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DISCRETE SERIES AND THE UNIPOTENT SUBGROUP

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Introduction

In this paper we give an explicit decomposition of the restriction of any irreducible discrete series complex representation of $GL(n, q)$ to the unipotent subgroup consisting of upper unitriangular matrices. The decomposition is as a tensor product of representations which are shown to be multiplicity free, and whose components are exhibited explicitly as representations induced from radical subgroups of a special type. The subgroups occurring correspond precisely to root subgroups, and it is hoped that the results presented here may lead to generalizations for other groups of Lie type.

A by-product is the result that there are certain representations of high degree in the restriction, and it is shown in the final section that these representations have maximal degree among the irreducible complex representations of the unipotent group. This leads in particular to the result that this maximal degree is a power $q^{\mu(n)}$ of q which depends only on n. Also it is shown that the representations of maximal degree are in some sense 'dense'.

The decomposition shall take place in three parts. Firstly, some general propositions are proved which relate to representations of semi-direct products. Then the restriction of the discrete series characters is defined by means of its values, and an inductive property is used to achieve the tensor product decomposition. The factors are then analysed, using the results of the first part, and tensor products are calculated using Mackey's theorem.

1. Representations of semi-direct products

PROPOSITION 1.1: *Let G be a finite group expressible as a semi-direct product $G = A \rtimes B$, and let ρ be a complex representation of B. Then if θ is the permutation representation of G on cosets of B, we have*

$$\rho^G = \theta \otimes \rho^*$$

where ρ^* denotes the lift (or 'pullback') of ρ from B to G.

PROOF: Let V_ρ be a CB module for the representation ρ. Then ρ^G has CG module $W = \oplus_{a\in A} aV_\rho$ (see Serre [7]) where G acts as follows: if $g \in G$, $G = ab(a \in A, \quad b \in B)$ then $g(a_0 v_0) = ab(a_0 v_0) = (a \cdot ba_0 b^{-1}) \cdot (bv_0) = a_1 v_1 (a_0, a_1 \in A; v_0, v_1 \in V_\rho)$ where $a_1 = a \cdot ba_0 b^{-1}$ and $v_1 = bv_0$ (b acting by means of the representation ρ of B on V_ρ). Thus $g = ab$ sends $a_0 v_0$ to $a_1 v_1$ where a_1 is the unique element of A in $ga_0 B$ and $v_1 = bv_0$. Now the elements of A are left coset representatives for B in G and the permutation $g : a_0 \mapsto a_1$ defined above is the permutation g defines on the left B-cosets in G. Moreover V_ρ may be regarded as a CG module for the representation ρ^*, where if $v \in V_\rho$, $g = ab(a \in A, b \in B)$ then $gv = bv$. But any element of W has a unique expression as a sum $\sum_{a\in A} a \cdot v_a (v_a \in V_\rho)$ and thus as C-vector space W is the tensor product of a space realising θ (viz the set of C-linear combinations of elements of a) and a space realising ρ^* (viz V_ρ). Finally the description of the G-action on W above shows that $\rho^G = \theta \otimes \rho^*$, where equality here denotes equivalence of representations.

The object of the next proposition is to show that the processes of lifting and of induction commute with each other.

PROPOSITION 1.2: *Let G be as in proposition 1.1 and suppose C is a subgroup of B. Let λ be a complex character of C, λ^* its lift to $C^* = A \cdot C = A \rtimes C$. Then $\lambda^{*G} = (\lambda^{B*})$, where (λ^{B*}) denotes the lift of λ^B from B to G.*

PROOF: We can choose a common set of representatives $b_1, \cdots, b_n$ ($b_i \in B$) for cosets of C in B and for cosets of C^* in $B^* = G$. Then if $g \in G$, $g = ab(a \in A, b \in B)$ we have

$$(\lambda^B)^*(g) = \lambda^B(b) = \sum_{i=1}^{n} \hat{\lambda}(b_i b b_i^{-1})$$

where $\hat{\lambda}(x) = \lambda(x)$ if $x \in C$ and $\hat{\lambda}(x) = 0$ otherwise.
Also

$$(\lambda_C^*)^G(g) = \sum_{i=1}^{n} \hat{\lambda}^*(b_i abb_i^{-1})$$

where $\hat{\lambda}^*(y) = \lambda^*(y)$ if $y \in C^*$ and $\hat{\lambda}^*(y) = 0$ otherwise. But $b_i abb_i^{-1} = b_i ab_i^{-1} \cdot b_i bb_i^{-1} \in C^* \Leftrightarrow b_i ab_i^{-1} \in C$. Hence $\hat{\lambda}^*(b_i abb_i^{-1}) = \hat{\lambda}(b_i bb_i^{-1})$ and the result follows.

