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# TRANSCENDENCE MEASURES OF EXPONENTIALS AND LOGARITHMS OF ALGEBRAIC NUMBERS

P. L. Cijsouw

## 1. Introduction

Let $\sigma$ be a transcendental number. A positive function $f$ of two integer variables $N$ and $H$ is called a *transcendence measure of* $\sigma$ if

$$|P(\sigma)| > f(N, H)$$

for all non-constant polynomials $P$ of degree at most $N$ and with integral coefficients of absolute values at most $H$.

The purpose of the present paper, which covers a part of the authors thesis [2], is to give transcendence measures for the numbers $e^{\alpha}$ ($\alpha$ algebraic, $\alpha \neq 0$) and $\log \alpha$ ($\alpha$ algebraic, $\alpha \neq 0, 1$, for any fixed value of the logarithm). These transcendence measures will be of the form

$$f(N, H) = \exp\{-C\, N^a S^b (1+\log N)^c (1+\log S)^d\},$$

where $S = N + \log H$, for an effectively computable constant $C > 0$ and for given constants $a$, $b$, $c$ and $d$. We try to obtain a small total degree in the exponent in $N$ and $S$ together, and to get a minimal contribution of $S$ within this total degree. Such measures are important for certain applications; see e.g. [1] and [14]. On the other hand, we do not try to determine the constant $C$ in the exponent as small as possible. In fact, $C$ will be chosen very large to keep the proof uncomplicated.

As far as we know, no transcendence measure for $e^{\alpha}$ which contains explicitly both the dependence on $N$ and $H$ was ever published. Earlier transcendence measures of similar types for the special case of the number $e$ and for $\log \alpha$ are given by N. I. FEL'DMAN, namely

$$\exp\{-C_1\, N^2 S (\log S)^3\}$$

for $e$, see [5],

$$\exp\{-C_2 N^2 \log(1+N)(1+N\log N+\log H)\log(2+N\log N+\log H)\}$$

for $\log \alpha$, see [3], and

$$\exp\{-C_3\, N^2 \log H (1+\log N)^2\} \text{ if } N < (\log H)^{\frac{1}{4}}$$

for $\log \alpha$, see [4].

Transcendence measures of other types are published by several authors. Generally speaking, in their results the height plays a more important rôle while the dependence on the degree is not explicitly given. However, in a recent paper, [7], A. I. GALOČKIN proved a measure for $e^\alpha$ of the form

$$\exp\{-(1+\varepsilon)N\log H\} \text{ if } N \leqq \log\log H,\ H \geqq H_0(\alpha, \varepsilon).$$

For more references and information, see [2], [8] and [11]. Finally, we remark that the transcendence of the considered numbers $e^\alpha$ and $\log \alpha$ was proved by F. LINDEMANN in [9].

## 2. Formulation of results

We shall prove the following theorems, where again $S = N + \log H$:

THEOREM 1: *Let $\alpha$ be a non-zero algebraic number. Then there exists an effectively computable number $C_4 = C_4(\alpha) > 0$ such that* $\exp\{-C_4 N^2 S\}$ *is a transcendence measure of $e^\alpha$.*

THEOREM 2: *Let $\alpha$ be algebraic, $\alpha \neq 0, 1$. Let* $\log \alpha$ *be any fixed value of the logarithm of $\alpha$. Then there exists an effectively computable number $C_5 = C_5(\alpha) > 0$ such that* $\exp\{-C_5 N^2 S(1+\log N)^2\}$ *is a transcendence measure of* $\log \alpha$.

The method of the proofs will be A. O. GEL'FOND'S method; this method was used too by N. I. FEL'DMAN in the quoted papers. From the nature of these proofs it is clear that the constants $C_4$ and $C_5$ are effectively computable, so we will make no further reference to this aspect.

## 3. Notations and lemmas

For any polynomial $P$ with complex coefficients

$$P(z) = a_n z^n + \cdots + a_1 z + a_0 \quad (a_n \neq 0)$$

we call $n$ the *degree* and

$$h = \max_{i=0,1,\cdots,n} |a_i|$$

the *height* of $P$. If $\alpha$ is an algebraic number, then we use the *degree* $d(\alpha)$ and the *height* $h(\alpha)$ as the degree and height of its minimal defining polynomial. We call $s(\alpha) = d(\alpha) + \log h(\alpha)$ the *size* of $\alpha$. **Q** will denote the field of the rational numbers. If $a$ is a real number, then $[a]$ is the greatest integer smaller than or equal to $a$.

**Lemma** 1: *Let $\alpha$ be algebraic of height $h(\alpha)$. Then*

$$|\alpha| < h(\alpha)+1. \tag{1}$$

*If moreover $\alpha \neq 0$, then we have*

$$|\alpha| > (h(\alpha)+1)^{-1}. \tag{2}$$

**Proof**: For the first part, see [11], Hilfssatz 1. For the second part, take into consideration that if

$$a_n z^n + \cdots + a_1 z + a_0$$

is the minimal polynomial of $\alpha$, then

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

is the minimal polynomial of $\alpha^{-1}$, apart from a factor $\pm 1$.

