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Transcendence measures of exponentials and logarithms of algebraic numbers


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TRANSCENDENCE MEASURES OF EXPONENTIALS AND LOGARITHMS OF ALGEBRAIC NUMBERS

P. L. Cijsouw

1. Introduction

Let \( \sigma \) be a transcendental number. A positive function \( f \) of two integer variables \( N \) and \( H \) is called a transcendence measure of \( \sigma \) if

\[
|P(\sigma)| > f(N, H)
\]

for all non-constant polynomials \( P \) of degree at most \( N \) and with integral coefficients of absolute values at most \( H \).

The purpose of the present paper, which covers a part of the authors thesis [2], is to give transcendence measures for the numbers \( e^\alpha \) (\( \alpha \) algebraic, \( \alpha \neq 0 \)) and \( \log \alpha \) (\( \alpha \) algebraic, \( \alpha \neq 0, 1 \), for any fixed value of the logarithm). These transcendence measures will be of the form

\[
f(N, H) = \exp \left\{-C N^a S^b (1 + \log N)^c (1 + \log S)^d \right\},
\]

where \( S = N + \log H \), for an effectively computable constant \( C > 0 \) and for given constants \( a, b, c \) and \( d \). We try to obtain a small total degree in the exponent in \( N \) and \( S \) together, and to get a minimal contribution of \( S \) within this total degree. Such measures are important for certain applications; see e.g. [1] and [14]. On the other hand, we do not try to determine the constant \( C \) in the exponent as small as possible. In fact, \( C \) will be chosen very large to keep the proof uncomplicated.

As far as we know, no transcendence measure for \( e^\alpha \) which contains explicitly both the dependence on \( N \) and \( H \) was ever published. Earlier transcendence measures of similar types for the special case of the number \( e \) and for \( \log \alpha \) are given by N. I. Fel’dman, namely

\[
\exp \left\{-C_1 N^2 S (\log S)^3 \right\}
\]

for \( e \), see [5],

\[
\exp \left\{-C_2 N^2 \log (1 + N) (1 + N \log N + \log H) \log (2 + N \log N + \log H) \right\}
\]

for \( \log \alpha \), see [3], and

\[
\exp \left\{-C_3 N^2 \log H (1 + \log N)^2 \right\} \text{ if } N < (\log H)^4
\]

for \( \log \alpha \), see [4].
Transcendence measures of other types are published by several authors. Generally speaking, in their results the height plays a more important rôle while the dependence on the degree is not explicitly given. However, in a recent paper, [7], A. I. GALOČKIN proved a measure for \( e^\alpha \) of the form

\[
\exp \{- (1+\varepsilon)N \log H \} \text{ if } N \lesssim \log \log H, \ H \gtrsim H_0(\alpha, \varepsilon).
\]

For more references and information, see [2], [8] and [11]. Finally, we remark that the transcendence of the considered numbers \( e^\alpha \) and \( \log \alpha \) was proved by F. LINDEMANN in [9].

2. Formulation of results

We shall prove the following theorems, where again \( S = N + \log H \):

**Theorem 1:** Let \( \alpha \) be a non-zero algebraic number. Then there exists an effectively computable number \( C_4 = C_4(\alpha) > 0 \) such that \( \exp \{- C_4 N^2 S \} \) is a transcendence measure of \( e^\alpha \).

**Theorem 2:** Let \( \alpha \) be algebraic, \( \alpha \neq 0, 1 \). Let \( \log \alpha \) be any fixed value of the logarithm of \( \alpha \). Then there exists an effectively computable number \( C_5 = C_5(\alpha) > 0 \) such that \( \exp \{- C_5 N^2 S(1+\log N)^2 \} \) is a transcendence measure of \( \log \alpha \).

The method of the proofs will be A. O. GEL’FOND’s method; this method was used too by N. I. FEL’DMAN in the quoted papers. From the nature of these proofs it is clear that the constants \( C_4 \) and \( C_5 \) are effectively computable, so we will make no further reference to this aspect.

3. Notations and lemmas

For any polynomial \( P \) with complex coefficients

\[
P(z) = a_n z^n + \cdots + a_1 z + a_0 \quad (a_n \neq 0)
\]

we call \( n \) the degree and

\[
h = \max_{i = 0, 1, \cdots, n} |a_i|
\]

the height of \( P \). If \( \alpha \) is an algebraic number, then we use the degree \( d(\alpha) \) and the height \( h(\alpha) \) as the degree and height of its minimal defining polynomial. We call \( s(\alpha) = d(\alpha) + \log h(\alpha) \) the size of \( \alpha \). \( \mathbb{Q} \) will denote the field of the rational numbers. If \( a \) is a real number, then \( [a] \) is the greatest integer smaller than or equal to \( a \).
LEMMA 1: Let $\alpha$ be algebraic of height $h(\alpha)$. Then

(1) \[ |\alpha| < h(\alpha) + 1. \]

If moreover $\alpha \neq 0$, then we have

(2) \[ |\alpha| > (h(\alpha) + 1)^{-1}. \]

PROOF: For the first part, see [11], Hilfssatz 1. For the second part, take into consideration that if

$$a_n z^n + \cdots + a_1 z + a_0$$

is the minimal polynomial of $\alpha$, then

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

is the minimal polynomial of $\alpha^{-1}$, apart from a factor $\pm 1$.

