approx}{\xleftarrow{p'_\#}} & \pi(CK, K; P'X_1, W) & \underset{\approx}{\xrightarrow{\delta}} & \pi(K, W)
\end{array}$$

where W bears the same relation to X_1, X_2 as D does to E_1, E_2. By [7] Theorem 6.3 the right-hand $f_\#$ is bijective if dim $K < n-1$, surjective if dim $K \leqq n-1$, and Theorem 4.1 follows.

The following collapsing theorem is immediate from Theorem 4.1 and [12] p. 487 Corollary 6.

Corollary (4.2): *Let (E_1, E_2) be an ex-space pair over B, where ρ_{E_1}, ρ_{E_2} and ρ_{E_1/E_2} are Serre fibrations. Let F_1, F_2 be the fibres of E_1, E_2 over some point of B. Suppose that F_2 is m-connected, $m \geqq 1$, and (F_1, F_2) has the homotopy type of an n-connected relative CW-complex, $n \geqq 2$. Then the function*

$$k_\# : \pi(CK, K; E_1, E_2) \to \pi(\Sigma K, E_1/E_2)$$

induced by the collapsing ex-map $k : (E_1, E_2) \to (E_1/E_2, \sigma_{E_1/E_2}(B)$ is bijective for dim $K < m+n$, surjective for dim $K \leqq m+n$.

We remark in passing that Theorem 4.1 in conjunction with [12] p. 484 Theorem 5 yields an ex-homotopy excision theorem.

5. The EHP-sequence

Our objective is to investigate the suspension functor Σ in the metastable range.

Proposition (5.1): (*Stability theorem*) (*c.f.* [7] *Theorem* 6.4)

Let (E, ρ, σ) be a Hurewicz ex-space over B, where σ is a cofibration, and let K be an ex-complex over B. If the fibre of E is m-connected, then the suspension function

$$\Sigma : \pi(K, E) \to \pi(\Sigma K, \Sigma E)$$

is injective if dim $K \leqq 2m$, *surjective if* dim $K \leqq 2m+1$.

PROOF: By [2] Corollary 3.4 the projection of the ex-space $\Omega\Sigma E$ is a fibration, and so [7] Theorem 6.3 may be applied in the same way as in the proof of [7] Theorem 6.4 to prove our proposition.

THEOREM (5.2) (EHP-sequence): *Let (E, ρ, σ) be a placid, distance-based ex-space whose fibre F is m-connected ($m \geqq 1$) and let K be a proper ex-complex of dimension k. Then there are pointed functions H, P (all H, P except the final P are homomorphisms between ex-homotopy groups) such that the following sequence is exact:*

$$\pi(\Sigma^{3m-k+1}K, E) \xrightarrow{\Sigma} \pi(\Sigma^{3m-k+2}K, \Sigma E) \xrightarrow{H} \pi(\Sigma^{3m-k+2}K, E * E) \xrightarrow{P} \cdots$$
$$\cdots \to \pi(\Sigma K, E) \xrightarrow{\Sigma} \pi\,(\Sigma^2 K, \Sigma E) \xrightarrow{H} \pi(\Sigma^2 K, E * E) \xrightarrow{P} \pi(K, E)$$
$$\xrightarrow{\Sigma} \pi(\Sigma K, \Sigma E).$$

PROOF: The restriction to the fibres of the inclusion ex-map: $E_2 \subset E_\infty$ (see Section 1 for notation) is the inclusion map : $F_2 \subset F_\infty$. The function inclusion$_\#$: $\pi_r(F_2) \to \pi_r(F_\infty)$ is bijective for $r < 3m+2$ and surjective for $r \leqq 3m+2$. By the comparison theorem ([7]Theorem 6.3) inclusion$_\#$: $\pi(K, E_2) \to \pi(K, E_\infty)$ is injective if dim $K < 3m+2$, surjective if dim $K \leqq 3m+2$. (Observe that the projections of E_2 and E_∞ are Hurewicz fibrations by a proof similar to that of [2] Corollary 3.4. Namely, if λ is a special lifting function for ρ with respect to σ then the composite map

$$\text{quotient} \cdot \lambda^n : \tilde{\Omega}_\rho n \to (E^n)^I \to (E_n)^I$$

factors through $\tilde{\Omega}_\rho n$.)