**Lemma** 2: *Let $\alpha_i$ be algebraic of degree $d_i$ and height $h_i$ $(i = 1, \cdots, n)$. Denote by $d$ the degree of $\mathbf{Q}(\alpha_1, \cdots, \alpha_n)$ over $\mathbf{Q}$. Let*

$$P(z_1, \cdots, z_n) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_n=0}^{N_n} p_{i_1 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n}$$

*be a polynomial with integral coefficients $p_{i_1} \cdots {}_{i_n}$, such that the sum of the absolute values of the coefficients is at most $B$. Then $P(\alpha_1, \cdots, \alpha_n) = 0$ or*

$$|P(\alpha_1, \cdots, \alpha_n)| \geqq B^{-d+1} \prod_{i=1}^{n} \{(d_i+1)h_i\}^{-N_i d/d_i} \tag{3}$$

**Proof**: See [6], Lemma 2.

For convenience we formulate the following consequence of Lemma 2, in which occurring empty sums should be omitted:

**Lemma** 3: *Let $\xi$ be algebraic of degree $N$ and size $S$. Let $n \geqq 0$ be an integer and let $\alpha_i$ be algebraic of degree $d_i$ and size $s_i$ $(i = 1, \cdots, n)$. Put $d = [\mathbf{Q}(\alpha_1, \cdots, \alpha_n) : \mathbf{Q}]$ if $n \geqq 1$ and $d = 1$ if $n = 0$. Let*

$$P(z_0, z_1, \cdots, z_n) = \sum_{i_0=0}^{N_0} \sum_{i_1=0}^{N_1} \cdots \sum_{i_n=0}^{N_n} p_{i_0 i_1 \cdots i_n} z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}.$$

*be a polynomial with integral coefficients whose sum of absolute values is at most $B$. Then $P(\xi, \alpha_1, \cdots, \alpha_n) = 0$ or*

$$|P(\xi, \alpha_1, \cdots, \alpha_n)| > B^{-dN} e^{-dN_0 S} \exp\left\{-dN \sum_{i=1}^{n} \frac{N_i s_i}{d_i}\right\}. \tag{4}$$

**Proof**: Apply Lemma 2 with $n$ replaced by $n+1$ and $\alpha_1, \cdots, \alpha_n$ replaced by $\xi, \alpha_1, \cdots, \alpha_n$. Use the inequalities

$$[\mathbf{Q}(\xi, \alpha_1, \cdots, \alpha_n) : \mathbf{Q}] \leqq dN,$$
$$(N+1)H < e^S$$

and

$$(d_i+1)h_i < e^{s_i} \quad (i = 1, \cdots, n).$$

LEMMA 4: *Let r and s be positive integers such that $s > 2r$. Then any set of r linear forms in s variables*

$$\sum_{\sigma=1}^{s} a_{\rho\sigma} x_\sigma \quad (\rho = 1, \cdots, r)$$

*with complex coefficients $a_{\rho\sigma}$ such that $|a_{\rho\sigma}| \leqq A$ ($\rho = 1, \cdots, r$; $\sigma = 1, \cdots, s$) has the following property: For every positive even integer C there exist integers $C_1, \cdots, C_s$, not all zero, with $|C_\sigma| \leqq C$ ($\sigma = 1, \cdots, s$) and*

$$(5) \qquad |\sum_{\sigma=1}^{s} a_{\rho\sigma} C_\sigma| \leqq \sqrt{2} \cdot sAC^{1-s/(2r)} \quad (\rho = 1, \cdots, r).$$

PROOF: See [11], Hilfssatz 28.

LEMMA 5: *Let*

$$P_{\rho\sigma}(z_1, \cdots, z_n) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_n=0}^{N_n} p_{\rho\sigma i_1 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n}$$

*($\rho = 1, \cdots, r$; $\sigma = 1, \cdots, s$) be polynomials with integral coefficients $p_{\rho\sigma i_1 \cdots i_n}$, such that the sum of the absolute values of the coefficients of each polynomial is at most B. Let $\alpha_i$ be algebraic of degree $d_i$ and height $h_{i_1}$ ($i = 1, \cdots, n$) and put $d = [\mathbf{Q}(\alpha_1, \cdots, \alpha_n) : \mathbf{Q}]$. Let C be a positive even integer. If*

$$(6) \qquad s > 2rd$$

*and*

$$(7) \qquad C^{s/(2r)-d} > \sqrt{2}(Bs)^d \prod_{i=1}^{n} \{(h_i+1)^{N_i}((d_i+1)h_i)^{N_i d/d_i}\}$$

*then there exist integers $C_1, \cdots, C_s$, not all zero, with $|C_\sigma| \leqq C$ for $\sigma = 1, \cdots, s$, such that*

$$(8) \qquad \sum_{\sigma=1}^{s} C_\sigma P_{\rho\sigma}(\alpha_1, \cdots, \alpha_n) = 0 \quad (\rho = 1, \cdots, r).$$

PROOF: From Lemma 1 we know that $|\alpha_i| < h_i+1$.

