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## ON AUTOMORPHISM GROUPS OF FINITE DIMENSIONAL MODULES

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The object of this paper is to obtain a representation for locally finite groups  $G$  of automorphisms of finite dimensional modules. This representation is one in which the group is expressed as a direct sum of a finite number of linear groups modulo a unipotent normal subgroup. As a consequence of this representation, several theorems on linear groups can be extended to a new setting.

The structure of the paper is as follows. First we recall some facts from ring theory which are going to be used throughout the paper. Then we prove the main theorems concerning the representation of  $G$  as a direct sum of a finite number of linear groups modulo a unipotent radical. Finally, general consequences are derived from the representation theorem.

### 1. Preliminaries on rings and modules

We begin with some preliminary results on rings, which have been lifted from [3].

If  $R$  is a ring (associative with identity 1), we let  $Rad R$  denote the Jacobson radical of  $R$ . This is the largest ideal  $K$  such that for all  $r$  in  $K$ ,  $1-r$  is a unit. Equivalently,  $Rad R$  is the intersection of all the maximal right ideals. Thus, the Jacobson radical contains every left, right and 2-sided nil ideal of  $R$ . A ring  $R$  is *semisimple* if  $Rad R = 0$ . For any arbitrary ring  $R$ , we have that  $R/Rad R$  is semisimple. A ring  $R$  is *Artinian (Noetherian)* if it is Artinian (Noetherian) as a right  $R$ -module; i.e., if it satisfies the descending (ascending) chain condition on right ideals. It can be seen that the Jacobson radical of an Artinian ring is a nilpotent ideal. The classical theorem of Wedderburn-Artin says that a ring  $R$  is a nonzero semisimple Artinian ring if and only if  $R$  is isomorphic to a direct sum of a finite number of rings, each of which is a complete matrix ring over some division ring.

A nonzero right ideal  $I$  of a ring  $R$  is *indecomposable* if it is not isomorphic to the direct sum of nonzero right ideals. Two nonzero idempotents  $e$  and  $f$  are *orthogonal* if  $ef = fe = 0$ . A nonzero idempotent  $e$  of

$R$  is called *primitive* if it cannot be written as the sum of two nonzero orthogonal idempotents. This is easily seen to be equivalent to the fact that the right ideal  $eR$  is indecomposable.

LEMMA 1.1: *An idempotent  $e \neq 0$  of  $R$  is primitive if and only if the ring  $eRe$  contains no idempotents other than 0 and  $e$ .*

PROOF: See [3]

Let  $N$  be an ideal of a ring  $R$ . We say that idempotents modulo  $N$  can be *lifted* if for every element  $u$  of  $R$  such that  $u^2 - u$  is in  $N$ , there exists an element  $e^2 = e$  in  $R$  such that  $e - u$  is in  $N$ .

LEMMA 1.2: *Let  $\text{Rad } R$  be a nil ideal. Then any finite or countable set of orthogonal nonzero idempotents modulo  $N$  can be lifted to an orthogonal set of nonzero idempotents of  $R$ .*

PROOF: See [3]

LEMMA 1.3: *Let  $R$  be a ring such that  $\text{Rad } R$  is a nil ideal. Then any primitive idempotent of  $R$  remains primitive modulo  $N = \text{Rad } R$ .*

PROOF: Let  $e$  be a primitive idempotent of  $R$ . By Lemma 1.1,  $eRe$  contains no idempotents other than 0 and  $e$ . For any  $r$  in  $R$ , let  $\bar{r}$  be the image of  $r$  under the canonical homomorphism of  $R$  onto  $\bar{R} = R/\text{Rad } R$ . To prove the proposition, it is enough to prove that  $\bar{e}\bar{R}\bar{e}$  does not contain any idempotent different from 0 or  $\bar{e}$ . Suppose, on the contrary, that  $\bar{e}\bar{r}\bar{e}$  is a nonzero idempotent of  $\bar{e}\bar{R}\bar{e}$  different from  $\bar{e}$ . Then  $\bar{1} - \bar{e}$  and  $\bar{e}\bar{r}\bar{e}$  are orthogonal idempotents in  $\bar{R}$ . Now we prove that they can be lifted to orthogonal idempotents of  $R$ , where one of them is precisely  $1 - e$ . By Lemma 1.2, we may lift  $\bar{e}\bar{r}\bar{e}$  to  $h = h^2$  such that  $h - ere$  is in  $N$ . It follows that  $h(1 - e)$  and  $(1 - e)h$  are in  $N$ . In particular,  $1 - h(1 - e)$  has an inverse in  $R$ . Consider the element