LEMMA 2: Let $\alpha_i$ be algebraic of degree $d_i$ and height $h_i$ ($i = 1, \cdots, n$). Denote by $d$ the degree of $\mathbb{Q}(\alpha_1, \cdots, \alpha_n)$ over $\mathbb{Q}$. Let

$$P(z_1, \cdots, z_n) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_n=0}^{N_n} p_{i_1 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n}$$

be a polynomial with integral coefficients $p_{i_1 \cdots i_n}$, such that the sum of the absolute values of the coefficients is at most $B$. Then $P(\alpha_1, \cdots, \alpha_n) = 0$ or

(3) \[ |P(\alpha_1, \cdots, \alpha_n)| \geq B^{-d+1} \prod_{i=1}^{n} \{(d_i+1)h_i\}^{-N/d/d_i} \]

PROOF: See [6], Lemma 2.

For convenience we formulate the following consequence of Lemma 2, in which occurring empty sums should be omitted:

LEMMA 3: Let $\xi$ be algebraic of degree $N$ and size $S$. Let $n \geq 0$ be an integer and let $\alpha_i$ be algebraic of degree $d_i$ and size $s_i$ ($i = 1, \cdots, n$). Put $d = [\mathbb{Q}(\alpha_1, \cdots, \alpha_n) : \mathbb{Q}]$ if $n \geq 1$ and $d = 1$ if $n = 0$. Let

$$P(z_0, z_1, \cdots, z_n) = \sum_{i_0=0}^{N_0} \sum_{i_1=0}^{N_1} \cdots \sum_{i_n=0}^{N_n} p_{i_0 \cdots i_n} z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$$

be a polynomial with integral coefficients whose sum of absolute values is at most $B$. Then $P(\xi, \alpha_1, \cdots, \alpha_n) = 0$ or

(4) \[ |P(\xi, \alpha_1, \cdots, \alpha_n)| > B^{-dN} e^{-dNs} \exp \left\{ -dN \sum_{i=1}^{n} \frac{N_i s_i}{d_i} \right\} \]

PROOF: Apply Lemma 2 with $n$ replaced by $n + 1$ and $\alpha_1, \cdots, \alpha_n$ replaced by $\xi, \alpha_1, \cdots, \alpha_n$. Use the inequalities
LEMMA 4: Let $r$ and $s$ be positive integers such that $s > 2r$. Then any set of $r$ linear forms in $s$ variables

$$
\sum_{\sigma=1}^{s} a_{\rho\sigma} x_{\sigma} \quad (\rho = 1, \cdots, r)
$$

with complex coefficients $a_{\rho\sigma}$ such that $|a_{\rho\sigma}| \leq A \ (\rho = 1, \cdots, r; \ \sigma = 1, \cdots, s)$ has the following property: For every positive even integer $C$ there exist integers $C_1, \cdots, C_s$, not all zero, with $|C_\sigma| \leq C \ (\sigma = 1, \cdots, s)$ and

$$
|\sum_{\sigma=1}^{s} a_{\rho\sigma} C_\sigma| \leq 2 \cdot sAC^{1-s/(2r)} \ (\rho = 1, \cdots, r).
$$

PROOF: See [11], Hilfssatz 28.

LEMMA 5: Let

$$
P_{\rho\sigma}(z_1, \cdots, z_n) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_n=0}^{N_n} p_{\rho\sigma i_1 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n}
$$

($\rho = 1, \cdots, r; \ \sigma = 1, \cdots, s$) be polynomials with integral coefficients $p_{\rho\sigma i_1 \cdots i_n}$ such that the sum of the absolute values of the coefficients of each polynomial is at most $B$. Let $c_i$ be algebraic of degree $d_i$ and height $h_i$ ($i = 1, \cdots, n$) and put $d = [\mathbb{Q}(c_1, \cdots, c_n) : \mathbb{Q}]$. Let $C$ be a positive even integer. If

$$
s > 2rd
$$

and

$$
C^{s/(2r)-d} > \sqrt{2(Bs)^d} \prod_{i=1}^{n} \{(h_i + 1)^{N_i}((d_i + 1)h_i)^{N_i d_i/|d_i|}\}
$$

then there exist integers $C_1, \cdots, C_s$, not all zero, with $|C_\sigma| \leq C$ for $\sigma = 1, \cdots, s$, such that

$$
\sum_{\sigma=1}^{s} C_\sigma P_{\rho\sigma}(c_1, \cdots, c_n) = 0 \quad (\rho = 1, \cdots, r).
$$

PROOF: From Lemma 1 we know that $|c_i| < h_i + 1$.