Hence,

$$|P_{\rho\sigma}(\alpha_1, \cdots, \alpha_n)| < B(h_1+1)^{N_1} \cdots (h_n+1)^{N_n}.$$

Define $Y_\rho$ for $\rho = 1, \cdots, r$ by

$$Y_\rho = \sum_{\sigma=1}^{s} C_\sigma P_{\rho\sigma}(\alpha_1, \cdots, \alpha_n).$$

From Lemma 4 we conclude that there exist integers $C_1, \cdots, C_s$, not all zero, with $|C_\sigma| \leqq C$ for $\sigma = 1, \cdots, s$ and

$$|Y_\rho| < \sqrt{2s}B(h_1+1)^{N_1} \cdots (h_n+1)^{N_n} C^{1-s/(2r)} \tag{9}$$

for $\rho = 1, \cdots, r$. From (7) and (9) it now follows that

$$|Y_\rho| < (BsC)^{-d+1} \prod_{i=1}^{n} \{(d_i+1)h_i\}^{-N_i d/d_i} \tag{10}$$

for $\rho = 1, \cdots, r$. However, $Y_\rho$ is a polynomial in $\alpha_1, \cdots, \alpha_n$, of degree at most $N_i$ in $\alpha_i$ and with sum of absolute values of its coefficients at most $BsC$. Therefore, according to Lemma 2, the inequality (10) implies that $Y_\rho = 0$ for $\rho = 1, \cdots, r$.

LEMMA 6: *Let $P_{\rho\sigma}(z_0, z_1, \cdots, z_n)$ for $\rho = 1, \cdots, r$ and $\sigma = 1, \cdots, s$ be polynomials with integral coefficients, such that the sum of the absolute values of the coefficients of each polynomial is at most $B$, and such that the degree in $z_i$ of each polynomial is at most $N_i$ $(i = 0, 1, \cdots, n)$. Let $\xi$ be algebraic of degree $N$ and size $S$. Let $\alpha_i$ be algebraic of degree $d_i$ and size $s_i$, $i = 1, \cdots, n$. Put $d = [\mathbf{Q}(\alpha_1, \cdots, \alpha_n) : \mathbf{Q}]$ if $n \geqq 1$ and $d = 1$ if $n = 0$, and let $C$ be a positive even integer. If*

$$s \geqq 4rd \tag{11}$$

*and*

$$C^N \geqq (Bs)^N e^{2(N_0+N)S} \exp\left\{2N \sum_{i=1}^{n} \frac{N_i s_i}{d_i}\right\}, \tag{12}$$

*then there exist integers $C_{\sigma\nu}$ $(\sigma = 1, \cdots, s;\ \nu = 0, 1, \cdots, N-1)$, not all zero, such that $|C_{\sigma\nu}| \leqq C$ for $\sigma = 1, \cdots, s$ and $\nu = 0, 1, \cdots, N-1$ and such that*

$$\sum_{\sigma=1}^{s} \sum_{\nu=0}^{N-1} C_{\sigma\nu} \xi^\nu P_{\rho\sigma}(\xi, \alpha_1, \cdots, \alpha_n) = 0 \tag{13}$$

*for $\rho = 1, \cdots, r$.*

PROOF: Define $P_{\rho\sigma\nu}$ for $\nu = 0, 1, \cdots, N-1$ by

$$P_{\rho\sigma\nu}(z_0, z_1, \cdots, z_n) = z_0^\nu P_{\rho\sigma}(z_0, z_1, \cdots, z_n).$$

Then $P_{\rho\sigma\nu}$ is of degree at most $N_0+N-1$ in $z_0$ and at most $N_i$ in $z_i$ for

$i = 1, \cdots, n$. The sum of the absolute values of the coefficients of each $P_{\rho\sigma\nu}$ is at most $B$. The equations (13) now reduce to

$$\sum_{\sigma=1}^{s} \sum_{\nu=0}^{N-1} C_{\sigma\nu} P_{\rho\sigma\nu}(\xi, \alpha_1, \cdots, \alpha_n) = 0 \tag{14}$$

for $\rho = 1, \cdots, r$.

We apply Lemma 5 to the polynomials $P_{\rho\sigma\nu}$; to this end we replace $z_1, \cdots, z_n$ by $z_0, z_1, \cdots, z_n$, where the degree in $z_0$ is at most $N_0+N-1$; $\alpha_1, \cdots, \alpha_n$ by $\xi, \alpha_1, \cdots, \alpha_n$; $s$ by $Ns$ and $d$ by a number that is at most $Nd$. For all positive integers $N$ we have

$$2\sqrt{2}N(N+1) < e^{2N}.$$

Hence,

$$\sqrt{2}N^{Nd}(H+1)^{N_0+N}\{(N+1)H\}^{(N_0+N)d} \leqq \{2\sqrt{2}N(N+1)H^2\}^{d(N_0+N)} < e^{2d(N_0+N)S}, \tag{15}$$

where $H$ denotes the height of $\xi$. Let $h_i$ be the height of $\alpha_i$. We have

$$2(d_i+1)h_i^2 \leqq e^{2d_i}h_i^2 = e^{2s_i}.$$

From this it follows for $i = 1, \cdots, n$ that

$$(h_i+1)^{N_i}\{(d_i+1)h_i\}^{N_iNd/d_i} \leqq \exp\{2N_i Nds_i/d_i\}. \tag{16}$$

The inequalities (11), (12), (15) and (16) imply that conditions (6) and (7) with the appropriate substitutions are satisfied. Hence, it follows from Lemma 5 that the integers $C_{\sigma\nu}$ can be chosen in the required way.