From the exact ex-homotopy sequence of the pair (E_∞, E_2) (Proposition 3.1) we have $\pi(CK, K; E_\infty, E_2) = 0$ for $\dim K < 3m+2$. By Proposition 3.2

$$\pi(C\Sigma K, \Sigma K; E_\infty, E_2) \xrightarrow{\delta} \pi(CK, K; E_2, E) \xrightarrow{\text{incln}\#} \# \ \pi(CK, K; E_\infty E)$$
$$\xrightarrow{\text{incln}} {}_\# \ \pi(CK, K; E_\infty, E_2)$$

is exact, and so

$$\text{(5.3)} \qquad \text{inclusion}_\# : \pi(CK, K; E_2, E) \to \pi(CK, K; E\text{w}, E)$$

is injective for dim $K < 3m+2$, and surjective for dim $K \leqq 3m+2$.

The collapsing ex-map: $(E_2, E) \to (E_2/E, \sigma_{E_2/E}(B))$ induces a pointed function

$$\text{(5.4)} \qquad \text{collapse}_\# : \pi(CK, K; E_2, E) \to \pi(\Sigma K, E_2/E) = \pi(\Sigma K, E \# E)$$

which by Corollary 4.2 is injective for dim $K < 3m+1$ and surjective for dim $K \leqq 3m+1$.

The pointed space $F \# F$ is $2m$–connected, and so by Propositions 5.1 and 2.1

(5.5) $\Sigma : \pi(K, E \# E) \to \pi(\Sigma K, \Sigma(E \# E)) \cong \pi(\Sigma K, E * E)$

is injective for dim $K \leqq 4m$, and surjective for dim $K \leqq 4m+1$.

Consider the ex-homotopy exact sequence (3.1) of the pair (E_∞, E)

(5.6) $$\cdots \xrightarrow{\delta} \pi(\Sigma K, E) \xrightarrow{i_\#} \pi(\Sigma K, E_\infty) \xrightarrow{j_\#} \pi(CK, K; E_\infty, E) \xrightarrow{\delta} \pi(K, E) \xrightarrow{i_\#} \pi(K, E_\infty)$$

where i, j are inclusion ex-maps. By Proposition 1.3 there is a bijection: $\pi(K, E_\infty) \approx \pi(K, \Omega\Sigma E)$ for all dim K, and (5.3), (5.4) and (5.5) provide a bijection: $\pi(CK, K; E_\infty, E) \to \pi(\Sigma^2 K, E * E)$ for dim $K < 3m+1$. Inserting these bijections in (5.6) and observing that the diagram

$$\begin{tikzcd} \pi(K, E) \arrow[r, "\text{inclusion}_\#"] \arrow[d, "\Sigma"'] & \pi(K, E_\infty) \arrow[d, phantom, "\wr\wr"] \\ \pi(\Sigma K, \Sigma E) \arrow[r, "\text{adjoint}", "\approx"'] & \pi(K, \Omega\Sigma E) \end{tikzcd}$$

commutes, one deduces Theorem 5.2.

When the fibre F of ρ is a sphere one can obtain information about Σ irrespective of the dimension of K.

Theorem (5.7.i): *Let (E, ρ, σ) be a placid distance-based ex-space over B with fibre S^m, m odd. Let K be a proper ex-complex over B. Then there are pointed functions H, P (homomorphisms except for the final P) such that the sequence*

$$\cdots \to \pi(\Sigma K, E) \xrightarrow{\Sigma} \pi(\Sigma^2 K, \Sigma E) \xrightarrow{H} \pi(\Sigma^2 K, E * E) \xrightarrow{P} \pi(K, E) \xrightarrow{\Sigma} \pi(\Sigma K, \Sigma E)$$

is exact.

Proof: Write $c : (E_2, E) \to (E \# E, B)$ for the collapsing ex-map, and $C : (E_\infty, E) \to ((E \# E)_\#, B)$ for the combinatorial extension of c see [8] p. 176 Lemma 2.5). By the theorem of I. M. James ([9] Theorem 1.2 or [14] Theorem 2.4), $(C|\text{fibre})_\# : \pi_i((S^m)_\infty, S^m) \to \pi_i((S^{2m})_\infty)$ is isomorphism for all i. By Theorem 4.1 $C_\# : \pi(CK, K; E_\infty, E) \to \pi(\Sigma K, (E \# E)_\infty)$ is bijective. Theorem 5.7 (i) follows from the ex-homotopy exact sequence of the pair (E_∞, E) as in the last paragraph of the proof of Theorem 5.2.