Hence,

$$
|P_{\rho\sigma}(c_1, \cdots, c_n)| < B(h_1 + 1)^{N_1} \cdots (h_n + 1)^{N_n}.
$$
Define $Y_\rho$ for $\rho = 1, \cdots, r$ by

$$Y_\rho = \sum_{\sigma=1}^{s} C_\sigma P_\rho(\alpha_1, \cdots, \alpha_n).$$

From Lemma 4 we conclude that there exist integers $C_1, \cdots, C_s$, not all zero, with $|C_\sigma| \leq C$ for $\sigma = 1, \cdots, s$ and

$$|Y_\rho| < \sqrt{2sB(h_1 + 1)^{N_1} \cdots (h_n + 1)^{N_n} C^{1-s/(2r)}}$$

for $\rho = 1, \cdots, r$. From (7) and (9) it now follows that

$$|Y_\rho| < (BsC)^{-d+1} \prod_{i=1}^{n} \{(d_i + 1)h_i\}^{-N_i/d_i}$$

for $\rho = 1, \cdots, r$. However, $Y_\rho$ is a polynomial in $\alpha_1, \cdots, \alpha_n$, of degree at most $N_i$ in $\alpha_i$ and with sum of absolute values of its coefficients at most $BsC$. Therefore, according to Lemma 2, the inequality (10) implies that $Y_\rho = 0$ for $\rho = 1, \cdots, r$.

**Lemma 6:** Let $P_{\rho\sigma}(z_0, z_1, \cdots, z_n)$ for $\rho = 1, \cdots, r$ and $\sigma = 1, \cdots, s$ be polynomials with integral coefficients, such that the sum of the absolute values of the coefficients of each polynomial is at most $B$, and such that the degree in $z_i$ of each polynomial is at most $N_i$ ($i = 0, 1, \cdots, n$). Let $z$ be algebraic of degree $N$ and size $S$. Let $z_i$ be algebraic of degree $d_i$ and size $s_i$, $i = 1, \cdots, n$. Put $d = [Q(\alpha_1, \cdots, \alpha_n) : Q]$ if $n \geq 1$ and $d = 1$ if $n = 0$, and let $C$ be a positive even integer. If

$$s \geq 4rd$$

and

$$C^N \geq (Bs)^{N_0 N_1} \exp \left(2N \sum_{i=1}^{n} \frac{N_i s_i}{d_i} \right),$$

then there exist integers $C_{\sigma v}$ ($\sigma = 1, \cdots, s; v = 0, 1, \cdots, N - 1$), not all zero, such that $|C_{\sigma v}| \leq C$ for $\sigma = 1, \cdots, s$ and $v = 0, 1, \cdots, N - 1$ and such that

$$\sum_{\sigma=1}^{s} \sum_{v=0}^{N-1} C_{\sigma v} z_0^v P_{\rho\sigma}(z_0, \alpha_1, \cdots, \alpha_n) = 0$$

for $\rho = 1, \cdots, r$.

**Proof:** Define $P_{\rho v}$ for $v = 0, 1, \cdots, N-1$ by

$$P_{\rho v}(z_0, z_1, \cdots, z_n) = z_0^v P_{\rho\sigma}(z_0, z_1, \cdots, z_n).$$

Then $P_{\rho v}$ is of degree at most $N_0 + N - 1$ in $z_0$ and at most $N_i$ in $z_i$ for
The sum of the absolute values of the coefficients of each $P_{\rho \sigma \nu}$ is at most $B$. The equations (13) now reduce to

$$\sum_{s=1}^{N-1} \sum_{v=0}^{N-1} C_{\sigma \nu} P_{\rho \sigma \nu}(\xi, \alpha_1, \cdots, \alpha_n) = 0$$

for $\rho = 1, \cdots, r$.

We apply Lemma 5 to the polynomials $P_{\rho \sigma \nu}$; to this end we replace $z_1, \cdots, z_n$ by $z_0, z_1, \cdots, z_n$, where the degree in $z_0$ is at most $N_0 + N - 1; \alpha_1, \cdots, \alpha_n$ by $\xi, \alpha_1, \cdots, \alpha_n; s$ by $Ns$ and $d$ by a number that is at most $Nd$. For all positive integers $N$ we have

$$2\sqrt{2N(N+1)} < e^{2N}.$$

Hence,

$$\sqrt{2N^d(H+1)^{N_0+N}((N+1)H)^{(N_0+N)d}} \leq \{2\sqrt{2N(N+1)H^2} \}^{d(N_0+N)} < e^{2d(N_0+N)^2},$$

where $H$ denotes the height of $\xi$. Let $h_i$ be the height of $\alpha_i$. We have

$$2(d_i+1)h_i^2 \leq e^{2d_i h_i^2} = e^{2s_i}.$$

From this it follows for $i = 1, \cdots, n$ that

$$(h_i+1)^{(d_i+1)h_i^{N_0Nd/d_i}} \leq \exp \{2N_i Nds_i/d_i\}.$$  

The inequalities (11), (12), (15) and (16) imply that conditions (6) and (7) with the appropriate substitutions are satisfied. Hence, it follows from Lemma 5 that the integers $C_{\sigma \nu}$ can be chosen in the required way.