LEMMA 7: *Let $F$ be an entire function and let $P$ and $T$ be integers and $R$ and $A$ be real numbers such that $R \geqq 2P$ and $A > 2$. Put*

$$M_r = \max_{|z| \leqq r} |F(z)| \quad (r > 0)$$

*and*

$$E_1 = \max_{\substack{t=0,1,\cdots,T-1 \\ p=0,1,\cdots,P-1}} \frac{1}{t!}|F^{(t)}(p)|.$$

*Then*

$$M_R \leqq 2M_{AR}\left(\frac{2}{A}\right)^{PT} + \left(\frac{9R}{P}\right)^{PT} E_1. \tag{17}$$

PROOF: By the maximum modulus principle we can choose a complex number $z$ with $|z| = R$ and $|F(z)| = M_R$. From the residue theorem of Cauchy we have the following well-known consequence:

$$(18)\quad F(z) = \frac{1}{2\pi i}\int_{|\zeta|=AR} \frac{F(\zeta)}{\zeta - z}\prod_{p=0}^{P-1}\left(\frac{z-p}{\zeta-p}\right)^T d\zeta + \\ -\frac{1}{2\pi i}\sum_{p=0}^{P-1}\sum_{t=0}^{T-1}\frac{F^{(t)}(p)}{t!}\int_{|\zeta-p|=\frac{1}{2}}\frac{(\zeta-p)^t}{\zeta-z}\prod_{q=0}^{P-1}\left(\frac{z-q}{\zeta-q}\right)^T d\zeta.$$

Let $p$ be one of the numbers $0, 1, \cdots, P-1$ and let $\zeta$ be a complex number with $|\zeta-p| = \frac{1}{2}$. Let $q_0, q_1, \cdots, q_{P-1}$ be the numbers $0, 1, \cdots, P-1$, rearranged in such a way that

$$|\zeta-q_0| \leqq |\zeta-q_1| \leqq \cdots \leqq |\zeta-q_{P-1}|.$$

Then

$$|\zeta-q_0| = \tfrac{1}{2} \text{ and } |\zeta-q_i| \geqq \tfrac{1}{2}i \text{ for } i = 1, \cdots, P-1.$$

Hence,

$$\prod_{q=0}^{P-1}|\zeta-q| = \prod_{i=0}^{P-1}|\zeta-q_i| \geqq \tfrac{1}{2}\prod_{i=1}^{P-1}\tfrac{1}{2}i = 2^{-P}(P-1)!.$$

The inequality $(P-1)! > (P/3)^P$ is easily checked for $P = 1, \cdots, 10$. For higher values of $P$ it can be proved by induction, using the inequality

$$\left(\frac{P+1}{P}\right)^{P+1} = \frac{P+1}{P}\cdot\left(\frac{P+1}{P}\right)^P < \frac{11}{10}e < 3.$$

It follows that

$$\prod_{q=0}^{P-1}|\zeta-q| > 2^{-P}(P/3)^P = (P/6)^P.$$

From (18) we now obtain the estimate

$$|F(z)| \leqq AR\frac{M_{AR}}{AR-R}\left(\frac{R+P}{AR-P}\right)^{PT} + \frac{1}{2\pi}PE_1\frac{\pi}{R-P}\left(\frac{R+P}{P/6}\right)^{PT}\sum_{t=0}^{T-1}(\tfrac{1}{2})^t$$
$$< AR\frac{M_{AR}}{\frac{1}{2}AR}\left(\frac{\frac{3}{2}R}{\frac{3}{4}AR}\right)^{PT} + \tfrac{1}{2}PE_1\frac{1}{P}\left(\frac{\frac{3}{2}R}{\frac{1}{6}P}\right)^{PT}\cdot 2 =$$
$$2M_{AR}\left(\frac{2}{A}\right)^{PT} + \left(\frac{9R}{P}\right)^{PT}E_1.$$

LEMMA 8: *Let*

$$F(z) = \sum_{k=0}^{K-1}\sum_{m=0}^{M-1} C_{km}z^m e^{\omega_k z}$$

*be an exponential polynomial with complex numbers* $C_{km}$ *and* $\omega_k$, *such that* $\omega_k \neq \omega_l$ *if* $k \neq l$. *Put*

$$\Omega = \max(1, \max_{k=0,1,\cdots,K-1}|\omega_k|)$$

*and*

$$\omega = \min(1, \min_{\substack{k,l=0,1,\cdots,K-1\\ k\neq l}} |\omega_k - \omega_l|).$$

*Let $T'$ and $P'$ be positive integers and put*

$$E = \max_{\substack{t=0,1,\cdots,T'-1\\ p=0,1,\cdots,P'-1}} |F^{(t)}(p)|.$$

*If*

$$T'P' \geqq 2KM + 13\Omega P', \tag{19}$$

*then*

$$|C_{km}| \leqq P' \left\{\frac{6}{\sqrt{K}} \frac{\Omega}{\omega} \max\left(6, \frac{KM}{\max(1, P'-1)}\right)\right\}^{KM} 72^{T'P'} E \tag{20}$$

*for $k = 0, 1, \cdots, K-1$ and $m = 0, 1, \cdots, M-1$.*

*Moreover, if in particular $\omega_k = k\theta$ $(k = 0, 1, \cdots, K-1)$ for some complex number $\theta$, then we may replace* (20) *by*

$$|C_{km}| \leqq P' \left\{\frac{6}{K} \frac{\Omega}{\omega} \max\left(6, \frac{KM}{\max(1, P'-1)}\right)\right\}^{KM} 72^{T'P'} E. \tag{21}$$

Proof: See [13], Theorem 2.

