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ENDS OF GROUPS AND BASELESS SUBGROUPS OF WREATH PRODUCTS

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Introduction

In this paper we investigate the relationship between Specker's theory [4] of ends of groups and Hartley's results [2] on baseless subgroups of wreath products. The wreath product $W = A \; Wr \; B$ of groups A and B is the split extension of the group $K = A^B$ of functions from B to A, by B, with $f^b(x) = f(xb^{-1})$, for $f \in A^B$, b, $x \in B$. We shall assume throughout that A and B are non-trivial. Let α be an infinite cardinal and, for $f \in K$, let supp (f) denote the support of f. We define K_α to consist of those $f \in K$ with $|\text{supp}\,(f)| < \alpha$ and let $W_\alpha = BK_\alpha \leqq W$. We note that in the case $\alpha = \aleph_0$, K_α consists of those functions with finite support and W_α is the restricted wreath product $A \; wr \; B$ of A and B.

Hartley [2] investigates baseless subgroups of W_α, that is, subgroups which have trivial intersection with the base group K_α. He gives conditions for baseless subgroups of W_α to be conjugate to subgroups of B and conditions for the existence of baseless subgroups which are maximal in some class of subgroups of W_α. He applies these results to construct locally finite groups with certain given sets of locally finite p-groups as Sylow p-subgroups. Our aim is to establish a connection between Hartley's results and the theory of ends, which we now consider.

For a group G, we define $Q_\alpha(G)$ to consist of those $S \subseteq G$ such that $|Sg \cap S'| < \alpha$, for all $g \in G$, where $S' = G \setminus S$. The set of α-ends of G consists of the ultrafilters of sets S in $Q_\alpha(G)$ with $|S| \geqq \alpha$. We denote the number of α-ends of G by $e_\alpha(G)$; in the case $\alpha = \aleph_0$, we omit mention of the cardinal. From now on we assume $|G| \geqq \alpha$, which is equivalent to the condition $e_\alpha(G) > 0$. Specker [4] shows that $e_\alpha(G) = 1$, 2, or is infinite and that $e_\alpha(G) = 2$ if and only if $\alpha = \aleph_0$ and G is infinite cyclic by finite. We note that $e_\alpha(G) = 1$ if and only if, for $S \subseteq G$, the condition $|Sg \cap S'| < \alpha$, for all $g \in G$, implies $|S| < \alpha$ or $|S'| < \alpha$.

The theory of ends of groups, with $\alpha = \aleph_0$, has been studied extensively (see Cohen [1], Stallings [5]); in particular, finitely generated groups with more than one end have been characterised. Various sufficient conditions for G to have 1 end are known and some may be extended to the case of a general cardinal α. We do not pursue this point, as the results in the case $\alpha = \aleph_0$ are far from complete. We require the following result.

THEOREM (1): (Specker [4, p. 173 (a)]). *If either* $|G| = \alpha > \aleph_0$ *or* $|G| = \alpha = \aleph_0$ *and* G *is locally finite, then* $e_\alpha(G)$ *is infinite.*

Baseless subgroups and ends

Let L be a baseless subgroup of $W = A \; Wr \; B$. Then, for a unique $C \leqq B$, $LA^B = CA^B$. Hartley shows [2, Lemma 3.2. (i)] that there is an element $g \in A^B$ such that $L = C^g$. It follows that the baseless subgroups of W_α are all the subgroups of W of the form C^g with $C \leqq B$, $g \in A^B$ and $C^g \leqq W_\alpha$. Throughout this section, let $S' = B \setminus S$, for $S \subseteq B$.