**Lemma 7:** Let $F$ be an entire function and let $P$ and $T$ be integers and $R$ and $A$ be real numbers such that $R \geq 2P$ and $A > 2$. Put

$$M_r = \max_{|z| \leq r} |F(z)| \quad (r > 0)$$

and

$$E_1 = \max_{t=0,1,\cdots,T-1, p=0,1,\cdots,P-1} \frac{1}{t!} |F^{(t)}(p)|.$$  

Then

$$M_R \leq 2M_{AR} \left(\frac{2}{A}\right)^{PT} + \left(\frac{9R}{P}\right)^{PT} E_1.$$

**Proof:** By the maximum modulus principle we can choose a complex number $z$ with $|z| = R$ and $|F(z)| = M_R$. From the residue theorem of Cauchy we have the following well-known consequence:
Let \( p \) be one of the numbers 0, 1, \cdots, \( P-1 \) and let \( \zeta \) be a complex number with \( |\zeta - p| = \frac{1}{2} \). Let \( q_0, q_1, \ldots, q_{P-1} \) be the numbers 0, 1, \cdots, \( P-1 \), rearranged in such a way that

\[
|\zeta - q_0| \leq |\zeta - q_1| \leq \cdots \leq |\zeta - q_{P-1}|.
\]

Then

\[
|\zeta - q_0| = \frac{1}{2} \text{ and } |\zeta - q_i| \geq \frac{1}{2}i \text{ for } i = 1, \cdots, P-1.
\]

Hence,

\[
\prod_{q=0}^{P-1} |\zeta - q| = \prod_{i=0}^{P-1} |\zeta - q_i| \geq \frac{1}{2} \prod_{i=1}^{P-1} \frac{1}{2}i = 2^{-P}(P-1)!.\]

The inequality \((P-1)! > (P/3)^P\) is easily checked for \( P = 1, \cdots, 10 \). For higher values of \( P \) it can be proved by induction, using the inequality

\[
\left( \frac{P+1}{P} \right)^{P+1} = \frac{P+1}{P} \cdot \left( \frac{P+1}{P} \right)^P < \frac{11}{10} e < 3.
\]

It follows that

\[
\prod_{q=0}^{P-1} |\zeta - q| > 2^{-P}(P/3)^P = (P/6)^P.
\]

From (18) we now obtain the estimate

\[
|F(z)| \leq AR \cdot M_{AR} \left( \frac{R+P}{AR-R} \right)^{PT} + \frac{1}{2\pi} PE_1 \frac{\pi}{R-P} \left( \frac{R+P}{P/6} \right)^{PT} \sum_{t=0}^{T-1} \left( \frac{1}{2} \right)^t
\]

\[
< AR \cdot M_{AR} \left( \frac{\frac{3}{4}R}{\frac{3}{4}AR} \right)^{PT} + \frac{1}{2\pi} PE_1 \frac{1}{P} \left( \frac{\frac{3}{4}R}{\frac{3}{4}P} \right)^{PT} \cdot 2 =
\]

\[
2M_{AR} \left( \frac{2}{A} \right)^{PT} + \left( \frac{9R}{P} \right)^{PT} E_1.
\]

**Lemma 8:** Let

\[
F(z) = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} C_{km} z^m e^{\omega_k z}
\]

be an exponential polynomial with complex numbers \( C_{km} \) and \( \omega_k \), such that \( \omega_k \neq \omega_l \) if \( k \neq l \). Put

\[
\Omega = \max \left( 1, \max_{k=0, 1, \cdots, K-1} |\omega_k| \right)
\]
and

$$\omega = \min \left( 1, \min_{k, l=0, 1, \ldots, K-1} |\omega_k - \omega_l| \right).$$

Let $T'$ and $P'$ be positive integers and put

$$E = \max_{t=0, 1, \ldots, T'-1 \atop p=0, 1, \ldots, P'-1} |F(t)(p)|.$$

If

$$T'P' \geq 2KM + 13\Omega P',$$

then

$$|C_{km}| \leq P' \left( \frac{6}{K\omega} \max \left( \frac{KM}{\max(1, P'-1)} \right) \right)^{KM} 72^{T'P'} E$$

for $k = 0, 1, \ldots, K-1$ and $m = 0, 1, \ldots, M-1$.

Moreover, if in particular $\omega_k = k\theta$ $(k = 0, 1, \ldots, K-1)$ for some complex number $\theta$, then we may replace (20) by

$$|C_{km}| \leq P' \left( \frac{6}{K\omega} \max \left( \frac{KM}{\max(1, P'-1)} \right) \right)^{KM} 72^{T'P'} E.$$

**PROOF:** See [13], Theorem 2.

**LEMMA 9:** Let $\phi(n, s)$ be a positive function defined for all positive integers $n$ and all $s \geq 1$ with the following properties:

(i) $\phi(n, s) \geq ns$

(ii) $\phi(n, s_1) \leq \phi(n, s_2)$ for all $n, s_1, s_2$ with $s_1 < s_2$

(iii) $\frac{\phi(n_1, s)}{n_1} \leq \frac{\phi(n_2, s)}{n_2}$ for all $n_1, n_2, s$ with $n_1 < n_2$.

If, for some transcendental number $\sigma$,

$$|\sigma - \xi| > \exp \left\{ -\phi(N, S) \right\}$$

for all algebraic numbers $\xi$, where $N$ and $S$ denote the degree and the size of $\xi$, then

$$|P(\sigma)| > \exp \left\{ -3\phi(N, 2S) \right\}$$

for all non-constant polynomials $P$ with integral coefficients, where $N$ and $H$ are the degree and height of $P$ and $S = N + \log H$.