Lemma 9: *Let $\phi(n, s)$ be a positive function defined for all positive integers $n$ and all $s \geqq 1$ with the following properties:*

(i) $\phi(n, s) \geqq ns$

(ii) $\phi(n, s_1) \leqq \phi(n, s_2)$ *for all $n, s_1, s_2$ with $s_1 < s_2$*

(iii) $\dfrac{\phi(n_1, s)}{n_1} \leqq \dfrac{\phi(n_2, s)}{n_2}$ *for all $n_1, n_2, s$ with $n_1 < n_2$.*

*If, for some transcendental number $\sigma$,*

$$|\sigma - \xi| > \exp\{-\phi(N, S)\} \tag{22}$$

*for all algebraic numbers $\xi$, where $N$ and $S$ denote the degree and the size of $\xi$, then*

$$|P(\sigma)| > \exp\{-3\phi(N, 2S)\} \tag{23}$$

*for all non-constant polynomials $P$ with integral coefficients, where $N$ and $H$ are the degree and height of $P$ and $S = N + \log H$.*

Proof: (compare [3], Proof of Theorem 2) If $P$ is irreducible, it follows by [3], Lemma 5 and by (i) that

$$|P(\sigma)| > \exp\{-\phi(N, S) - 2NS\} \geqq \exp\{-3\phi(N, S)\}.$$

In the general case, write $P = aP_1 \cdots P_m$ where $P_i$ is a non-constant irreducible polynomial and $a > 0$ an integer. Denote degree and height of $P_i$ by $N_i$ and $H_i$ and put $S_i = N_i + \log H_i (i = 1, \cdots, m)$. Then clearly

$$|P_i(\sigma)| > \exp\{-3\phi(N_i, S_i)\} \quad (i = 1, \cdots, m).$$

By e.g. GEL'FOND's well-known inequality on the height of a product of polynomials (see [8] p. 135, Lemma II; see also [12], Lemma 3 and [10]) we have $H_i \leqq e^N H$ and thus,

$$S_i \leqq 2S \quad (i = 1, \cdots, m).$$

Using (ii) and (iii) it follows that

$$|P_i(\sigma)| > \exp\left\{-3N_i \frac{\phi(N_i, 2S)}{N_i}\right\} \geqq \exp\left\{-3N_i \frac{\phi(N, 2S)}{N}\right\}$$

for $i = 1, \cdots, m$. By multiplying these inequalities we obtain the required expression (23), since $N_1 + \cdots + N_m = N$ and $a \geqq 1$.

## 4. Proof of Theorem 1

First we prove

THEOREM 3: *Let $\alpha \neq 0$ be an algebraic number of size $s(\alpha)$. Then there exists an effectively computable number $S_1 = S_1(\alpha)$ such that*

$$(24) \qquad |e^\alpha - \xi| > \exp\{-5.10^8 e^{6s(\alpha)} N^2 S\}$$

*for all algebraic numbers $\xi$ of degree $N$ and size $S \geqq S_1$.*

PROOF: All estimates occurring in this proof hold for $S$ sufficiently large. $S_1$ can be chosen as the maximum of the finitely many bounds thus obtained. Suppose that

$$(25) \qquad |e^\alpha - \xi| \leqq \exp\{-5.10^8 e^{6s(\alpha)} N^2 S\}$$

for some algebraic number $\xi$ of degree $N$ and size $S$. It will be shown that this assumption leads to a contradiction if $S \geqq S_1$. Observe that (25) implies that $|\xi| < |e^\alpha| + 1$.

Choose the following positive integers:

$$K = [10^3 e^{2s(\alpha)} N] \qquad T = \left[3.10^4 e^{3s(\alpha)} \frac{NS}{\log S}\right]$$

$$M = \left[15.10^4 e^{4s(\alpha)} \frac{NS}{\log S}\right] \qquad P = [10^3 e^{2s(\alpha)} N]$$

$$C = 2[\tfrac{1}{2} \exp\{4.10^6 e^{4s(\alpha)} NS\}] \quad T' = \left[45.10^4 e^{4s(\alpha)} \frac{NS}{\log S}\right].$$

We use the auxiliary exponential polynomial

$$F(z) = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} \sum_{v=0}^{N-1} C_{kmv} \xi^v z^m e^{\alpha k z}$$

where the $C_{kmv}$ are integers of absolute values at most $C$. Later we shall specify them further.

For $t = 0, 1, 2, \cdots$ and $p = 0, 1, 2, \cdots$ we have

$$F^{(t)}(p) = \sum_k \sum_m \sum_v C_{kmv} \xi^v \sum_{\tau=0}^{m} \binom{t}{\tau} \frac{m!}{(m-\tau)!} p^{m-\tau} k^{t-\tau} \alpha^{t-\tau} (e^\alpha)^{kp}.$$

Define $\Phi_{tp}$ for the same $t$ and $p$ by

$$\Phi_{tp} = \sum_k \sum_m \sum_v C_{kmv} \xi^v \sum_{\tau=0}^{m} \binom{t}{\tau} \frac{m!}{(m-\tau)!} p^{m-\tau} k^{t-\tau} \alpha^{t-\tau} \xi^{kp}.$$

Then $\Phi_{tp}$ is an algebraic number, approximating $F^{(t)}(p)$ very closely. In fact,

$$|(e^\alpha)^{kp} - \xi^{kp}| \leqq KP(|e^\alpha|+1)^{KP-1} |e-\xi| \leqq \exp\{-4.5 \times 10^8 e^{6s(\alpha)} N^2 S\}$$

for $k = 0, 1, \cdots, K-1$ and $p = 0, 1, \cdots, P-1$. Hence,

$$(26) \quad \begin{aligned} &|F^{(t)}(p) - \Phi_{tp}| \\ &\leqq KMNC(|e^\alpha|+1)^N 2^{T'} M^M P^M K^{T'} \{\max(1, |\alpha|)\}^{T'} \\ &\times \exp\{-4.5 \times 10^8 e^{6s(\alpha)} N^2 S\} \\ &\leqq \exp\{-4.10^8 e^{6s(\alpha)} N^2 S\} \end{aligned}$$

for $t = 0, 1, \cdots, T'-1$ and $p = 0, 1, \cdots, P-1$.