LEMMA (2): *Suppose* $C \leqq B$ *and* $g \in A^B$. *Let* $g(B) = \{a_i : i \in I\}$ *and* $B_i = \{b \in B : g(b) = a_i\}$. *Then*
(i) $C^g \leqq W_\alpha$ *if and only if*

$$|\bigcup_{i \in I} B_i c \cap B_i'| < \alpha,$$

for all $c \in C$,
(ii) if $|C| \geqq \alpha$, C^g *is conjugate to* C *in* W_α *if and only if there is a subset* J *of* I *and a partition* $\{D_j : j \in J\}$ *of* B *with* $D_j C = D_j$ *such that*

$$|\bigcup_{j \in J} D_j \cap B_j'| < \alpha.$$

PROOF: (i) If $c \in C$, $c^g = g^{-1}cg = c(g^{-1})^c g$ and so $c^g \in W_\alpha$ if and only if $|\mathrm{supp}\ ((g^{-1})^c g| < \alpha$. Now $((g^{-1})^c g)(x) = (g(xc^{-1}))^{-1} g(x)$, so

$$\mathrm{supp}\,((g^{-1})^c g) = \bigcup_{i \in I} B_i c \cap B_i'.$$

Thus $C^g \leqq W_\alpha$ if and only if

$$|\bigcup_{i \in I} B_i c \cap B_i'| < \alpha, \text{ for all } c \in C.$$

(ii) Suppose $C^g = C^h$ with $h \in K_\alpha$. Then $z = gh^{-1}$ is in the centraliser of C and hence is constant on each left coset bC of C in B. Since $|\mathrm{supp}\ (z^{-1}g)| = |\mathrm{supp}\,(h)| < \alpha$ and $|C| \geqq \alpha$, $z(B) \subseteq g(B)$. Let $J = \{i \in I : a_i \in z(B)\}$ and let $D_j = \{b \in B : z(b) = a_j\}$, for $j \in J$. Then $\{D_j : j \in J\}$ is a partition of B with $D_j C = D_j$ and

$$\bigcup_{j \in J} D_j \cap B_j' = \mathrm{supp}\,(z^{-1}g) = \mathrm{supp}\,(h)$$

so

$$|\bigcup_{j \in J} D_j \cap B_j'| < \alpha.$$

Conversely, suppose the given condition is satisfied and define $z \in A^B$ by $z(D_j) = (g(B_j))^{-1}$. Then $C^g = C^{zg}$ and

$$\operatorname{supp}(zg) = \bigcup_{j \in J} D_j \cap B'_j,$$

so $|\operatorname{supp}(zg)| < \alpha$.

THEOREM (3): *Suppose L is a baseless subgroup of W_α with $LA^B = CA^B$, $C \leqq B$. If $e_\alpha(C) = 1$, then L is conjugate to C in W_α.*

PROOF: We have $L = C^g$ where the partition associated with g satisfies

$$|\bigcup_{i \in I} B_i c \cap B'_i| < \alpha,$$

for $c \in C$. For $i \in I$, $b \in B$ we have, for all $c \in C$,

$$B_i c \cap B'_i \supseteq (bC \cap B_i)c \cap (bC \cap B'_i) = b((C \cap b^{-1}B_i)c \cap (C \cap b^{-1}B'_i))$$

and hence $|(C \cap b^{-1}B_i)c \cap (C \setminus (C \cap b^{-1}B_i))| < \alpha$. Since $e_\alpha(C) = 1$, $|C \cap b^{-1}B_i| < \alpha$ or $|C \cap b^{-1}B'_i| < \alpha$ and so $|bC \cap B_i| < \alpha$ or $|bC \cap B'_i| < \alpha$. Let $M = \{(bC, i) : 0 < |bC \cap B_i| < \alpha\}$.

We first suppose $|C| > \alpha$ and take $N \subseteq M$, with $|N| \leqq \alpha$. Then

$$|\bigcup_{(bC, i) \in N} (bC \cap B_i)^{-1}(bC \cap B_i)| \leqq \alpha,$$

so there is an element $c \in C$ such that, for $(bC, i) \in N$, $c \notin (bC \cap B_i)^{-1}(bC \cap B_i)$ and $(bC \cap B_i)c \subseteq B'_i$. Then

$$\begin{aligned} |\bigcup_{(bC, i) \in N} bC \cap B_i| &= |\bigcup_{(bC, i) \in N} (bC \cap B_i)c| \\ &= |\bigcup_{(bC, i) \in N} (bC \cap B_i)c \cap B'_i| \\ &\leqq |\bigcup_{i \in I} B_i c \cap B'_i| < \alpha. \end{aligned}$$

