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Compositio Mathematica, tome 28, n° 1 (1974), p. 33-36

<http://www.numdam.org/item?id=CM_1974__28_1_33_0>

COMPOSITIO MATHEMATICA, Vol. 28. Fasc. 1, 1974, pag. 33–36
Noordhoff International Publishing
Printed in the Netherlands

ON THE FUNDAMENTAL GROUP OF A MAPPING SPACE. AN EXAMPLE

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1. Introduction

The purpose of this paper is to provide an example of a mapping space between connected, compact polyhedra having finitely generated homotopy groups in all dimensions, such that at least one component in the mapping space has infinitely generated fundamental group.

Throughout K is a connected, compact polyhedron. Mostly $K = S^1$, the unit circle. For any connected, topological space X, X^K shall denote the space of continuous maps of K into X equipped with the compact-open topology. In particular X^{S^1} denotes the free loop space on X. We adopt the terminology that an aspherical space is a connected, compact polyhedron X, which is also an Eilenberg-MacLane space of type $(\pi, 1)$, i.e. $\pi_1(X) = \pi$ and $\pi_i(X) = 0$ for all $i \geqq 2$.

THEOREM: *There exists an aspherical space X with finitely generated fundamental group, such that the free loop space X^{S^1} contains a component with infinitely generated fundamental group.*

A similar phenomenon cannot happen in higher dimensions. It is, in fact, easy to prove that if X has finitely generated homotopy groups in dimensions $\geqq 2$, then all the homotopy groups of X^K in dimensions $\geqq 2$ are also finitely generated.

Results on the qualitative structure of the homotopy groups of a mapping space were obtained among others by Federer [1] and Thom [4]. The example in the theorem above shows that Thom's statement ([4], Theorem 4) is incorrect for the fundamental group.

Finally, I want to thank J. Eells, A. C. Robinson, B. Hartly and D. B. A. Epstein for various helpful remarks during the preparation of this paper.

2. Free loop spaces. Their fundamental groups

In this section X is a connected, topological space with base point $*$. It is well-known that the (path-)components in the mapping space X^{S^1} can be enumerated by $\pi_1(X, *)$. The fundamental group of a component in X^{S^1} depends normally on the component in question. To describe how, recall that the centralizer of an element g in a group G is the subgroup of G defined by $C_g(G) = \{g' \in G | g'g = gg'\}$. We have then

PROPOSITION 1: *Suppose that $\pi_2(X, *) = 0$ and let $f : S^1 \to X$ be an arbitrary base point preserving map representing the homotopy class $[f] \in \pi_1(X, *)$. Then*

$$\pi_1(X^{S^1}, f) = C_{[f]}(\pi_1(X, *)).$$

PROOF: Let ΩX denote the ordinary loop space on X, i.e. the space of based maps of S^1 into X equipped with the compact-open topology. It is well-known that the map $p : X^{S^1} \to X$ defined by evaluation at the base point of S^1, also denoted $*$, is a Hurewicz fibration with fibre ΩX. See e.g. Hu ([2], Theorem 13.1 p. 83). Consider now the following part of the homotopy sequence of that fibration,

$$\pi_1(\Omega X, f) \to \pi_1(X^{S^1}, f) \xrightarrow{p_*} \pi_1(X, *).$$

Since $\pi_1(\Omega X, f) \cong \pi_2(X, *) = 0$, p_* is a monomorphism. Hence $\pi_1(X^{S^1}, f)$ is isomorphic to the image of p_*. To determine this image observe that $\alpha \in \pi_1(X, *)$ is in the image of p_* if and only if there exists a map $S^1 \times S^1 \to X$ such that $S^1 \times \{*\} \to X$ represents α and $\{*\} \times S^1 \to X$ represents $[f]$. On the other hand such a map exists if and only if the Whitehead product of α and $[f]$ is the identity element of $\pi_1(X, *)$. Since a Whitehead product of elements in a fundamental group coincides with the corresponding commutator product, we conclude, that α is in the image of p_* if and only if $\alpha[f]\alpha^{-1}[f]^{-1} = 1$ or equivalently that $\alpha \in C_{[f]}(\pi_1(X, *))$. This proves the proposition.

3. The example

Let X be the connected, compact, 2-dimensional polyhedron obtained from a cylinder over an oriented circle with base point by pinching one boundary circle into a figure eight and then identifying the resulting three boundary circles according to orientation. See Figure 1.