We are going to choose the integers $C_{kmv}$ such that $\Phi_{tp} = 0$ for $t = 0, 1, \cdots, T-1$ and $p = 0, 1, \cdots, P-1$. To this end we apply Lemma 6 with $n = 1$, $\alpha_1 = \alpha$, $r = TP$ and $s = KM$ to the polynomials

$$P_{tpkm}(z_0, z_1) = \sum_{\tau=0}^{m} \binom{t}{\tau} \frac{m!}{(m-\tau)!} p^{m-\tau} k^{t-\tau} z_1^{t-\tau} z_0^{kp}$$

($t = 0, 1, \cdots, T-1$; $p = 0, 1, \cdots, P-1$; $k = 0, 1, \cdots, K-1$; $m = 0, 1, \cdots, M-1$). Using the notations of Lemma 6, we have

$d = d(\alpha) \leqq s(\alpha) \leqq e^{s(\alpha)}$, $N_0 + N \leqq (K-1)(P-1) + N \leqq KP$, $N_1 \leqq T$

and

$$B \leqq 2^T M^M P^M K^T \leqq \exp\{10^6 e^{4s(\alpha)} NS\}.$$

By means of these inequalities one easily verifies conditions (11) and (12) of Lemma 6. According to this lemma, we now choose the integers $C_{kmv}$, not all zero, with $|C_{kmv}| \leqq C$ such that

$$\Phi_{tp} = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} \sum_{v=0}^{N-1} C_{kmv} \xi^v P_{tpkm}(\xi, \alpha) = 0 \tag{27}$$

for $t = 0, 1, \cdots, T-1$ and $p = 0, 1, \cdots, P-1$. In this way, our function $F$ is completely fixed.

From (26) we obtain with (27)

$$|F^{(t)}(p)| \leqq \exp\{-4.10^8 e^{6s(\alpha)} N^2 S\} \tag{28}$$

for $t = 0, 1, \cdots, T-1$ and $p = 0, 1, \cdots, P-1$. From (1) it follows that $|\alpha| \leqq e^{s(\alpha)}$. Using this inequality we see from the definition of $F$ that

$$\max_{|z| \leqq 2PS} |F(z)| \leqq \exp\{7.10^6 e^{5s(\alpha)} N^2 S\}. \tag{29}$$

We apply Lemma 7 with $R = 2P$ and $A = S$. We obtain by (28) and (29)

$$\begin{aligned} \max_{|z|<2P} |F(z)| &\leqq \exp\{-2.1 \times 10^7 e^{5s(\alpha)} N^2 S\} \\ &\quad + \exp\{-3.10^8 e^{6s(\alpha)} N^2 S\} \leqq \exp\{-2.10^7 e^{5s(\alpha)} N^2 S\}. \end{aligned} \tag{30}$$

Hence, for $t = 0, 1, \cdots, T'-1$ and $p = 0, 1, \cdots, P-1$ we have

$$\begin{aligned} |F^{(t)}(p)| &= \left| \frac{t!}{2\pi i} \int_{|z|=2P} \frac{F(z)}{(z-p)^{t+1}} dz \right| \\ &\leqq T'^{T'} \cdot 2P \cdot \max_{|z| \leqq 2P} |F(z)| \leqq \exp\{-1.5 \times 10^7 e^{5s(\alpha)} N^2 S\} \end{aligned} \tag{31}$$

and for the same values of $t$ and $p$, using (26),

$$|\Phi_{tp}| \leqq \exp\{-10^7 e^{5s(\alpha)} N^2 S\}. \tag{32}$$

But $\Phi_{tp}$ is a polynomial with integral coefficients in $\xi$ and $\alpha$, of degree at most $(K-1)(P-1)+N-1 \leqq KP$ in $\xi$ and less than $T'$ in $\alpha$. The sum of the absolute values of its coefficients is not greater than

$$KMNC 2^{T'} M^M P^M K^{T'} \leqq \exp\{6.10^6 e^{4s(\alpha)} NS\}.$$

From Lemma 3 it follows that $\Phi_{tp} = 0$ or

$$|\Phi_{tp}| > \exp\{-10^7 e^{5s(\alpha)} N^2 S\} \tag{33}$$

for $t = 0, 1, \cdots, T'-1$ and $p = 0, 1, \cdots, P-1$. Since (32) and (33) are incompatible, it follows that $\Phi_{tp} = 0$ for $t = 0, 1, \cdots, T'-1$ and $p = 0, 1, \cdots, P-1$. From (26) we now obtain that

$$|F^{(t)}(p)| \leqq \exp\{-4.10^8 e^{6s(\alpha)} N^2 S\} \tag{34}$$

for $t = 0, 1, \cdots, T'-1$ and $p = 0, 1, \cdots, P-1$.