The final proposition of this section describes the irreducible representations of the group G of proposition 1.1 in case A is abelian.

PROPOSITION 1.3:

(i) *With G as in proposition 1.1, let A be abelian. Then each irreducible complex representation of G is of the form $(\chi\phi)^G$ where χ is an irreducible* *representation of A and ϕ is an irreducible representation of the centralizer B_χ of χ in B.*

(ii) *In (i) above, χ may be replaced by any representation conjugate to it under the action of B.*

PROOF: (i) is a theorem of Mackey ([6]) and (ii) is a simple consequence (see [3]).

2. Representations of the unipotent group

In this section we recall briefly (cf. [3]) the consequences of applying proposition 1.3 to the unipotent group. Let $G = U_n$ be the group of upper unitriangular matrices of size n with coefficients in $GF(q)$. Let A be the subgroup of G consisting of matrices all of whose non-diagonal elements are zero except for the last column, and take for B the set of matrices whose non-diagonal entries in the last column are zero. Then $G = A \rtimes B$, $A \cong (GF(q)^+)^{n-1}$ is abelian and may be regarded as an $(n-1)$-dimensional vector space over $GF(q)$ on which B acts:

$$\left[\begin{array}{c|c} X & 0 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} I & v \\ \hline & 1 \end{array}\right]\left[\begin{array}{c|c} X^{-1} & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} I & Xv \\ \hline 0 & 1 \end{array}\right]$$

In fact $B \cong U_{n-1}$ and by an obvious identification B acts as U_{n-1} on column vectors of length $n-1$ which are regarded as the elements of A.

An irreducible character χ of A is given by a set of $n-1$ characters χ_i of $GF(q)^+$, where

$$\chi\begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \end{bmatrix} = \prod_{i=1}^{n-1} \chi_i(v_i),$$

and if $b \in B$, $\chi^b(v) = \chi(bvb^{-1})$. Henceforth let χ_0 be a fixed non-trivial irreducible character of $GF(q)^+$; then any irreducible character of $GF(q)^+$ is of the form $a\chi_0$, where $a\chi_0(f) = \chi_0(af)$ $(a, f \in GF(q))$, and thus corresponds to an element of $GF(q)$. Hence characters $\chi = \chi_1 \cdots \chi_{n-1}$ also can be regarded as row vectors over $GF(q)$.

With these identifications it is clear that if $b \in B$ and $\chi = (\chi_1, \cdots, \chi_{n-1})$ then $\chi^b = (\chi_1, \cdots, \chi_{n-1})(b_{ij})$ where (b_{ij}) is the leading $(n-1)\times(n-1)$ part of b.

Let k be the least index such that $\chi_k \neq 0$, if $\chi \neq (0, 0, \cdots, 0)$; then the B-orbit of χ contains precisely all characters of the form $(0, 0, \cdots, 0, \chi_k, \mu_{k+1}, \cdots \mu_{n-1})$ where the μ_l are arbitrary. In particular this orbit contains the character $\chi_c = (0, 0, \cdots, 0, \chi_k, 0, \cdots, 0)$ which will be referred to as the canonical character in the orbit of χ. The B-orbits in the character group $\hat{A}$ of A are thus represented by the $(n-1)(q-1)$ canonical characters χ_c, together with the zero (identity) character.

The centralizer B_{χ_c} consists of matrices in B whose non-diagonal entries in the k^{th} row are zero. It depends, therefore, only on the index k which is referred to as the *type* of χ. The identity is canonical of type 0.