**PROOF:** (compare [3], Proof of Theorem 2) If $P$ is irreducible, it follows by [3], Lemma 5 and by (i) that

$$|P(\sigma)| > \exp \left\{ -\phi(N, S) - 2NS \right\} \geq \exp \left\{ -3\phi(N, S) \right\}.$$
In the general case, write \( P = aP_1 \cdots P_m \) where \( P_i \) is a non-constant irreducible polynomial and \( a > 0 \) an integer. Denote degree and height of \( P_i \) by \( N_i \) and \( H_i \) and put \( S_i = N_i + \log H_i \) \((i = 1, \cdots, m)\). Then clearly

\[
|P_i(\sigma)| > \exp\left\{ -3\phi(N_i, S_i) \right\} \quad (i = 1, \cdots, m).
\]

By e.g. GEL'FOND's well-known inequality on the height of a product of polynomials (see [8] p. 135, Lemma II; see also [12], Lemma 3 and [10]) we have \( H_i \leq e^{N_i}H \) and thus,

\[
S_i \leq 2S \quad (i = 1, \cdots, m).
\]

Using (ii) and (iii) it follows that

\[
|P_i(\sigma)| > \exp\left\{ -3N_i \frac{\phi(N_i, 2S)}{N_i} \right\} \geq \exp\left\{ -3N_i \frac{\phi(N, 2S)}{N} \right\}
\]

for \( i = 1, \cdots, m \). By multiplying these inequalities we obtain the required expression (23), since \( N_1 + \cdots + N_m = N \) and \( a \geq 1 \).

4. Proof of Theorem 1

First we prove

**Theorem 3:** Let \( \alpha \neq 0 \) be an algebraic number of size \( s(\alpha) \). Then there exists an effectively computable number \( S_1 = S_1(\alpha) \) such that

\[
|e^\alpha - \xi| > \exp\left\{ -5.10^8 e^{6s(\alpha)N^2S} \right\}
\]

for all algebraic numbers \( \xi \) of degree \( N \) and size \( S \geq S_1 \).

**Proof:** All estimates occurring in this proof hold for \( S \) sufficiently large. \( S_1 \) can be chosen as the maximum of the finitely many bounds thus obtained. Suppose that

\[
|e^\alpha - \xi| \leq \exp\left\{ -5.10^8 e^{6s(\alpha)N^2S} \right\}
\]

for some algebraic number \( \xi \) of degree \( N \) and size \( S \). It will be shown that this assumption leads to a contradiction if \( S \geq S_1 \). Observe that (25) implies that \( |\xi| < |e^\alpha| + 1 \).

Choose the following positive integers:

\[
K = \left[ 10^3 e^{2s(\alpha)N} \right] \quad T = \left[ 3.10^4 e^{3s(\alpha) \frac{NS}{\log S}} \right]
\]

\[
M = \left[ 15.10^4 e^{4s(\alpha) \frac{NS}{\log S}} \right] \quad P = \left[ 10^3 e^{2s(\alpha)N} \right]
\]

\[
C = 2^{\frac{1}{2}} \exp\left\{ 4.10^6 e^{4s(\alpha)NNS} \right\} \quad T' = \left[ 45.10^4 e^{4s(\alpha) \frac{NS}{\log S}} \right].
\]
We use the auxiliary exponential polynomial

\[ F(z) = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} \sum_{v=0}^{N-1} C_{kmv} \zeta^v z^m e^{\xi k} \]

where the \( C_{kmv} \) are integers of absolute values at most \( C \). Later we shall specify them further.

For \( t = 0, 1, 2, \ldots \) and \( p = 0, 1, 2, \ldots \) we have

\[ F^{(t)}(p) = \sum_k \sum_m \sum_v C_{kmv} \zeta^v \sum_{\tau=0}^{m} \binom{t}{\tau} \frac{m!}{(m-\tau)!} p^{m-\tau} t^{t-\tau} \xi^{kp}. \]

Define \( \Phi_{tp} \) for the same \( t \) and \( p \) by

\[ \Phi_{tp} = \sum_k \sum_m \sum_v C_{kmv} \zeta^v \sum_{\tau=0}^{m} \binom{t}{\tau} \frac{m!}{(m-\tau)!} p^{m-\tau} t^{t-\tau} \xi^{kp}. \]

Then \( \Phi_{tp} \) is an algebraic number, approximating \( F^{(t)}(p) \) very closely. In fact,

\[ |(e^{ix})^{kp} - \zeta^{kp}| \leq KP(|e^{ix}| + 1)^{KP-1} |e - \zeta| \leq \exp \{-4.5 \times 10^8 e^{6s(a)} N^2 S \} \]

for \( k = 0, 1, \ldots, K-1 \) and \( p = 0, 1, \ldots, P-1 \). Hence,

\[ |F^{(t)}(p) - \Phi_{tp}| \leq \frac{K M NC (|e^{ix}| + 1)^{NT} M^M P^M K^T \{\max (1, |x|)\}^T \times \exp \{ -4.5 \times 10^8 e^{6s(a)} N^2 S \}}{\leq \exp \{ -4.10^8 e^{6s(a)} N^2 S \}} \]

for \( t = 0, 1, \ldots, T' -1 \) and \( p = 0, 1, \ldots, P-1 \).