$$h' = (1 - h(1 - e))^{-1}h(1 - h(1 - e)).$$

This is an idempotent such that  $h'(1 - e) = 0$ . Furthermore,

$$(1 - h(1 - e))h' = h(1 - h(1 - e)),$$

implies that

$$h' - h(1 - e)h' = h - h(1 - e),$$

so that

$$h' - h = h(1 - e)h' - h(1 - e);$$

since the right hand side is in  $N$ , we have that  $h$  is congruent to  $h'$  modulo  $N$ . Now let

$$f = h' - (1 - e)h' = eh'.$$

Then  $(1-e)f = f(1-e) = 0$ . Also,  $f$  is congruent to  $ere$  modulo  $N$ , for we have that  $f-eh$  is in  $N$  and  $eh-h$  is in  $N$ . Moreover,

$$f^2 = (eh')(eh') = e(h'e)h' = e(h')h' = eh' = f.$$

Thus  $1-e$  and  $f$  are orthogonal idempotents that lift  $\bar{1}-\bar{e}$  and  $\bar{e}\bar{r}\bar{e}$  respectively. Since  $(1-e)f = f(1-e) = 0$  we have that

$$f = ef = ef(fe) = efe.$$

Thus  $f$  is an idempotent in  $eRe$  different from 0 and  $e$ , which is a contradiction. Q.E.D.

LEMMA 1.4: *If  $e$  is an idempotent of  $R$ , and  $N = \text{Rad } R$ , then*

$$\text{Rad}(eRe) = eNe.$$

PROOF: See [3].

LEMMA 1.5: *Let  $R$  be a semisimple ring and let  $e^2 = e \neq 0$  be in  $R$ . Then  $eR$  is a minimal right ideal if and only if  $eRe$  is a division ring.*

PROOF: See [3].

LEMMA 1.6: *Let  $R$  be an Artinian ring and let  $I$  be a non-nilpotent right ideal of  $R$ . Then  $I$  contains a nonzero idempotent.*

PROOF: See [3].

The argument for the next proposition is standard.

PROPOSITION 1.7: *Let  $R$  be an arbitrary ring and let  $R_0$  be an Artinian subring. Then  $R_0 \cap \text{Rad } R$  is a nil ideal.*

PROOF: See [3, Theorem 1, p. 38].

We say that a ring is *locally finite* if every finite set of elements generates a finite subring.

The next lemma is an easy consequence of a well-known theorem of Wedderburn.

LEMMA 1.8: *Any locally finite division ring  $D$  is commutative.*

Finally, we consider finite dimensional modules, and discuss a property of their rings of endomorphisms.

Finite dimensional modules were first considered by A. Goldie in connection with the study of noncommutative rings with maximum condition. The motivation for their study was to generalize and combine the basic theories of Artinian rings and commutative Noetherian rings.

DEFINITION: Let  $M$  be a module over an arbitrary ring  $R$ . Then  $M$  is a *finite dimensional* module if every direct sum of nonzero submodules of  $M$  has but a finite number of terms.

PROPOSITION 1.9: *Let  $M$  be an  $R$ -module of finite dimension. Then there is an integer  $n \geq 0$  such that every direct sum of nonzero submodules of  $M$  has at most  $n$  terms.*

PROOF: See [1, Theorem (3.3)].

The preceding result allows for the following definition.