DEFINITION: If the irreducible character χ of $A \lhd U_n$ is of type k, we write $Z_{k,n-1} = B_{\chi_c}$ and $U_{k,n} = A \rtimes Z_{k,n-1}$

Proposition 1.3 applied here gives

LEMMA 2.1: *Any irreducible representation* v *of* U_n *is of the form* $v = (\chi\phi)^{U_n}_{U_{k,n}}$ *where* χ *is a canonical character of type* k *and* ϕ *is an irreducible representation of* B_χ.

To conclude this section we introduce some notation. For each integer m, $1 \leqq m \leqq n$ we regard the group U_m as the specific subgroup of U_n consisting of matrices in U_n with all non-diagonal entries zero, except those in the leading $m \times m$ square. Then U_m has an obvious normal complement C_m and $U_n = C_m \rtimes U_m$.

DEFINITION:

(i) If H is a subgroup of U_m we write H^* for $H^* = C_m \rtimes H$.

(ii) $A_m = C_{m-1} \cap U_m$ is the normal complement of U_{m-1} in U_m (e.g. A_n is the A of the discussion above).

3. The discrete series

There is a family of distinguished irreducible representations of $GL(n, q)$ which have degree $(q-1)(q^2-1) \cdots (q^{n-1}-1)$ and whose importance is outlined in [8]. An explicit description of the values of the characters of these representations is given in Green's famous paper [2], and can also be found in [4]. It transpires that all discrete series representations have the same restriction to U_n since the character values are the same on U_n.

DEFINITION: For $u \in U_n$, define the rational integer $\delta_n(u)$ by $\delta_n(u) = (-1)^{n-1}(1-q)(1-q^2) \cdots (1-q^{s(u)})$ where $s(u) = (n-1)-r(u)$ and $r(u) =$ rank of the matrix $1-u$.

Then δ_n is the character of the restriction to U_n of any irreducible discrete series representation of $GL(n, q)$; we denote this representation of U_n by Δ_n and it is with its decomposition that we are concerned. The first step is the remark

PROPOSITION 3.1: *We have* $\delta_n = \delta_{n-1}^{U_n} - \delta_{n-1}^*$, *where* δ_{n-1} *is the discrete series character of* U_{n-1} *as defined in the previous section.*

PROOF: This is simple to verify, using the formula for calculating induced characters; the explicit calculation may be found in [5]. This fact has been remarked on by several authors, including Ennola [1].

Proposition 1.1 enables us to exploit this inductive property of Δ_n. We define the representation ρ_n as follows: Let Γ_n be the set of cosets of U_{n-1} in U_n and let $\mathscr{F}_n$ be the set of functions $f : \Gamma_n \to \boldsymbol{C}$. Then U_n acts naturally on the q^{n-1}-dimensional space $\mathscr{F}_n$, and this is simply the permutation representation θ_n of U_n on the cosets of U_{n-1}. Let

$$\overline{\mathscr{F}}_n = \{f \in \mathscr{F}_n | \sum_{\gamma \in \Gamma_n} f(\gamma) = 0\}.$$

Then U_n stabilizes $\overline{\mathscr{F}}_n$, and we define ρ_n to be the $(q^{n-1}-1)$-dimensional representation of U_n on $\overline{\mathscr{F}}_n$.

PROPOSITION 3.2: *We have* $\Delta_n = \rho_n \otimes \Delta^*_{n-1}$

PROOF: Let $f_0 \in \mathscr{F}_n$ be the function taking the value 1 at each element of Γ_n, and let $\langle f_0 \rangle$ be the 1-dimensional space spanned by f_0. Then $\mathscr{F}_n = \overline{\mathscr{F}}_n \oplus \langle f_0 \rangle$ and both components are stable under U_n. But the representation of U_n on $\langle f_0 \rangle$ is the identity, and so we have $\theta_n = \rho_n \oplus 1$.

By proposition 1.1 $\Delta^{U_n}_{n-1} = \theta_n \otimes \Delta^*_{n-1}$ and this by the above is equal to $(\rho_n \oplus 1) \otimes \Delta^*_{n-1}$. Hence $\Delta^{U_n}_{n-1} = \rho_n \otimes \Delta^*_{n-1} \oplus \Delta^*_{n-1}$. The result is now immediate from proposition 3.1.