So $|N| < \alpha$ and we deduce that $|M| < \alpha$ and

$$|\bigcup_{(bC, i) \in M} bC \cap B_i| < \alpha.$$

In the other case, $|C| = \alpha$. Since we are assuming $e_\alpha(C) = 1$, Theorem 1 implies that $\alpha = \aleph_0$ and C is not locally finite. Let $D = \langle d_1, \cdots, d_n \rangle$ be a finitely generated infinite subgroup of C. Since, for $j = 1, \cdots, n$,

$$\bigcup_{i \in I} B_i d_j \cap B'_i$$

is finite, almost all B_i are fixed under right multiplication by $d_1, \cdots, d_n$. So $H = \{i : B_i \neq B_i D\}$ is finite. If $(bC, i) \in M$ then $bC \cap B_i$ is finite and non-empty. Thus $bC \cap B_i \neq (bC \cap B_i)D = bC \cap B_i D$, so $B_i \neq B_i D$

and $i \in H$. Also, for some d_j, $(bC \cap B_i)d_j \cap B'_i \neq \emptyset$ so, for some $c \in C$, $bc \in B_i \cap B'_i d_j^{-1}$ and $b \in (B_i \cap B'_i d_j^{-1})C$. Thus if $(bC, i) \in M$, then $i \in H$ and $bC \subseteq FC$, where F is the finite set

$$\bigcup_{j=1}^{n} (\bigcup_{i \in I} B_i d_j \cap B'_i) d_j^{-1}.$$

So M is finite and hence

$$\bigcup_{(bC,\, i) \in M} bC \cap B_i$$

is also finite.

In each case, we now have

$$|\bigcup_{(bC,\, i) \in M} bC \cap B_i| < \alpha.$$

Furthermore,

$$|C| \geqq \alpha \text{ and } bC = \bigcup_{i \in I} bC \cap B_i.$$

Thus, for $b \in B$, $|bC \cap B_i| \geqq \alpha$ for some i and then $|bC \cap B_j| < \alpha$ for $\neq i$. For $i \in I$, let $T_i = \{bC : |bC \cap B_i| \geqq \alpha\}$ and let

$$D_i = \bigcup_{bC \in T_i} bC.$$

Put $J = \{i \in I : D_i \neq \emptyset\}$. Then $\{D_j : j \in J\}$ is a partition of B with $D_j = D_j C$. For $j \in J$,

$$D_j \cap B'_j \subseteq \bigcup_{(bC,\, i) \in M} bC \cap B_i$$

so

$$|\bigcup_{j \in J} D_j \cap B'_j| \leqq |\bigcup_{(bC,\, i) \in M} bC \cap B_i| < \alpha.$$

Then Lemma 2 (ii) implies L is conjugate to C in W_α.

Theorem 1 describes a class of groups C with $e_\alpha(C)$ infinite. We now consider these groups in more detail.

LEMMA (4): *Suppose $|C| = \alpha > \aleph_0$ or $|C| = \alpha = \aleph_0$ and C is locally finite. Then there is a partition $\{C_1, C_2\}$ of C such that $|C_1| = |C_2| = \alpha$ and $|sC_1 t \cap C_2| < \alpha$, for all $s, t \in C$.*

PROOF: Consider α as an ordinal equivalent to none of its predecessors. Under the conditions of the theorem,

$$C = \bigcup_{\lambda < \alpha} H_\lambda$$

for some system of subgroups such that $|H_\lambda| < \alpha$ and $H_\lambda < H_\mu$ if $\lambda < \mu$.

Let

$$C_1 = \bigcup_{\lambda<\alpha} H_{2\lambda+1}\backslash H_{2\lambda}.$$

Then

$$C_2 = C\backslash C_1 = H_0 \cup \bigcup_{\lambda<\alpha} H_{2\lambda+2}\backslash H_{2\lambda+1}$$

and $|C_1| = |C_2| = \alpha$. Suppose $s, t \in C$. For some ordinal $\mu < \alpha$, we have $s, t \in H_\lambda$, for $\lambda \geqq \mu$, and so $sH_\lambda t = H_\lambda$ and $s(H_{\lambda+1} \setminus H_\lambda)t = H_{\lambda+1} \setminus H_\lambda$. Thus $sC_1 t \cap C_2 \subseteq sH_\mu t$ and $|sC_1 t \cap C_2| < \alpha$.