DEFINITION: A finite dimensional  $R$ -module  $M$  has *dimension  $n$*  if  $M$  contains a direct sum of  $n$  nonzero submodules and each direct sum of nonzero submodules of  $M$  has at most  $n$  terms. Let  $\text{Dim } M$  denote the dimension of  $M$ .

The purpose of the next lemma is to prove that any subring of the endomorphism ring of any finite dimensional module is a direct sum of a finite number of indecomposable right ideals.

LEMMA 1.10: *Let  $M$  be a finite dimensional module over an arbitrary ring  $R$  with  $\text{Dim } M = n$ . Let  $T$  be any subring of the full ring of endomorphisms  $\text{End}_R M$ . Then  $T$  is the sum of indecomposable right ideals of the form  $e_i T$ , where  $i \leq n$  and  $e_i$  is a primitive idempotent of  $T$ .*

PROOF: Let  $E$  be any set of nonzero orthogonal idempotents of  $T$ . Then

$$\sum_{e_i \in E} M e_i$$

is a direct sum of nonzero submodules of  $M$ . Since  $\text{Dim } M = n$ , there can only be at most  $n$  idempotents in  $E$ . If  $e_1, \dots, e_m$  is a maximal set of orthogonal idempotents of  $T$ , we can easily see that  $1 = e_1 + \dots + e_m$  and that each  $e_i, i \leq m \leq n$ , is a primitive idempotent. Thus

$$T = \sum_{i=1}^n e_i T$$

and the lemma is proved.

Q.E.D.

## 2. The radical $\text{Rad } G$ of the group $G$ and the quotient $G/\text{Rad } G$

Throughout this section  $G$  will denote a locally finite group of automorphisms of a finite dimensional module  $M$  over a ring  $R$  of arbitrary nonzero characteristic. The purpose of this section is to study a certain unipotent normal subgroup of  $G$ , to be defined below. This subgroup will be denoted by  $\text{Rad } G$ , and is constructed in  $G$  from the Jacobson radical of a certain ring. The subgroup  $\text{Rad } G$  is also studied under stronger conditions for  $M$ .

We remind the reader that a group  $G$  is said to be nilpotent of *class  $c$*  if the upper central series has the form

$$1 = Z_0(G) \leq \dots \leq Z_c(G) = G.$$

Also, a group  $G$  is said to be locally nilpotent if every finitely generated subgroup is nilpotent.

Let  $R_0$  denote the subring of  $R$  generated by the identity. Since  $R$  has non-zero characteristic,  $R_0$  is a finite ring. Throughout this section, we will let  $R_0G$  denote the  $R_0$ -span of  $G$  in  $\text{End}_R M$ . Thus, the elements of  $R_0G$  consist of finite sums of the form

$$r_1 g_1 + \cdots + r_n g_n,$$

where  $r_1, \dots, r_n$  are elements in  $R_0$  and  $g_1, \dots, g_n$  are elements in  $G$ . Thus, only a finite number of members of  $G$  intervene in the representation of any element of  $R_0G$ . Since  $G$  is assumed to be locally finite, the elements  $g_1, \dots, g_n$  generate a finite group  $G_0$ . Let  $R_0G_0$  denote the  $R_0$ -span of the subgroup  $G_0$  in  $\text{End}_R M$ . Then  $R_0G_0$  is a finite ring. Thus, every element of  $R_0G$  is a member of an Artinian subring of  $R_0G$ .

DEFINITION: The *radical of  $G$* , denoted by  $\text{Rad } G$ , is defined to be the subset of  $G$  of the form

$$G \cap (1 + \text{Rad } R_0G)$$

Where  $R_0$  is the subring of  $R$  generated by the identity element.

First we observe a crucial property of  $\text{Rad } R_0G$ .