An immediate corollary is

THEOREM 3.3: *Let ρ_i be the $(q^{i-1}-1)$-dimensional representation of U_i defined as above. Then*

$$\Delta_n = \rho_n \otimes \rho_{n-1} \otimes \ldots \otimes \rho_2$$

where ρ_i is identified with its lift from U_i to U_n.

We now turn our attention to the ρ_i.

4. The representation ρ_n

LEMMA 4.1: *The restriction of ρ_n to A_n is the sum of all the irreducible representations (characters) of A_n except for the identity representation, each one occurring with multiplicity one.*

PROOF: The action of A_n on Γ_n is the permutation action of A_n on its own elements by left translation since the elements of A_n form a set of coset representatives. Hence the representation of A_n on $\mathscr{F}_n$ is the regular representation of A_n, which is the sum of all the irreducible representations of A_n, since A_n is abelian. But the representation of A_n on $\langle f_0 \rangle$ is the identity representation, and so the representation of A_n on $\overline{\mathscr{F}}_n$ (which is $\rho_n|_{A_n}$) is the sum of the non-identity representations of A_n.

COROLLARY 4.1: *ρ_n is a multiplicity free representation of U_n.*

This is clear, since its restriction to A_n is multiplicity free.

LEMMA 4.2: *There is a 1–1 correspondence $\chi \to \alpha(\chi)$ between canonical non-trivial characters χ of A_n and irreducible constituents $\alpha(\chi)$ of ρ_n such that if χ is of type k, degree $(\alpha(\chi)) = q^{n-k-1}(k = 1, 2, \cdots, n-1)$.*

PROOF: Let α be an irreducible constituent of ρ_n. By Clifford's theorem together with lemma 4.1, the restriction $\alpha|_{A_n}$ is the sum of the characters of A_n in a U_{n-1}-orbit of non-trivial characters. By the discussion in Section 2, these orbits are represented by canonical characters. Let χ be a canonical character of A_n, of type k. Then if α is the unique constituent of ρ_n containing χ in its restriction to A_n, we have that the degree of α is the number of characters of A_n in the orbit of χ under U_{n-1}. By the discussion preceding lemma 2.1 this is q^{n-k-1}.

In view of this correspondence we introduce the following notation.

DEFINITION:

(i) Let χ be a non-trivial character of $GF(q)^+$. Denote by $\chi^{(k)}$ the type k canonical character $(0, \cdots, 0, \chi, 0, \cdots, 0)$ of A_n.

(ii) Write $\alpha_{kn}(\chi)$ for the irreducible constituent of ρ_n containing $\chi^{(k)}$ in its restriction to A_n.

COROLLARY 4.2: *We have $\alpha_{kn}(\chi) = (\chi^{(k)} \cdot \phi)_{U_{kn}}^{U_u}$ where ϕ is a representation of degree one of $Z_{k,n-1}$.*

PROOF: By lemma 2.1 $\alpha_{kn}(\chi)$ is of the form stated; ϕ has degree one by Lemma 4.2.

We next show that ϕ can in fact be taken as the identity representation of $Z_{k,n-1}$ in each case.

LEMMA 4.3: *The restriction $\rho_n|^{U_{kn}}$ contains the one-dimensional representations $\chi^{(k)} \cdot 1$ for each non-trivial character χ of $GF(q)^+$*

PROOF: We construct a subspace $\overline{\mathscr{F}}_n(\chi^{(k)})$ of $\overline{\mathscr{F}}_n$ which realizes the representation $\chi^{(k)} \cdot 1$ of $U_{k,n}$.