A refinement of this argument gives a proof of Theorem 1. For a positive integer n,

$$C_i = \bigcup_{\lambda<\alpha} H_{n\lambda+i}\backslash H_{n\lambda+i-1}$$

is in $Q_\alpha(C)$ and $\{C_1, \cdots, C_{n-1}, C_n \cup H_0\}$ is a partition of C. Thus the number of α-ends of C is unbounded.

Suppose $e_\alpha(C) = 2$. Then $\alpha = \aleph_0$ and C has an infinite cyclic normal subgroup $E = \langle e \rangle$ of finite index. The centraliser H of E in C has index 1 or 2 and so $C = H \cup Hd = E(F \cup Fd)$, where $d = 1$ or $d \in C \setminus H$ and the finite set F is a set of coset representatives for E in H. Let $P = \{e^i : i > 0\}$, $C_1 = P(F \cup Fd)$ and $C_2 = C \setminus C_1$. For $s \in H$, $t \in C$, $s(F \cup Fd)t$ is a set of coset representatives for E in C and $sC_1 t = Ps(F \cup Fd)t = P\{e^{i(f)}f : f \in F \cup Fd\}$, for some finite set of integers $\{i(f)\}$. Then $sC_1 t \cap C_2$ is finite, and similarly $sC_2 t \cap C_1$ is finite, for $s \in H$, $t \in C$. Thus we have proved the following result.

LEMMA (5): *Suppose C has 2 ends. Then there is a partition $\{C_1, C_2\}$ of C, with $C_1 \in Q(C)$, such that the sets $\{c \in C_2 : cC_1 \cap C_2$ is finite$\}$ and $\{c \in C_1 : cC_2 \cap C_1$ is finite$\}$ are infinite.*

Once again we consider C as a subgroup of B in W_α, but now assuming $e_\alpha(C) > 1$. We recall that this implies that either $e_\alpha(C)$ is infinite or $\alpha = \aleph_0$ and $e_\alpha(C) = e(C) = 2$.

THEOREM (6): *Suppose $C \leqq B$ with $e_\alpha(C) > 1$. Then there is a baseless subgroup L of W_α satisfying the following conditions:*

(i) $LA^B = CA^B$,

(ii) L is not conjugate to C in W_α,

(iii) if M is a baseless subgroup of W_α with $M > L$ then $C = N_D(C)$ where $MA^B = DA^B$ with $D \leqq B$. Furthermore, if either $|C| = \alpha > \aleph_0$ or otherwise $|C| = \alpha = \aleph_0$ and C is locally finite or has 2 ends, then the assumptions of (iii) imply the following stronger result:

(iv) $|C \cap uCv^{-1}| < \alpha$ for all $u, v \in D \setminus C$.

PROOF: Since $e_\alpha(C) > 1$, there is a partition $\{C_1, C_2\}$ of C with $|C_i| \geqq \alpha$ and $|C_1 c \cap C_2| < \alpha$, for all $c \in C$. Then $|C_2 c \cap C_1| = |C_2 \cap C_1 c^{-1}| < \alpha$, for all $c \in C$. Take $a \in A$, $a \neq 1$, and define $g \in A^B$ by $g(C_1) = a$, $g(C_1') = 1$, where C_1' denotes $B \setminus C_1$. For $c \in C$, $|(C_1 c \cap C_1') \cup (C_1' c \cap C_1)| = |(C_1 c \cap C_2) \cup (C_2 c \cap C_1)| < \alpha$, so Lemma 2(i) implies $C^g \leqq W_\alpha$. Let $\{D_1, D_2\}$ be a partition of B with $D_i C = D_i$ and suppose $C \subseteq D_1$ (here we allow $D_2 = \emptyset$). Then $D_1 \cap C_1' \supseteq C_2$ and $D_1 \cap C_1 = C_1$ so, from Lemma 2 (ii), C^g is not conjugate to C in W_α.

