COMPOSITIO MATHEMATICA

R. TIJDEMAN

On the maximal distance between integers composed of small primes

Compositio Mathematica, tome 28, nº 2 (1974), p. 159-162
<http://www.numdam.org/item?id=CM_1974__28_2_159_0>

COMPOSITIO MATHEMATICA, Vol. 28, Fasc. 2, 1974, pag. 159–162
Noordhoff International Publishing
Printed in the Netherlands

ON THE MAXIMAL DISTANCE BETWEEN INTEGERS COMPOSED OF SMALL PRIMES

R. Tijdeman

Let $p_1, \cdots, p_r$ be fixed primes, $r \geqq 2$. Let $n_1 = 1 < n_2 < \cdots$ be the sequence of all positive integers composed of these primes. In [3] we proved the existence of an effectively computable constant $C_1 > 0$ such that

$$n_{i+1} - n_i > \frac{n_i}{(\log n_i)^{C_1}} \quad \text{for } n_i \geqq 3.$$

In this note we shall prove the existence of effectively computable constants $C_2 > 0$ and N such that

$$n_{i+1} - n_i < \frac{n_i}{(\log n_i)^{C_2}} \quad \text{for } n_i \geqq N. \tag{1}$$

The average order of the difference $n_{i+1} - n_i$ is about $n_i/(\log n_i)^{r-1}$. Hence, $C_1 \geqq r-1$, $C_2 \leqq r-1$. (Compare [3].)

In the proof of (1) we use some elementary properties of continued fractions (See for example [2, Ch. V]) and a result of N.I. Fel'dman. All constants $c, c_1, c_2, \cdots$ will be positive and effectively computable. They only depend on the fixed primes $p_1, \cdots, p_r$ or p, q.

1

LEMMA: *Let p, q be fixed primes, $p \neq q$. Let $h_0/k_0, h_1/k_1, \cdots$ be the sequence of convergents of* $\log p/\log q$. *Then there exists an effectively computable constant c such that*

$$k_{j+1} < k_j^c \log q \quad \textit{for } j = 2, 3, \cdots.$$

PROOF: One has $k_j \geqq 2$ for $j \geqq 2$.
Since

$$\left| \frac{h_j}{k_j} - \frac{\log p}{\log q} \right| < \frac{1}{k_j k_{j+1}} \quad \text{for } j = 0, 1, 2, \cdots,$$

we have

$$|h_j \log q - k_j \log p| < \frac{\log q}{k_{j+1}}. \tag{2}$$

On the other hand, Fel'dman's result [1] implies

$$|h_j \log q - k_j \log p| > \exp\{-c_1(1+\log H_0)\},$$

where $H_0 = \max(1+h_j,\ 1+k_j)$ and c_1 is a constant. Since h_j/k_j is bounded by $1+\log p/\log q$, one has $1+\log H_0 \leqq c_2 \log k_j$ for $j \geqq 2$. So we obtain a constant c such that

$$(3) \qquad |h_j \log q - k_j \log p| > k_j^{-c} \quad \text{for } j \geqq 2.$$

The lemma follows from (2) and (3).

2

In order to prove (1) we may assume $r = 2$ without loss of generality. Hence, it suffices to prove

THEOREM: *Let p and q be primes, $p \neq q$. Let $n_1 = 1 < n_2 < \cdots$ be the sequence of all positive integers composed of these primes. Then there exist effectively computable constants C and N such that*

$$n_{i+1} - n_i < \frac{n_i}{(\log n_i)^C} \quad \textit{for } n_i \geqq N.$$

PROOF: Let $n = n_i = p^u q^v \geqq N$. It is no restriction to assume that $p^u \geqq \sqrt{n}$, and, hence,

$$(4) \qquad u \geqq \frac{\log n}{2 \log p}.$$

Let h_0/k_0, h_1/k_1, $\cdots$ be the convergents of $\log p/\log q$. Then $k_1, k_2, \cdots$ is a monotonic increasing sequence. Take j such that $k_j \leqq u < k_{j+1}$. We suppose that N is so large that both $n \geqq 3$ and $j \geqq 2$. We distinguish cases (a) and (b).