PROPOSITION 2.1: *The Jacobson radical of  $R_0G$  is a nil ideal.*

PROOF: Let  $r$  be an element of  $\text{Rad } R_0G$ . Then  $r = r_1 g_1 + \cdots + r_n g_n$ , and the elements  $g_1, \dots, g_n$  generate a finite subgroup  $G_0$  of  $G$ . Consequently,  $r$  is an element of

$$N = R_0G_0 \cap \text{Rad } R_0G.$$

Since  $R_0G_0$  is Artinian, we have by Lemma 1.7 that  $N$  is a nil ideal of  $R_0G_0$ . Therefore, the element  $r$  is nilpotent and  $\text{Rad } R_0G$  is a nil ideal.

Q.E.D.

PROPOSITION 2.2: *The radical  $\text{Rad } G$  of  $G$  is a unipotent, locally nilpotent, normal subgroup of  $G$ .*

PROOF: We first observe that  $1 + \text{Rad } R_0G$  is a group under multiplication. Indeed, by Proposition 2.1,  $\text{Rad } R_0G$  is a nil ideal. Thus, if  $r$  is an element of  $\text{Rad } R_0G$ , we have that  $r^{n+1} = 0$ . Then

$$\begin{aligned} (1+r)(1-r+r^2-\cdots\pm r^n) &= 1-r+\cdots\pm r^n+r-r^2+\cdots\mp r^n\pm r^{n+1} \\ &= 1. \end{aligned}$$

Since  $r-r^2+\cdots\pm r^n$  is an element of  $\text{Rad } R_0G$ , we see that  $1 + \text{Rad } R_0G$  is a group. Thus,  $\text{Rad } G = G \cap (1 + \text{Rad } R_0G)$  is a subgroup of  $G$ . Furthermore, since  $\text{Rad } R_0G$  is an ideal,  $\text{Rad } G$  is a normal subgroup.

Since every element of  $\text{Rad } G$  has the form  $1+r$ , where  $r$  is a nilpotent element, it is clear that  $G$  is a unipotent group. We are left to prove only that  $\text{Rad } G$  is locally nilpotent. Let  $g_1, \dots, g_m$  be the elements of a finite subgroup  $G'$  of  $\text{Rad } G$ . Then, for every  $i$ ,  $1 \leq i \leq m$ , we have that  $g_i = 1+n_i$ , where  $n_i$  is a member of  $\text{Rad } R_0 G$ . Furthermore

$$n_i = r_{1i}g_{1i} + \dots + r_{ni}g_{ni},$$

and the elements  $g_{1i}, \dots, g_{ni}$ , for all  $i$ ,  $1 \leq i \leq m$ , generate a finite group, say  $G_0$ . The Jacobson radical of the Artinian ring  $R_0 G_0$  is nilpotent. Also, by Lemma 1.7

$$R_0 G_0 \cap \text{Rad } R_0 G \leq \text{Rad } R_0 G_0;$$

consequently, by a theorem of P. Hall [see 2, p. 17]

$$1 + (R_0 G_0 \cap \text{Rad } R_0 G)$$

is a nilpotent group. Since

$$G'_0 \leq 1 + (R_0 G_0 \cap \text{Rad } R_0 G),$$

we have that  $G'_0$  is a nilpotent group. Thus,  $\text{Rad } G$  is locally nilpotent. Q.E.D.

We remind the reader that a module  $M$  has finite length  $n$  if it has a composition series of length  $n$ .

**THEOREM 2.3:** *Let  $M$  be a module of finite length  $n$  over a ring  $R$  of arbitrary nonzero characteristic. Then  $\text{Rad } G$  is a nilpotent group with nilpotency class less than or equal to  $n$ .*

**PROOF:** By Proposition 2.1,  $\text{Rad } R_0 G$  is a nil ideal of  $R_0 G$ . Consequently,  $\text{Rad } R_0 G$  consists of nilpotent endomorphisms of the Artinian and Noetherian module  $M$ . Thus we have that  $\text{Rad } R_0 G$  is a nilpotent set, with index of nilpotency less than or equal to  $n$ . By the theorem of P. Hall,  $1 + \text{Rad } R_0 G$  is a nilpotent group with nilpotency class less than or equal to  $n$ . Since

$$\text{Rad } G = G \cap (1 + \text{Rad } R_0 G)$$

we see that  $\text{Rad } G$  is also a nilpotent group with nilpotency class less than or equal to  $n$ . Q.E.D.