We consider the elements of Γ_n as identified with A_n, and let f be the function in $\overline{\mathscr{F}}_n$ such that

$$f(a) = \chi^{(k)}(a) \ (a \in A_n)$$

and take $\overline{\mathscr{F}}_n(\chi^{(k)})$ to be the 1-dimensional space spanned by f. Then A_n clearly acts on $\overline{\mathscr{F}}_n(\chi^{(k)})$ according to $\chi^{(k)}$. Moreover $Z_{k,n-1}$ acts on f according to

$$zf(a) = f(zaz^{-1}) = \chi^{(k)}(zaz^{-1}) = \chi^{(k)}(a) = f(a), \ (z \in Z_{k,n-1}),$$

since $Z_{k,n-1}$ centralizes the character $\chi^{(k)}$ of A_n. Hence $Z_{k,n-1}$ fixes the function f and $\overline{\mathscr{F}}_n(\chi^{(k)})$ is a module for $U_{k,n}$ realizing the representation $\chi^{(k)} \cdot 1$.

This gives us the following explicit form for $\alpha_{k,n}(\chi)$.

COROLLARY 4.3: *We have* $\alpha_{k,n}(\chi) = (\chi^{(k)} \cdot 1)_{U_{k,n}}^{U_n}$

PROOF: The right hand side is an irreducible representation of U_n which occurs in ρ_n by lemma 4.3 and Frobenius reciprocity. Moreover it (the R.H.S.) contains $\chi^{(k)}$ in its restriction to A_n. Hence the result.

Collecting together the last three results we have

THEOREM 4.4: *The representation* ρ_n *has additive decomposition*

$$\rho_n = \bigoplus_{\substack{k=1 \\ \chi}}^{n-1} \alpha_{k,n}(\chi)$$

where χ *runs over the non-trivial characters of* $GF(q)^+$ *and* $\alpha_{k,n}(\chi)$ *is induced from the* 1*-dimensional representation* $\chi^{(k)} \cdot 1$ *of* $U_{k,n}$.

Theorem 4.4 of course immediately yields the decomposition of ρ_i for each i (see theorem 3.3), but using proposition 1.2 we can obtain the constituents of ρ_i directly as induced representations.

PROFOSITION 4.5: *Let* $\lambda_{ki}(\chi)$ *be the character of* U_{ki}^* *(for notation see Section 2) given by* $\lambda_{ki}(\chi)(u) = \chi(u_{ki})$ *where* u_{ki} *is the* (k, i) *matrix coefficient of* $u(u \in U_{ki}^*)$. *Then if* $\alpha_{ki}(\chi)$ *is the representation induced from* $\lambda_{ki}(\chi)$ *we have*

$$\rho_i = \bigoplus_{\substack{k=1 \\ \chi}}^{i-1} \alpha_{ki}(\chi)$$

where χ *runs over the non-trivial characters of* $GR(q)^+$, *and* ρ_i *is as in* 3.3.

PROOF: This follows from Theorem 4.4 and proposition 1.2, which says that lifting the analogues of the $\alpha_{kn}(\chi)$ from U_i to U_n gives the same result as first lifting the 1-dimensional representation of U_{ki} and then inducing.

In summary, we have the following decomposition of Δ_n:

THEOREM 4.6: *We have* $\Delta_n = \rho_n \otimes \rho_{n-1} \otimes \cdots \otimes \rho_2$ *where*

$$\rho_i = \bigoplus_{\substack{k=1 \\ \chi}}^{i-1} \alpha_{ki}(\chi)$$

and the $\alpha_{ki}(\chi)$ *are induced from the* 1*-dimensional representations* $\lambda_{ki}(\chi)$ *of* U_{ki}^*, *so that degree* $(\alpha_{ki}(\chi)) = q^{i-k-1}$.

5. Tensor products

Theorem 4.6 shows that to obtain an additive decomposition of Δ_n, it is necessary to work out tensor products of the form

$$\alpha_{k_2,2}(\chi_2) \otimes \alpha_{k_3,3}(\chi_3) \otimes \cdots \otimes \alpha_{k_n,n}(\chi_n).$$

The object of the present section is to show that many such tensor products are irreducible.