$$\text{(a)} \qquad \frac{h_j}{k_j} > \frac{\log p}{\log q}.$$

Put $n' = p^{u-k_j} q^{v+h_j}$. Hence, $n' > n$. We have

$$\frac{h_j}{k_j} - \frac{\log p}{\log q} < \frac{h_j}{k_j} - \frac{h_{j+1}}{k_{j+1}} = \frac{1}{k_j k_{j+1}}.$$

It follows that

$$\log \frac{n'}{n} = \log \frac{q^{h_j}}{p^{k_j}} = h_j \log q - k_j \log p < \frac{\log q}{k_{j+1}}.$$

Using (4) and $u < k_{j+1}$ we obtain

$$\log\frac{n'}{n} < \frac{\log q}{k_{j+1}} < \frac{\log q}{u} \leqq \frac{2\log p \log q}{\log n}.$$

We see that n'/n has an upper bound only depending on p and q. We therefore have

$$\log\frac{n'}{n} > c_3\left(\frac{n'}{n}-1\right)$$

for some constant c_3. The combination of these inequalities yields

$$\frac{n'}{n}-1 < \frac{c_4}{\log n},$$

and, hence,

(5) $$n_{i+1} \leqq n' < n + \frac{c_4 n}{\log n}.$$

(b) $$\frac{h_j}{k_j} < \frac{\log p}{\log q}.$$

Then $h_{j-1}/k_{j-1} > \log p/\log q$. Put $n' = p^{u-k_{j-1}}q^{v+h_{j-1}}$. Hence, $n' > n$. We have

$$\frac{h_{j-1}}{k_{j-1}} - \frac{\log p}{\log q} < \frac{h_{j-1}}{k_{j-1}} - \frac{h_j}{k_j} = \frac{1}{k_{j-1}k_j}.$$

It follows that

$$\log\frac{n'}{n} = \log\frac{q^{h_{j-1}}}{p^{k_{j-1}}} = h_{j-1}\log q - k_{j-1}\log p < \frac{\log q}{k_j}.$$

We know from the lemma that

$$k_j > \left(\frac{k_{j+1}}{\log q}\right)^{1/c}.$$

Using (4) and $u < k_{j+1}$ we obtain

$$\log\frac{n'}{n} < \frac{\log q}{k_j} < \frac{(\log q)^{1+1/c}}{k_{j+1}^{1/c}} \leqq \frac{(2\log p)^{1/c}(\log q)^{1+1/c}}{(\log n)^{1/c}}.$$

Hence,

$$\log\frac{n'}{n} > c_5\left(\frac{n'}{n}-1\right)$$

for some constant c_5. It follows that

(6) $$n_{i+1} \leqq n' < n + \frac{c_6 n}{(\log n)^{1/C}}.$$

We have in both cases, from (5) and (6),

$$n_{i+1} \leqq n_i + \frac{c_7 n_i}{(\log n_i)^{C_8}} \quad \textit{for } n_i \geqq N.$$

For N sufficiently large this implies

$$n_{i+1} < n_i + \frac{n_i}{(\log n_i)^{C_9}}, \quad \textit{for } n_i \geqq N.$$

This completes the proof.

REFERENCES

[1] N. I. Fel'dman: Improved estimate for a linear form of the logarithms of algebraic numbers. *Mat. Sb.* (*N.S.*) *77* (119) (1968) 423–436. Transl. *Math. USSR Sbornik* *6* (1968) 393–406.

[2] I. Niven: Irrational numbers. *Carus Math. Monographs, 11.* Math. Assoc. America. Wiley, New York, 1956.

[3] R. Tijdeman: On integers with many small prime factors. *Compositio Mathematica* *26* (1973) 319–330.

(Oblatum 5–X–1973)

Mathematical Institute
University of Leiden
Leiden, Netherlands.