We will prove later that  $\text{Rad } G$  is the maximal normal unipotent subgroup of  $G$ , but before that we must consider the quotient  $G/\text{Rad } G$ .

Let again  $G$  denote a locally finite group of automorphisms of a finite dimensional module  $M$  over the ring  $R$ . The results that follow show that  $G/\text{Rad } G$  can be represented as a direct sum of a finite number of linear

groups over locally finite fields. This reduces further considerations of  $G/\text{Rad } G$  to countable groups. To obtain these results,  $R_0G/\text{Rad } R_0G$  is proved to be a semisimple Artinian ring. Thus, the Wedderburn-Artin Theorem is applicable, and the ring  $R_0G/\text{Rad } R_0G$  is isomorphic to a direct sum of complete rings of matrices over division rings. These division rings turn out to be locally finite, and consequently commutative fields. The isomorphism between  $R_0G/\text{Rad } R_0G$  and the direct sum of rings of matrices over fields extends to a group homomorphism from  $G$  to a direct sum of linear groups. The kernel of this homomorphism is precisely  $\text{Rad } G$ .

**THEOREM 2.4:** *The quotient  $R_0G/\text{Rad } R_0G$  is isomorphic to a direct sum of a finite number of complete matrix rings over locally finite fields.*

**PROOF:** Since  $R_0G$  is a subring of  $\text{End}_R M$ , by Lemma 1.10 we have that  $R_0G$  is a sum of a finite number of indecomposable right ideals. Furthermore, these ideals have the form  $e_1R_0G, \dots, e_nR_0G$  where, for each  $i$ ,  $1 \leq i \leq n$ ,  $e_i$  is a primitive idempotent.

We claim that  $R_0G/\text{Rad } R_0G$  is a semisimple Artinian ring. By Proposition 2.1, we have that  $\text{Rad } R_0G$  is a nil ideal. Thus, by Lemma 1.3, primitive idempotents remain primitive modulo  $\text{Rad } R_0G$ . Then

$$R_0G/\text{Rad } R_0G = \sum_1^n \bar{e}_i(R_0G/\text{Rad } R_0G),$$

where  $\bar{e}_i$ ,  $1 \leq i \leq n$ , denotes the image of  $e_i$  under the canonical homomorphism from  $R_0G$  onto  $R_0G/\text{Rad } R_0G$ . This is a decomposition of  $R_0G/\text{Rad } R_0G$  onto a direct sum of a finite number of indecomposable right ideals. We prove that they are actually minimal right ideals. Since  $R_0G/\text{Rad } R_0G$  is a semisimple ring, Lemma 1.5 implies that

$$\bar{e}_i(R_0G/\text{Rad } R_0G)$$

is a minimal right ideal if and only if

$$\bar{e}_i(R_0G/\text{Rad } R_0G)\bar{e}_i$$

is a division ring. By Lemma 1.4 it is enough to prove that

$$(e_iR_0Ge_i)/\text{Rad } (e_iR_0Ge_i)$$

is a division ring. Consider any right ideal  $I$  of  $e_iR_0Ge_i$  whose image  $\bar{I}$  under the canonical homomorphism of  $e_iR_0Ge_i$  onto  $(e_iR_0Ge_i)/\text{Rad } (e_iR_0Ge_i)$  is not zero. Then  $I$  contains an element  $r$  which is not nilpotent. Let  $r = e_it e_i$ , where

$$t = r_1g_1 + \dots + r_n g_n.$$

The elements  $g_1, \dots, g_n$  generate a finite group  $G_0$ , so that  $r$  is an element of  $(e_i R_0 G_0 e_i) \cap I = I_1$ . Since  $I$  is an ideal, we have that  $I_1$  is an ideal of the finite ring  $e_i R_0 G_0 e_i$ . By Lemma 1.6,  $I_1$  contains an idempotent  $e$ , which is also an idempotent of  $I$ . Since  $e_i$  is a primitive idempotent, we have that  $e = e_i$ , so that  $I_1 = e_i R_0 G_0 e_i$  and  $I = e_i R_0 G e_i$ . Consequently,

$$\bar{I} = \bar{e}_i(R_0 G/\text{Rad } R_0 G)\bar{e}_i$$

and this ring has no nonzero proper ideals. Therefore

$$\bar{e}_i(R_0 G/\text{Rad } R_0 G)\bar{e}_i$$

is a division ring. With this, we have proved the claim.