LEMMA 5.1: *Suppose $i < j$ and $k > l$; then $\alpha_{ki}(\chi) \otimes \alpha_{lj}(\chi')$ is irreducible.*

PROOF: The tensor product above is the lift of a corresponding tensor product from U_j to U_n. Hence we may assume that $j = n$ and $i < n$.
We have $\alpha_{ki}(\chi) \otimes \lambda_{ln}(\chi') = \lambda_{ki}(\chi)^{U_n} \otimes \lambda_{ln}(\chi')^{U_n}$ where $\lambda_{ki}(\chi)$ and $\lambda_{ln}(\chi')$ are one-dimensional representations of U^*_{ki} and U_{ln} respectively. By Mackey's formula for tensor products, we have, since $U^*_{ki} \cdot U_{ln} = U_n$ that

$$\alpha_{ki}(\chi) \otimes \alpha_{ln}(\chi') = (\lambda_{ki}(\chi) \cdot \lambda_{ln}(\chi')_{U^*_{ki} \cap U_{ln}})^{U_n}.$$

Let $P = U_{i-1} \cap U_{ln}$, $Q = A_i \cap U_{ln}$ and $R = PQ < Z_{l,n-1}$. Then $R = U_i \cap U_{ln} = Q \rtimes P$ with Q abelian, and proposition 1.3 applies.
Now we have

$$(\lambda_{ki}(\chi) \cdot \lambda_{ln}(\chi')^{U_n}_{U_{ki} \cap U_{ln}} = \{[(\lambda_{ki}(\chi) \cdot 1)^{R}_{U^*_{ki} \cap U_{ln}}]^*_{Zl,n+1} \lambda_{ln}\}^{U}_{U_{ln}}$$

where * denotes the lift to $Z_{l,n-1}$. Moreover since $k > l$ an easy calculation shows that $Z_{k,i-1} \cap U_{ln} < P$ is the full centralizer of the character $\lambda_{ki}(\chi)$ of Q. Hence the first factor above is an irreducible representation of $Z_{l,n-1}$ by proposition 1.3, and by another application of 1.3, so is the representation $\alpha_{ki}(\chi) \otimes \alpha_{ln}(\chi')$.
We have almost as a corollary

THEOREM 5.2: *Suppose $i_1 < i_2 < \cdots < i_r$ and $k_1 > k_2 > \cdots > k_r (1 < r < n)$. Then*

$$\alpha_{k_1,i_1}(\chi_1) \otimes \alpha_{k_2,i_2}(\chi_2) \otimes \cdots \otimes \alpha_{k_r,i_r}(\chi_r)$$

is an irreducible representation of U_n.

PROOF: Here we have

$$U^*_{k_{j+1},i_{j+1}} \cdot (U^*_{k_j,i_j} \cap \cdots \cap U^*_{k_1,i_1}) = U_n$$

since the k_j are distinct. We may also assume as in 5.1 that $i_r = n$. Then Mackey's theorem shows that

$$\alpha_{k_1,i_1}(\chi_1) \otimes \cdots \otimes \alpha_{k_r,i_r}(\chi_r) = \lambda_{k_1,i_1}(\chi_1) \cdots \lambda_{k_r i_r}(\chi_r)^{U_n}_{U_{k1,i_1} \cap \cdots \cap U_{k_r,i_r}}$$

The proof now proceeds inductively; the induced representation on the right is expressed as a representation induced in stages through subgroups of the form R of 5.1, and the irreducibility of the result of each step has the same proof as lemma 5.1. The details are left to the reader.

COROLLARY 5.2′: *Let $\mu(n) = (n-2)+(n-4)+\cdots$. Then there are irreducible components of Δ_n which have degree q^c for each integer c such that $0 \leqq c \leqq \mu(n)$.*

PROOF: The degree of $\alpha_{ki}(\chi)$ is q^{i-k-1}. Hence

$$\text{degree } (\alpha_{k_1,i_1}(\chi_1) \otimes \cdots \otimes \alpha_{k_r,i_r}(\chi_r))$$

is q^c where

$$c = \sum_{j=1}^{r} (i_j - k_j - 1).$$

Thus the corrollary amounts to finding sequences $(k_1, k_2, \ldots)$ such that

$$0 < k_i < i,\ k_{i+1} < k, \text{ and} \sum_{i=1}^{n} (1-k_i-1)$$

is specified between 0 and $\mu(n)$. Taking $k_n = 1$, $k_{n-1} = 2, \ldots,$ $k_{[n/2]+1} = [n/2]-1$, $k_i = i-1$ for $i < [n/2]+1$ we obtain corresponding degree $q^{\mu(n)}$, since if $k_i = i-1$ then $\alpha_{k_i,i}(\chi)$ has degree one. Taking $k_i = i-1$ for each i we obtain corresponding degree q^0. It is clear that by perturbing these choices for k_i we can obtain irreducible representations of degree q^c for each c such that $0 \leqq c \leqq \mu(n)$.