Subsequently we apply Lemma 8 to our exponential polynomial $F$, with

$$C_{km} = \sum_{\nu=0}^{N-1} C_{km\nu}\,\xi^{\nu},\ \omega_k = \alpha k$$

and $P' = P$. Using the notations of Lemma 8 we have, by $|\alpha| \leqq e^{s(\alpha)}$, the inequality

$$\Omega < 10^3\, e^{3s(\alpha)} N.$$

Thus,

$$T'P \geqq 4.10^8\, e^{6s(\alpha)} \frac{N^2 S}{\log S} \geqq 2KM + 13\Omega P$$

so that condition (19) is satisfied. Further, by Lemma 1 we have $|\alpha| \leqq e^{s(\alpha)}$ and $|\alpha| \geqq e^{-s(\alpha)}$. Hence

$$\frac{\Omega}{\omega} \leqq \frac{\max(1, K|\alpha|)}{\min(1, |\alpha|)} \leqq \frac{K\,e^{s(\alpha)}}{e^{-s(\alpha)}} = K\,e^{2s(\alpha)}.$$

Thus, it follows from (21) and (34) that

$$(35)\quad |\sum_{\nu=0}^{N-1} C_{km\nu}\,\xi^{\nu}| \leqq \exp\{3.5\times 10^8\, e^{6s(\alpha)} N^2 S\}$$
$$\times \exp\{-4.10^8\, e^{6s(\alpha)} N^2 S\} = \exp\{-5.10^7\, e^{6s(\alpha)} N^2 S\}$$

for $k = 0, 1, \cdots, K-1$ and $m = 0, 1, \cdots, M-1$.

But

$$\sum_{\nu=0}^{N-1} C_{km\nu}\,\xi^{\nu}$$

is a polynomial in $\xi$ of degree less than $N$ and with sum of the absolute values of its coefficients at most $NC$. It follows from Lemma 3 that

$$\sum_{\nu=0}^{N-1} C_{km\nu}\,\xi^{\nu} = 0$$

or

$$(36)\qquad |\sum_{\nu=0}^{N-1} C_{km\nu}\,\xi^{\nu}| > \exp\{-5.10^6\, e^{4s(\alpha)} N^2 S\}$$

for $k = 0, 1, \cdots, K-1$ and $m = 0, 1, \cdots, M-1$. Hence,

$$\sum_{\nu=0}^{N-1} C_{km\nu}\,\xi^{\nu} = 0$$

for all these $k$ and $m$. Since $\xi$ is algebraic of degree $N$, the numbers $1, \xi, \cdots, \xi^{N-1}$ are linearly independent over $\mathbf{Q}$. Thus, $C_{km\nu} = 0$ for $k = 0, 1, \cdots, K-1$, $m = 0, 1, \cdots, M-1$ and $\nu = 0, 1, \cdots, N-1$. This contradicts the choice of the integers $C_{km\nu}$ and by this contradiction Theorem 3 is proved.

We now complete the proof of Theorem 1 in the following way:

There are only finitely many algebraic numbers $\xi$ of size $S < S_1$. Since $e^{\alpha}-\xi \neq 0$ and since $N^2 S > 0$ for all of these numbers, there exists a number $C_6 > 0$ such that

$$|e^{\alpha}-\xi| > \exp\{-C_6 N^2 S\}$$

for all algebraic numbers $\xi$ of degree $N$ and size $S < S_1$. Then $C_7 = \max(C_6, 5.10^8 e^{6s(\alpha)})$ will have the property

$$|e^{\alpha}-\xi| > \exp\{-C_7 N^2 S\}$$

for all algebraic $\xi$ of degree $N$ and size $S$. From Lemma 9 we obtain that

$$|P(e^{\alpha})| > \exp\{-C_4 N^2 S\}$$

for all polynomials $P$ with integral coefficients, of degree $N$ and height $H$, with $S = N+\log H$ and $C_4 = 6C_7$. By reasons of monotony, the same inequality holds for polynomials of degree *at most* $N$ and height *at most* $H$. Hence, $\exp\{-C_4 N^2 S\}$ is a transcendence measure for $e^{\alpha}$ and Theorem 1 has been proved.

## 5. Proof of Theorem 2

Let $\log \alpha$ be an arbitrary but fixed value of the logarithm of the algebraic number $\alpha$. N. I. FEL'DMAN proved the following assertion: (see Theorem 1 of [4], with $m = 1$)

*There exists an effectively computable positive number $C_8 = C_8(\log \alpha)$, such that*

$$|\log \alpha-\xi| > \exp\{-C_8 N^2 \log H(\log(N+2))^2\} \tag{37}$$

*for all algebraic numbers $\xi$ of degree $N$ and height $H$, provided that $N < (\log H)^{\frac{1}{4}}$.*

From this, it easily follows that

$$|\log \alpha-\xi| > \exp\{-2C_8 N^2 S(1+\log N)^2\} \tag{38}$$

for all algebraic numbers $\xi$ of degree $N$ and with size $S$, provided that $N < (\frac{1}{2}S)^{\frac{1}{4}}$.