Let $\mathscr{C}$ be the Serre class of torsion abelian groups of odd order.

Theorem (5.7ii): *Let (E, ρ, σ) be a placid distance- based ex-space over a finite CW-complex B with fibre S^m, m even. Let K be a proper placid ex-complex over B with compact total space. Then there is a $\mathscr{C}$-exact sequence*

$$\cdots \to \pi(\Sigma^3K, E) \xrightarrow{\Sigma} \pi(\Sigma^4K, \Sigma E) \xrightarrow{H} \pi(\Sigma^4K, E * E) \xrightarrow{P} \pi(\Sigma^2K, E) \xrightarrow{\Sigma} \pi(\Sigma^3k, \Sigma E).$$

PROOF: We use the notation in Theorem 5.7(i). We know that $(C|\text{fibre})_\# : \pi_i((S^m)_\infty, S^m) \to \pi_i((S^m \# S^m)_\infty, *)$ is a $\mathscr{C}$-isomorphism for all i (by [9] Theorem 1.3 or [14] Theorem 2.4). To deduce that

$$C_\# : \pi(C\Sigma^2K, \Sigma^2K; E_\infty, E) \to \pi(\Sigma^3K, (E \# E)_\infty)$$

is a $\mathscr{C}$-isomorphism one uses [3] Theorem 3.1 made relative by the argument of Theorem 4.1.

6. Some calculations

Let E^3 denote the fibre suspension of the Hopf bundle over S^2, and regard E^3 as an ex-space over S^2 by choosing a cross-section (in one of the obvious ways). For $r > 3$ inductively define the ex-space E^r over S^2 by $E^r = \Sigma E^{r-1}$. (By [7] Theorem 6.1 E^r is ex-homotopically equivalent to the sphere bundle (with section) associated to the Whitney sum of the canonical complex line bundle over CP^1, regarded as a real vector bundle, and the product bundle over CP^1 with fibre $\boldsymbol{R}^{r-2}$.)

(6.1) $\qquad \pi(E^6, E^5) \approx Z \oplus Z_{24}, \pi(E^7, E^6) \approx Z_2 \oplus Z_{24}.$

By [6] Theorem (1.6), (1.8) there is an exact sequence

$$\cdots \to \pi_6 S^4 \xrightarrow{\Psi} \pi_7 S^4 \xrightarrow{\Theta} \pi(E^6, E^3) \xrightarrow{\varphi'} \pi_5 S^4 \xrightarrow{\Psi'} \pi_6 S^4$$

where

$$\Psi(\alpha) = \alpha \cdot S^4J(\beta) - S^2J(\beta) \cdot S\alpha \qquad \alpha \in \pi_6 S^4$$
$$\Psi'(\alpha') = \alpha' \cdot S^3J(\beta) - S^2J(\beta) \cdot S\alpha' \quad \alpha' \in \pi_5 S^4$$

where $\beta \in \pi_1(\mathrm{O}_2)$ is the classifying element for the Hopf bundle. We adopt standard notation (Toda [14]); then $\pi_6 S^4 \approx Z_2$ is generated by $\langle \eta \cdot \eta \rangle$ and $\pi_5 S^4 \approx Z_2$ is generated by $\langle \eta \rangle$. Since $\Psi(\eta \cdot \eta) = 0$ and $\Psi'(\eta) = 0$ we deduce that $\pi(E^6, E^5)$ is an extension of $Z \oplus Z_{12}$ by Z_2, but at this stage we do not know which. Similarly $\pi(E^7, E^6)$ is either $Z_2 \oplus Z_{24}$ or Z_{48}.

Apply Theorem 5.2 (the EHP-sequence) with "K" $= E^5$, "E" $= E^5$, "F" $= S^4$, "m" $= 3$, "B" $= S^2$, "k" $= 4+2 = 6$. We have $3m-k+1 = 4$ and so the following sequence is exact

$$\pi(E^8, E^6) \xrightarrow{H} \pi(E^8, \Sigma(E^5 \# E^5)) \xrightarrow{P} \pi(E^6, E^5) \to \pi(E^7, E^6)$$
$$\xrightarrow{H} \pi(E^7, \Sigma(E^5 \# E^5)) \text{ i.e. } Z_2 \xrightarrow{H} Z \xrightarrow{P} \pi(E^6, E^5) \xrightarrow{\Sigma} \pi(E^7, E^6) \xrightarrow{H} 0$$

Comparing this sequence with the previous results we obtain (6.1).