COROLLARY 5.2″: *The group U_n has at least $(q-1)^{[n/2]}$ non-isomorphic irreducible representations of degree $q^{\mu(n)}$, and the sum of the squares of their degrees is an integer polynomial in q with leading term $q^{\frac{1}{2}n(n-1)}$.*

PROOF: It is easy to show by induction on n that the $(q-1)^{[n/2]}$ irreducible representations

$$\alpha_{1,n}(\chi_1) \otimes \alpha_{2,n-1}(\chi_2) \otimes \cdots \otimes \alpha_{[n/2],[(n+1)/2]+1}(\chi_{[n/2]})$$

(where the χ_i range over all combinations of non-trivial characters of $GF(q)^+$) are mutually non-isomorphic, which explicitly constructs the required number of irreducible representations of degree $q^{\mu(n)}$. The conclusion about the sum of squares of the degrees follows from the integer identity $2\mu(n)+[n/2] = \frac{1}{2}n(n-1)$ which may be directly verified.

6. Representations of maximal degree

PROPOSITION 6.1: *Let $G = A \rtimes B$ be a semi-direct product with A abelian. Then the degree of any irreducible complex representation of G is not greater than $|B|$.*

PROOF: Let ψ be an irreducible representation of G. By proposition 1.3, ψ is of the form $\psi = (\chi\phi)^G_{A \times) B_\chi}$ where χ is a character of A, and B_χ is its centralizer in B. Then degree $(\psi) = (\text{degree } \phi) \cdot [B : B_\chi]$. But ϕ is an irreducible representation of B_χ and so has degree $\leqq |B_\chi|$. The result follows.

THEOREM 6.2: *The degree of any irreducible complex representation of U_n of maximal degree is $q^{\mu(n)}$.*

PROOF: By corollary 5.2″ it remains to show only that any irreducible representation has degree $\leqq q^{\mu(n)}$. For this let A be the abelian normal subgroup of U_n given by matrices whose non-diagonal entries are zero except for those in the upper right $[n/2] \times [(n+1)/2]$ rectangle, and let B be its natural complement $U_{[n/2]} \times U_{[n+1/2]}$. Then $|A| = q^l$ where $l = [n/2] \cdot [(n+1)/2]$ and $U_n = A \rtimes B$. By proposition 6.1 any irreducible representation χ of U_n has degree $\leqq |B|$. But $|B| = q^{N-l}$ where $N = \frac{1}{2}\, n(n-1)$, and we complete the proof by observing that $N - l = \mu(n)$ which is easily verified using the relations $2\mu(n) + [n/2] = N$ and $2l - [n/2] = N$.

We conclude with a conjecture on the representations of U_n, related to that made at the ICM in Moscow by Professor J. G. Thompson.

CONJECTURE 6.3:

(i) *U_n has irreducible complex representations only of degree q^c for $0 \leqq c \leqq \mu(n)$.*

(ii) *The number of irreducible representations of degree q^c is an integer polynomial in q.*

Part (i) in fact follows from theorem 6.2 and corollary 5.2′ if we assume a result recently communicated privately to Professor Thompson by E. Goutkin of Moscow. The result states that all irreducible complex representations of U_n have degree a power of q.

If we assume that q is prime, then we obtain

COROLLARY 6.4: *For q prime the degrees of the irreducible complex representations of U_n are polynomials in q, depending only on n.*

The author wishes to thank G. Lusztig for enlightening conversations and in particular for drawing his attention to another way of viewing the representations ρ_i.

It has also come to the attention of the author that the results in Section 2 have been independently obtained by P. V. Lambert and G. van Dijk [9].

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