The Wedderburn-Artin Theorem now implies that  $R_0 G/\text{Rad } R_0 G$  is isomorphic to a direct sum of a finite number of rings  $R_j$ ,  $1 \leq j \leq m$ , each of which is a complete matrix ring over a corresponding division ring. Thus, let  $R_j = (D^j)_{n_j \times n_j}$ , where  $D^j$  is a division ring. The local finiteness of  $R_0 G/\text{Rad } R_0 G$  implies that  $D^j$  is a locally finite division ring. By Lemma 1.8,  $D_j$  is commutative, and the theorem is proved.

Q.E.D.

The preceding theorem yields immediately a representation for the locally finite group  $G$ .

**THEOREM 2.5:** *The quotient  $G/\text{Rad } G$  can be embedded as a direct sum of a finite number of linear groups over locally finite fields.*

**PROOF:** The isomorphism between  $R_0 G/\text{Rad } R_0 G$  and the finite direct sum of complete matrix rings over algebraically closed fields extends to a group homomorphism on  $G$  whose kernel is precisely  $\text{Rad } G$ . Q.E.D.

Collecting together Proposition 2.2 and Theorem 2.5 we have the following extension of a theorem of D. Winter [6].

**THEOREM 2.6:** *Let  $M$  be a finite dimensional module over a ring  $R$  of arbitrary nonzero characteristic. If  $G$  is a locally finite group of automorphisms of  $M$ , then  $G$  is a countable extension of a unipotent locally nilpotent group.*

**PROOF:** Any locally finite field is obviously algebraic over its prime field. Also, any algebraic field over a finite field is countable. Hence any group of matrices over such a field is countable, and the result follows.

Q.E.D.

We are now in a position to study further the radical of  $G$ .

**DEFINITION:** For any subgroup  $H$  of  $G$ , let  $\langle 1-H \rangle$  be the ideal of  $R_0 G$  generated by all elements of the form  $1-h$ , where  $h$  is an element of  $H$ .

**THEOREM 2.7:** *The group  $\text{Rad } G$  contains every unipotent normal subgroup of  $G$ .*

**PROOF:** Let  $H$  be a unipotent normal subgroup of  $G$ . The equation

$$(1-h)g = g(1-g^{-1}hg)$$

implies that  $\langle 1-H \rangle$  is the  $R_0$  span of the elements of the form

$$g(1-h_1) \cdots (1-h_n)$$

where each  $h_i$ ,  $1 \leq i \leq n$ , is an element of  $H$ . Since  $H$  is unipotent, by Proposition 2.4 and Kolchin's Theorem [see 4], the homomorphic image  $\bar{H}$  of  $H$  on  $R_0G/\text{Rad } R_0G$  is unitriangular. Hence, for some positive integer  $m$ ,  $(1-\bar{H})^m = 0$ . Thus  $(1-H)^m$  is contained in  $\text{Rad } R_0G$ . By Proposition 2.1,  $\text{Rad } R_0G$  is a nil ideal, so that  $\langle 1-H \rangle$  is nil. Consequently  $\langle 1-H \rangle$  is contained in the Jacobson radical of  $R_0G$ . Thus

$$H \leq (1 + \text{Rad } R_0G) \cap G = \text{Rad } G. \quad \text{Q.E.D.}$$

**COROLLARY 2.8:** *If  $G$  is unipotent, then  $G$  is locally nilpotent.*

**PROOF:** If  $G$  is unipotent, then  $G = \text{Rad } G$  by Theorem 2.7. Consequently,  $G$  is locally nilpotent by Proposition 2.2. Q.E.D.