(6.2) $\underline{\pi(E^5, E^4) \approx Z_{24}.}$

From the exact sequence of [6]

$$\cdots \to \pi_5 S^3 \xrightarrow{0} \pi_6 S^3 \longrightarrow \pi(E^5, E^4) \longrightarrow \pi_4 S^3 \xrightarrow{0} \pi_5 S^3$$

we deduce $\pi(E^5, E^4) \approx Z_{12} \oplus Z_2$ or Z_{24}. By the EHP-sequence with "K" $= E^4$, "E" $= E^4$, "m" $= 2$, "k" $= 5$, $3m-k+1 = 2$ the sequence

$$\cdots \xrightarrow{H} \pi(E^7, \Sigma(E^4 \# E^4)) \xrightarrow{P} \pi(E^5, E^4) \xrightarrow{\Sigma} \pi(E^6, \pi(E^4 \# E^4)) \xrightarrow{P} \pi(E^4, E^4) \to \pi(E^5, E^5)$$

is exact. This sequence is:

$$\cdots \xrightarrow{H} 0 \xrightarrow{P} \pi(E^5, E^4) \xrightarrow{\Sigma} \pi(E^6, E^5) \xrightarrow{H} Z \xrightarrow{P} Z_2 \oplus Z \xrightarrow{\Sigma} Z_2 \oplus Z.$$

The final Σ is surjective since it is the last ex-suspension before the stable range, and by the specific nature of the groups it is an isomorphism. From (6.1) and the exact sequence we deduce that $\pi(E^5, E^4) \approx Z_{24}$.

REMARK: All auxiliary ex-homotopy groups used in this calculation can be computed using [6] and standard results on the homotopy groups of spheres.

Further remarks:

Let D, E, X be ex-spaces, and let $u : \Sigma D \to X$, $v : \Sigma E \to X$ be ex-maps. We define an ex-map $[u, v] : D{*}E \to X$ as follows. Regard u, v as ex-maps of pairs

$$u : (CD, D) \to (X, \sigma_*(B)) \qquad v : (CE, E) \to (X, \sigma_* B).$$

The map $u \times v$ determines an ex-map : $CD \times CE \to X \times X$, and the restriction to the subex-space $CD \times E\ D \times CE = D{*}E$ maps into $X_V X$ and determines an ex-map : $D{*}E \to X_V X$. The composite of this ex-map with the folding ex-map: $X_V X \to X$ is denoted $[u, v]$. If the ex-maps u, u' : $\Sigma D \to X$ are ex-homotopic then so are $[u, v]$ and $[u', v]$, and the analogous statement holds for v. Thus a pairing,

$$\pi(\Sigma D, X) \times \pi(\Sigma E, X) \xrightarrow{[\,,\,]} \pi(D{*}E, X)$$

called the *Whitehead product*, is induced at the ex-homotopy level.

As one would expect, the function P in Theorem 5.2 is related to the Whitehead product. Suppose in Theorem 5.2 that $E = \Sigma E'$ where E' is a placid ex-space. Then one can show by a naturality argument (see [4] Theorem 6.5) that the diagram

$$\begin{tikzcd}
\pi(\Sigma^2 K, \Sigma^3(E' \# E')) \arrow[r, "P"] & \pi(K, \Sigma E') \\
\pi(K, \Sigma(E' \# E')) \arrow[u, "\approx"] \arrow[ur, "{[i,\, i] \circ (\ )}"'] &
\end{tikzcd}$$

commutes, where $i : \Sigma E' \to \Sigma E'$ is the identity ex-map. One corollary of this result is that, with appropriate conditions on the ex-spaces E_1 and E_2, if $\alpha \in \pi(\Sigma E_1, X)$ and $\beta \in \pi(\Sigma E_2, X)$ thef $P(\cdot[\alpha, \beta])$ is precisely the join $\alpha * \beta \in \pi(\Sigma E_1 * \Sigma E_2, X * X)$.

One can easily derive many properties of the Whitehead product along the lines of [1] by introducing an ex-homotopy theory analogue of the Samelson product (see [4]). However at later stages difficulties arise, some of which may be discussed in a future paper.

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