We are going to choose the integers \( C_{kmv} \) such that \( \Phi_{tp} = 0 \) for \( t = 0, 1, \ldots, T-1 \) and \( p = 0, 1, \ldots, P-1 \). To this end we apply Lemma 6 with \( n = 1, x_1 = x \), \( r = TP \) and \( s = KM \) to the polynomials

\[ P_{tpkm}(z_0, z_1) = \sum_{\tau=0}^{m} \binom{t}{\tau} \frac{m!}{(m-\tau)!} p^{m-\tau} t^{t-\tau} z_1^{t-\tau} z_0^{kp} \]

\((t = 0, 1, \ldots, T-1; p = 0, 1, \ldots, P-1; k = 0, 1, \ldots, K-1; m = 0, 1, \ldots, M-1)\). Using the notations of Lemma 6, we have

\[ d = d(x) \leq s(x) \leq e^{s(x)}, N_0 + N \leq (K-1)(P-1) + N \leq KP, N_1 \leq T \]

and

\[ B \leq 2^T M^M P^M K^T \leq \exp \{ 10^6 e^{4s(a)NS} \}. \]

By means of these inequalities one easily verifies conditions (11) and (12) of Lemma 6. According to this lemma, we now choose the integers \( C_{kmv} \), not all zero, with \( |C_{kmv}| \leq C \) such that
for \( t = 0, 1, \cdots, T-1 \) and \( p = 0, 1, \cdots, P-1 \). In this way, our function \( F \) is completely fixed.

From (26) we obtain with (27)

\[
|F^{(t)}(p)| \leq \exp \left\{-4.10^8 e^{6s(a)}N^2S\right\}
\]

for \( t = 0, 1, \cdots, T ' -1 \) and \( p = 0, 1, \cdots, P-1 \). From (1) it follows that \( |\alpha| \leq e^{s(a)} \). Using this inequality we see from the definition of \( F \) that

\[
\max_{|z| \leq 2PS} |F(z)| \leq \exp \{7.10^6 e^{5s(a)}N^2S\}.
\]

We apply Lemma 7 with \( R = 2P \) and \( A = S \). We obtain by (28) and (29)

\[
\max_{|z| < 2P} |F(z)| \leq \exp \left\{-2.1 \times 10^7 e^{5s(a)}N^2S\right\}
\]

\[
+ \exp \left\{-3.10^8 e^{6s(a)}N^2S\right\} \leq \exp \left\{-2.10^7 e^{5s(a)}N^2S\right\}.
\]

Hence, for \( t = 0, 1, \cdots, T ' -1 \) and \( p = 0, 1, \cdots, P-1 \) we have

\[
|F^{(t)}(p)| = \left| \frac{t!}{2\pi i} \int_{|z| = 2P} \frac{F(z)}{(z-p)^{t+1}} \frac{dz}{z} \right| \leq T'T' \cdot 2P \cdot \max_{|z| \leq 2P} |F(z)| \leq \exp \left\{-1.5 \times 10^7 e^{5s(a)}N^2S\right\}
\]

and for the same values of \( t \) and \( p \), using (26),

\[
|\Phi_{tp}| \leq \exp \left\{-10^7 e^{5s(a)}N^2S\right\}.
\]

But \( \Phi_{tp} \) is a polynomial with integral coefficients in \( \xi \) and \( \alpha \), of degree at most \( (K-1)(P-1)+N-1 \leq KP \) in \( \xi \) and less than \( T' \) in \( \alpha \). The sum of the absolute values of its coefficients is not greater than

\[
KMNC 2^T M^M P^M K'T' \leq \exp \{6.10^6 e^{4s(a)}NS\}.
\]

From Lemma 3 it follows that \( \Phi_{tp} = 0 \) or

\[
|\Phi_{tp}| > \exp \left\{-10^7 e^{5s(a)}N^2S\right\}
\]

for \( t = 0, 1, \cdots, T' -1 \) and \( p = 0, 1, \cdots, P-1 \). Since (32) and (33) are incompatible, it follows that \( \Phi_{tp} = 0 \) for \( t = 0, 1, \cdots, T' -1 \) and \( p = 0, 1, \cdots, P-1 \). From (26) we now obtain that

\[
|F^{(t)}(p)| \leq \exp \left\{-4.10^8 e^{6s(a)}N^2S\right\}
\]

for \( t = 0, 1, \cdots, T' -1 \) and \( p = 0, 1, \cdots, P-1 \).

Subsequently we apply Lemma 8 to our exponential polynomial \( F \), with
and $P' = P$. Using the notations of Lemma 8 we have, by $|x| \leq e^{s(x)}$, the inequality

$$\Omega < 10^3 e^{3s(x)} N.$$

Thus,

$$T'P \geq 4.10^8 e^{6s(x)} \frac{N^2 S}{\log S} \geq 2KM + 13\Omega P$$

so that condition (19) is satisfied. Further, by Lemma 1 we have $|x| \leq e^{s(x)}$ and $|x| \geq e^{-s(x)}$. Hence

$$\frac{\Omega}{\omega} \leq \frac{\max (1, K|x|)}{\min (1, |x|)} \leq \frac{Ke^{s(x)}}{e^{-s(x)}} = Ke^{2s(x)}.$$