Under the assumptions of (iii) we have $C^g \leqq D^h \leqq W_\alpha$ for some $h \in A^B$. Then $C^g = C^h$ so hg^{-1} centralises C and the parts of the partition of B corresponding to hg^{-1} consist of unions of left cosets bC. Suppose $hg^{-1}(C) = a_1$ and so $h(C_1) = a_1 a$, $h(C_2) = a_1$, and $h(b) = hg^{-1}(b)$ for $b \notin C$. Let $B_1 = \{b : h(b) = a_1 a\}$, $B_2 = \{b : h(b) = a_1\}$. Then

$$B_1 = C_1 \cup \bigcup_{bC \in T_1} bC, \quad B_2 = C_2 \cup \bigcup_{bC \in T_2} bC,$$

where T_1, T_2 are disjoint sets of cosets distinct from C. From Lemma 2(i), $|B_i d \cap B_i'| < \alpha$, for $d \in D$ and $i = 1, 2$. If $d \in N_D(C) \setminus C$, then $|C_i d \cap dC| \geqq \alpha$ so $dC \cap B_1 \neq \emptyset \neq dC \cap B_2$ and $dC \in T_1 \cap T_2 = \emptyset$. Thus $N_D(C) = C$.

We now suppose that either $|C| = \alpha > \aleph_0$ or otherwise $|C| = \alpha = \aleph_0$ and C is locally finite or has 2 ends. We can then assume that the partition $\{C_1, C_2\}$ has been chosen as described in Lemma 4 or 5. Given $u, v \in D \setminus C$, we have $|C_i u \cap B_i'| < \alpha$, $|C_i v \cap B_i'| < \alpha$. Thus for some $G_i \subseteq C_i$ with $|C_i \setminus G_i| < \alpha$, we have $G_i u \cup G_i v \subseteq B_i$. Then

$$G_i u \cup G_i v \subseteq \bigcup_{bC \in T_i} bC.$$

Thus, for $c_1 \in G_1$, $c_2 \in G_2$, $c_1 uC \neq c_2 vC$ and $c_1 vC \neq c_2 uC$ and so $c_1^{-1} c_2$, $c_2^{-1} c_1 \notin uCv^{-1}$. From Lemmas 4 and 5, since $|C_1 \setminus G_1| < \alpha$, we may choose $c_1 \in G_1$ such that $|c_1 C_2 \cap C_1| < \alpha$ and hence $|c_1^{-1} C_1 \cap C_2| < \alpha$. Now $\{c_2 \in C_2 : c_2 \notin uCv^{-1}\} \supseteq c_1^{-1} G_2 \cap C_2$ so $C_2 \cap uCv^{-1} \subseteq C_2 \cap c_1^{-1}(C \setminus G_2) \subseteq (C_2 \cap c_1^{-1} C_1) \cup c_1^{-1}(C_2 \setminus G_2)$ and hence $|C_2 \cap uCv^{-1}| < \alpha$. Similarly, $|C_1 \cap uCv^{-1}| < \alpha$ and so $|C \cap uCv^{-1}| < \alpha$.

We note a consequence of Theorems 3 and 6. This generalises a result in [3] that if $A^{(B)}$ is the group of functions from B to A with finite support and B acts as in the wreath product, then the first cohomology set $H^1(B, A^{(B)})$ is trivial if and only if B has 1 end. However $H^1(B, A^{(B)})$ is precisely the set of conjugacy classes of complements of the base group $A^{(B)}$ in $A \text{ wr } B = W_{\aleph_0}$.

COROLLARY (7): *Suppose $C \leqq B$ with $|C| \geqq \alpha$. Every baseless subgroup L of W_α such that $LA^B = CA^B$ is conjugate to C in W_α if and only if $e_\alpha(C) = 1$.*

For certain subgroups C of B, Theorem 6 gives sufficient conditions for the existence of baseless subgroups L, with $LA^B = CA^B$, which are maximal in certain classes. Hartley's Theorem B, the first part of Theorem A and the second part of Theorem D [2] may be deduced. Using Corollary 7, the other parts of his Theorems A and D lead to new sufficient conditions for a group to have 1 end. Thus, unless they are infinite cyclic by finite, radical groups with non-periodic Hirsch-Plotkin radical have 1 end and so also do uncountable locally finite groups satisfying the normaliser condition. In this connection we mention the conjecture that uncountable locally finite groups have 1 end.

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