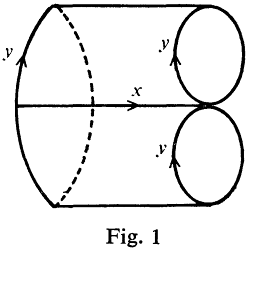

Fig. 1

We shall show that this polyhedron satisfies the requirements in the theorem. This will follow easily from Proposition 1 and Propositions 2 and 3 below.

The base point from the circle in the cylinder leaves X with a base point. Denote the fundamental group of X by π, and consider the elements x and y in π generated by the two loops indicated in Figure 1. For each $n = 0, 1, 2, \cdots$ we put $y_n = x^n y x^{-n}$. In particular $y_0 = y$.

PROPOSITION 2: *π is generated by the elements x and y subject to the single relation $x^{-1}yx = y^2$.*

The centralizer of $y \in \pi$, $C_y(\pi)$, is infinitely generated with $y_0, y_1, \cdots, y_n, \cdots$ as a system of generators.

PROOF: X can be obtained from the wedge of the two loops generating x and y by adding on a single 2-cell according to a map with the homotopy class $x^{-1}yxy^{-2}$. Hence the statement about π is well-known, see e.g. Spanier ([3], Theorem 10, p. 147).

x is an element of infinite order in π. Let F be the infinite cyclic subgroup of π generated by x and let C be the subgroup of π generated by the elements $y_0, y_1, \cdots, y_n, \cdots$. Clearly $F \cap C$ contains only the neutral element. One proves easily that any word in π can be expressed in the form $x^m w$, where m is an integer and w is a word in C. Hence π is generated by F and C. Altogether π is therefore a semidirect product of the subgroups F and C.

Using the relation $x^{-1}yx = y^2$ it is easy to prove that $y_{n+1}^2 = y_n$, and hence that $y_n = (y_{n+k})^{2^k}$. Having this information one verifies that C is abelian.

Since non-trivial powers of x never commute with $y = y_0$, it is now clear that $C_y(\pi) = C$.

Clearly C is infinitely generated, since one can never produce the element y_{n+1} out of a system of elements in C involving only $y_0, \cdots, y_n$.

This proves Proposition 2.

PROPOSITION 3: *X is an Eilenberg-MacLane space of type $(\pi, 1)$.*

PROOF: Let X' denote the polyhedron obtained from the space in Figure 1 by identifying this time only the two right-hand circles. Consider then the double infinite telescope X'' obtained by glueing together copies of X' in both ends, always such that the left-hand circle in any copy of X' is identified with the pair of identified right-hand circles in the neighbouring copy of X' to the left in the telescope.

Pushing one stage from the left to the right in the telescope defines in the obvious way an action of the integers $\mathbb{Z}$ on X''. The quotient space for this action can clearly be identified with X. Furthermore, the quotient map $X'' \to X$ is a covering projection. To prove that X is an Eilenberg-MacLane space of type $(\pi, 1)$, it suffices therefore to prove that $\pi_n(X'') = 0$ for all $n \geqq 2$.

For that purpose, let $f : S^n \to X''$ be an arbitrary based map of the n-sphere S^n for $n \geqq 2$ into X''. By a compactness argument the image of this map lies in a finite stage of the telescope. Therefore it is easy to see that we can homotope f into a map $g : S^n \to X''$, which maps one hemisphere of S^n into a long string, and maps the other hemisphere of S^n into a circle separating two copies of X' sufficiently far to the right in the telescope. Furthermore, we can arrange that g maps the common boundary of the two hemispheres into a single point. It is then clear that g, and hence also f, is homotopic to the constant map. Therefore $\pi_n(X'') = 0$ for all $n \geqq 2$, and as already remarked this completes the proof of Proposition 3.

We are now ready for the

PROOF OF THE THEOREM: By Propositions 2 and 3, X is an aspherical space with finitely generated fundamental group.

Let $f_y : S^1 \to X$ be a based map representing the homotopy class $[f_y] = y \in \pi = \pi_1(X, *)$. By Proposition 1, $\pi_1(X^{S^1}, f_y) = C_y(\pi)$, which is infinitely generated by Proposition 2. Hence the component of X^{S^1}, which contains the map f_y, has infinitely generated fundamental group. Therefore X satisfies the requirements in the theorem.

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(Oblatum: 4–X–1972 & 1–VIII–1973)

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