Thus, it follows from (21) and (34) that

$$| \sum_{v=0}^{N-1} C_{kmv} \xi^v | \leq \exp \left\{ 3.5 \times 10^8 e^{6s(x)} N^2 S \right\}$$

$$\times \exp \left\{ -4.10^8 e^{6s(x)} N^2 S \right\} = \exp \left\{ -5.10^7 e^{6s(x)} N^2 S \right\}$$

for $k = 0, 1, \cdots, K-1$ and $m = 0, 1, \cdots, M-1$. But

$$\sum_{v=0}^{N-1} C_{kmv} \xi^v$$

is a polynomial in $\xi$ of degree less than $N$ and with sum of the absolute values of its coefficients at most $NC$. It follows from Lemma 3 that

$$\sum_{v=0}^{N-1} C_{kmv} \xi^v = 0$$

or

$$| \sum_{v=0}^{N-1} C_{kmv} \xi^v | > \exp \left\{ -5.10^6 e^{4s(x)} N^2 S \right\}$$

for $k = 0, 1, \cdots, K-1$ and $m = 0, 1, \cdots, M-1$. Hence,

$$\sum_{v=0}^{N-1} C_{kmv} \xi^v = 0$$

for all these $k$ and $m$. Since $\xi$ is algebraic of degree $N$, the numbers $1, \xi, \cdots, \xi^{N-1}$ are linearly independent over $Q$. Thus, $C_{kmv} = 0$ for $k = 0, 1, \cdots, K-1, m = 0, 1, \cdots, M-1$ and $v = 0, 1, \cdots, N-1$. This contradicts the choice of the integers $C_{kmv}$ and by this contradiction Theorem 3 is proved.
We now complete the proof of Theorem 1 in the following way:

There are only finitely many algebraic numbers $\xi$ of size $S < S_1$. Since $e^\alpha - \xi \neq 0$ and since $N^2 S > 0$ for all of these numbers, there exists a number $C_6 > 0$ such that

$$|e^\alpha - \xi| > \exp \{-C_6 N^2 S\}$$

for all algebraic numbers $\xi$ of degree $N$ and size $S < S_1$. Then $C_7 = \max (C_6, 5.10^8 \ e^{6s(\alpha)})$ will have the property

$$|e^\alpha - \xi| > \exp \{-C_7 N^2 S\}$$

for all algebraic $\xi$ of degree $N$ and size $S$. From Lemma 9 we obtain that

$$|P(e^\alpha)| > \exp \{-C_4 N^2 S\}$$

for all polynomials $P$ with integral coefficients, of degree $N$ and height $H$, with $S = N + \log H$ and $C_4 = 6C_7$. By reasons of monotony, the same inequality holds for polynomials of degree at most $N$ and height at most $H$. Hence, $\exp \{-C_4 N^2 S\}$ is a transcendence measure for $e^\alpha$ and Theorem 1 has been proved.

5. Proof of Theorem 2

Let $\log \alpha$ be an arbitrary but fixed value of the logarithm of the algebraic number $\alpha$. N. I. Fel'dman proved the following assertion: (see Theorem 1 of [4], with $m = 1$)

There exists an effectively computable positive number $C_8 = C_8(\log \alpha)$, such that

$$|\log \alpha - \xi| > \exp \{-C_8 N^2 \log (\log (N+2))^2\}$$

for all algebraic numbers $\xi$ of degree $N$ and height $H$, provided that $N < (\log H)^4$.

From this, it easily follows that

$$|\log \alpha - \xi| > \exp \{-2C_8 N^2 S(1 + \log N)^2\}$$

for all algebraic numbers $\xi$ of degree $N$ and with size $S$, provided that $N < (\frac{1}{2}S)^4$.

For the complementary case $N \geq (\frac{1}{2}S)^4$ we prove the following theorem:

**Theorem 4:** Let $\alpha \neq 0, 1$ be an algebraic number of size $s(\alpha)$ and let $\log \alpha$ be an arbitrary, but fixed, value of the logarithm of $\alpha$. Then there exists an effectively computable number $S_2 = S_2(\log \alpha)$, such that

$$|\log \alpha - \xi| > \exp \{-3.10^9(s(\alpha))^7 N^2 S\}$$
for all algebraic numbers $\xi$ of degree $N$ and size $S \geq S_2$, provided that $N \geq (\frac{1}{4}S)^{\frac{3}{4}}$.

**Proof:** Since the structure of the proof is the same as that of Theorem 3, only a shorted proof is given. Suppose that

$$(40) \quad | \log \alpha - \xi | \leq \exp \{ -3.10^9(s(\alpha))^7N^2S \}$$

for some algebraic number $\xi$ of degree $N$ and size $S$, such that $N \geq (\frac{1}{4}S)^{\frac{3}{4}}$. We shall derive a contradiction in the case of large $S$.

Choose the integers

$$K = \lfloor 10^3(s(\alpha))^2N \rfloor \quad \quad M = \left\lfloor 3.10^5(s(\alpha))^5 \cdot \frac{NS}{\log S} \right\rfloor$$

$$C = 2\left\lfloor \frac{1}{4} \exp \{4.10^6(s(\alpha))^5NS\} \right\rfloor \quad \quad T = \left\lfloor 7.10^4(s(\alpha))^4 \cdot \frac{N^2}{\log S} \right\rfloor$$

$$P = \lfloor 10^3(s(\alpha))^2S \rfloor \quad \quad T' = \left\lfloor 7.10^5(s(\alpha))^5 \cdot \frac{N^2}{\log S} \right\rfloor.$$ 