**COROLLARY 2.8a:**  *$\text{Rad } G$  is the maximal normal unipotent subgroup of  $G$ .*

It is now easy to see how a large number of classical theorems on linear groups can be extended to a new setting. As an example, we prove the following theorem.

**THEOREM 2.9:** *Let  $G$  be a locally solvable torsion group of  $R$ -automorphisms of a module  $M_R$  of finite length  $n$ . Then there exists an integer valued function  $f$  which depends only on  $n$  such that  $G$  contains a normal subgroup  $H$  with*

$$|G : H| \leq f(n)$$

*and with nilpotent derived group  $H'$ :*

**PROOF:** Since  $G$  is a locally solvable torsion group, we have that  $G$  is locally finite. By Theorem 2.4,  $R_0G/\text{Rad } R_0G$  is isomorphic to a direct sum of complete matrix rings over fields, say

$$\sum_{j=1}^t (F^j)_{n_j \times n_j}.$$

Since  $R_0G$  cannot contain a set of orthogonal idempotents with more than  $n$  elements and idempotents modulo  $\text{Rad } R_0G$  can be lifted, we have that

$$\sum_{j=1}^t n_j^2 \leq n.$$

Let  $F_j$  denote the ring homomorphism from  $R_0G$  onto each summand  $(F^j)_{n_j \times n_j}$ . Since  $(G)F_j = G_j$  is a locally solvable linear group, it is solvable, by Zassenhaus' Theorem [see 7]. Furthermore, by Malcev's Theorem [see 5], there is a function  $\bar{f}(n_j)$  which depends only on  $n_j$ , such that for some triangularizable normal subgroup  $H_j$  of  $G_j$  we have  $|G_j : H_j| \leq \bar{f}(n_j)$ . Consider the subgroup

$$H = \bigcap_{j=1}^t (H_j)F_j^{-1}.$$

This is a normal subgroup of finite index, and, in fact, for some function  $f(n)$ , which depends only on  $n$ , we have that  $|G : H| \leq f(n)$ . Furthermore,  $H'_j$  is a unipotent group; therefore,  $H'$  is unipotent, for we have that

$$H' \subseteq \bigcap_{j=1}^t (H'_j)F_j^{-1}.$$

By Theorem 2.3, the subgroup  $H'$  is nilpotent, and the theorem has been proved. Q.E.D.

As a particular case of the above, let now  $V$  be an  $n$ -dimensional vector space over a division ring  $D$  of nonzero characteristic. Then  $V$  is a finite dimensional module, also of dimension  $n$ ; besides, the characteristic of  $D$  is necessarily a prime number  $p$ . Let  $G$  be a locally finite group of automorphisms of  $V_D$ . In this case,  $\text{Rad } G$  is necessarily a  $p$ -group. Indeed, if  $g$  is an element of  $\text{Rad } G$  such that  $(1-g)^m = 0$ , then for some power of  $p$ , say  $p^e$ , we have that  $(1-g)^{p^e} = 0$ , and consequently  $g^{p^e} = 1$ . The quotient  $G/\text{Rad } G$ , as we have seen, is a direct sum of linear groups. But in this case, the characteristic of the fields involved in the representation is the same, so that  $G/\text{Rad } G$  is isomorphic to a group of  $n \times n$  matrices over the algebraic closure of the prime field with  $p$  elements. Thus, we can immediately extend to this setting known theorems on linear groups. For example, if  $G$  is a locally finite group of matrices over a division ring  $D$  of characteristic  $p$ , then the Sylow  $p$ -subgroups are conjugate. Again, if  $G$  is a locally finite group of matrices over a division ring  $D$  of characteristic  $p$ , and  $G$  does not contain elements of order  $p$ , then  $G$  has an Abelian normal subgroup  $H$  such that  $|G : H| \leq f(n)$ , where  $f$  is an integer valued function of  $n$  only.

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## ADDENDUM

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