Put

$$F(z) = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} \sum_{\nu=0}^{N-1} C_{km} \xi^\nu \zeta^m z^{k(log \alpha)z},$$

where the numbers $C_{km\nu}$ are integers of absolute values at most $C$; they will be specified later. We have for $t, p = 0, 1, 2, \cdots$

$$F^{(t)}(p) = \sum_{k} \sum_{m} \sum_{\nu} C_{km} \xi^\nu \sum_{t=0}^{m} \binom{t}{\tau} \frac{m!}{(m-\tau)!} p^{m-\tau} \alpha^{-\tau t} \kappa^k.$$ 

Put

$$\Phi_{tp} = \sum_{k} \sum_{m} \sum_{\nu} C_{km} \xi^\nu \sum_{t=0}^{m} \binom{t}{\tau} \frac{m!}{(m-\tau)!} p^{m-\tau} \alpha^{-\tau t} \kappa^k.$$ 

For $t = 0, 1, \cdots, T' - 1$ and $p = 0, 1, \cdots, P - 1$ it easily follows from the inequality $|\alpha| \leq e^{s(\alpha)}$ that

$$(41) \quad |F^{(t)}(p) - \Phi_{tp}| \leq \exp \{ -2.10^9(s(\alpha))^7N^2S \}.$$ 

Let $P_{tpkm}$ for $t = 0, 1, \cdots, T - 1, p = 0, 1, \cdots, P - 1, k = 0, 1, \cdots, K - 1$ and $m = 0, 1, \cdots, M - 1$ be the polynomials (taken in the obvious way) such that

$$\Phi_{tp} = \sum_{k} \sum_{m} \sum_{\nu} C_{km} \xi^\nu P_{tpkm}(\xi, \alpha).$$ 

We now choose the numbers $C_{km\nu}$ according to Lemma 6, applied to the polynomials $P_{tpkm}$, as integers, not all zero, of absolute values at most $C$, such that $\Phi_{tp} = 0$ for $t = 0, 1, \cdots, T - 1$ and $p = 0, 1, \cdots, P - 1$. It follows from (41) that
Since and since \( N \approx (1 + S)^4 \) we see from Lemma 7 applied with \( R = 2P \) and \( A = S^{1/5} \) that

\[
|F^{(t)}(p)| \leq \exp \{ -2.10^9 (s(\alpha))^7 N^2 S \}
\]

for \( t = 0, 1, \cdots, T-1 \) and \( p = 0, 1, \cdots, P-1 \).

Hence,

\[
Hence,
\]

\[
\max \{ |F(z)| \} \leq \exp \{ 3.10^6 (s(\alpha))^4 |\log \alpha| N S^{6/5} \}
\]

and since \( N \geq (1 + S)^4 \) we see from Lemma 7 applied with \( R = 2P \) and \( A = S^{1/5} \) that

\[
\max \{ |F(z)| \} \leq \exp \{ -10^7 (s(\alpha))^6 N^2 S \}
\]

(43)

Hence,

\[
|F^{(t)}(p)| \leq \exp \{ -9.10^6 (s(\alpha))^6 N^2 S \}
\]

and

\[
|\Phi_{tp}| \leq \exp \{ -8.10^6 (s(\alpha))^6 N^2 S \}
\]

for \( t = 0, 1, \cdots, T'-1 \) and \( p = 0, 1, \cdots, P-1 \). From Lemma 3 we obtain \( \Phi_{tp} = 0 \) or

(45)

for the same values of \( t \) and \( p \). Thus, \( \Phi_{tp} = 0 \) and, from (41),

(46)

for \( t = 0, 1, \cdots, T'-1 \) and \( p = 0, 1, \cdots, P-1 \).

Condition (19) with \( P' = P \) and \( \Omega \leq 10^3 (s(\alpha))^2 |\log \alpha| N \) is satisfied. Further,

\[
\frac{\Omega}{\alpha} \leq \frac{\max (1, K |\log \alpha|)}{\min (1, |\log \alpha|)} \leq K \frac{\max (1, |\log \alpha|)}{\min (1, |\log \alpha|)}.
\]

Using this inequality and (46) we obtain from (21)

(47)

\[
|\sum_{v=0}^{N-1} C_{kmv} \xi^v| \leq \exp \{ -10^9 (s(\alpha))^7 N^2 S \}
\]

for \( k = 0, 1, \cdots, K-1 \) and \( m = 0, 1, \cdots, M-1 \).

From Lemma 3 it now follows that

\[
\sum_{v=0}^{N-1} C_{kmv} \xi^v = 0
\]

for all \( k \) and \( m \). This implies that all integers \( C_{kmv} \) are zero. By the contradiction to the choice of these integers, Theorem 4 has been proved.
We proceed to prove Theorem 2. From (39) it follows that there exists an effectively computable number \( C_9 = C_9(\log z) > 0 \) such that

\[
|\log z - \xi| > \exp \{-C_9 N^2 S\}
\]

for all algebraic numbers \( \xi \) of degree \( N \) and size \( S \) with \( N \geq (\frac{1}{2}S)^4 \), since there are only finitely many algebraic numbers of size smaller than \( S_2 \). Taking \( C_{10} = \max (2C_8, C_9) \) we see from (38) and (48) that

\[
|\log z - \xi| > \exp \{-C_{10} N^2 S(1 + \log N)^2\}
\]

for all algebraic numbers \( \xi \) of degree \( N \) and size \( S \). As before, the application of Lemma 9 completes the proof.

REFERENCES