For the complementary case $N \geqq (\frac{1}{2}S)^{\frac{1}{4}}$ we prove the following theorem:

THEOREM 4: *Let $\alpha \neq 0, 1$ be an algebraic number of size $s(\alpha)$ and let* $\log \alpha$ *be an arbitrary, but fixed, value of the logarithm of $\alpha$. Then there exists an effectively computable number $S_2 = S_2(\log \alpha)$, such that*

$$|\log \alpha-\xi| > \exp\{-3.10^9(s(\alpha))^7 N^2 S\} \tag{39}$$

*for all algebraic numbers $\xi$ of degree $N$ and size $S \geqq S_2$, provided that $N \geqq (\frac{1}{2}S)^{\frac{1}{4}}$.*

PROOF: Since the structure of the proof is the same as that of Theorem 3, only a shorted proof is given. Suppose that

$$|\log \alpha - \xi| \leqq \exp\{-3.10^9(s(\alpha))^7 N^2 S\} \tag{40}$$

for some algebraic number $\xi$ of degree $N$ and size $S$, such that $N \geqq (\frac{1}{2}S)^{\frac{1}{4}}$. We shall derive a contradiction in the case of large $S$.

Choose the integers

$$K = [10^3(s(\alpha))^2 N] \qquad M = \left[3.10^5(s(\alpha))^5 \frac{NS}{\log S}\right]$$

$$C = 2[\tfrac{1}{2}\exp\{4.10^6(s(\alpha))^5 NS\}] \quad T = \left[7.10^4(s(\alpha))^4 \frac{N^2}{\log S}\right]$$

$$P = [10^3(s(\alpha))^2 S] \qquad T' = \left[7.10^5(s(\alpha))^5 \frac{N^2}{\log S}\right].$$

Put

$$F(z) = \sum_{k=0}^{K-1} \sum_{m=0}^{M-1} \sum_{v=0}^{N-1} C_{kmv} \xi^v z^m e^{k(\log \alpha) z},$$

where the numbers $C_{kmv}$ are integers of absolute values at most $C$; they will be specified later. We have for $t, p = 0, 1, 2, \cdots$

$$F^{(t)}(p) = \sum_k \sum_m \sum_v C_{kmv} \xi^v \sum_{\tau=0}^{m} \binom{t}{\tau} \frac{m!}{(m-\tau)!} p^{m-\tau} k^{t-\tau} (\log \alpha)^{t-\tau} \alpha^{kp}.$$

Put

$$\Phi_{tp} = \sum_k \sum_m \sum_v C_{kmv} \xi^v \sum_{\tau=0}^{m} \binom{t}{\tau} \frac{m!}{(m-\tau)!} p^{m-\tau} k^{t-\tau} \xi^{t-\tau} \alpha^{kp}.$$

For $t = 0, 1, \cdots, T'-1$ and $p = 0, 1, \cdots, P-1$ it easily follows from the inequality $|\alpha| \leqq e^{s(\alpha)}$ that

$$|F^{(t)}(p) - \Phi_{tp}| \leqq \exp\{-2.10^9(s(\alpha))^7 N^2 S\}. \tag{41}$$

Let $P_{tpkm}$ for $t = 0, 1, \cdots, T-1$, $p = 0, 1, \cdots, P-1$, $k = 0, 1, \cdots, K-1$ and $m = 0, 1, \cdots, M-1$ be the polynomials (taken in the obvious way) such that

$$\Phi_{tp} = \sum_k \sum_m \sum_v C_{kmv} \xi^v P_{tpkm}(\xi, \alpha).$$

We now choose the numbers $C_{kmv}$ according to Lemma 6, applied to the polynomials $P_{tpkm}$, as integers, not all zero, of absolute values at most $C$, such that $\Phi_{tp} = 0$ for $t = 0, 1, \cdots, T-1$ and $p = 0, 1, \cdots, P-1$. It follows from (41) that

(42) $$|F^{(t)}(p)| \leqq \exp\{-2.10^9(s(\alpha))^7 N^2 S\}$$

for t $= 0, 1, \cdots, T-1$ and $p = 0, 1, \cdots, P-1$.
Since

$$\max_{|z| \leqq 2PS^{1/5}} |F(z)| \leqq \exp\{3.10^6(s(\alpha))^4 |\log \alpha| NS^{6/5}\}$$

and since $N \geqq (\frac{1}{2}S)^{\frac{1}{4}}$ we see from Lemma 7 applied with $R = 2P$ and $A = S^{1/5}$ that

(43) $$\max_{|z| \leqq 2P} |F(z)| \leqq \exp\{-10^7(s(\alpha))^6 N^2 S\}.$$

Hence,

$$|F^{(t)}(p)| \leqq \exp\{-9.10^6(s(\alpha))^6 N^2 S\}$$

and

(44) $$|\Phi_{tp}| \leqq \exp\{-8.10^6(s(\alpha))^6 N^2 S\}$$

for $t = 0, 1, \cdots, T'-1$ and $p = 0, 1, \cdots, P-1$.

From Lemma 3 we obtain $\Phi_{tp} = 0$ or

(45) $$|\Phi_{tp}| \geqq \exp\{-7.10^6(s(\alpha))^6 N^2 S\}$$

for the same values of $t$ and $p$. Thus, $\Phi_{tp} = 0$ and, from (41),

(46) $$|F^{(t)}(p)| \leqq \exp\{-2.10^9(s(\alpha))^7 N^2 S\}$$

for $t = 0, 1, \cdots, T'-1$ and $p = 0, 1, \cdots, P-1$.

Condition (19) with $P' = P$ and $\Omega \leqq 10^3(s(\alpha))^2 |\log \alpha| N$ is satisfied. Further,

$$\frac{\Omega}{\omega} \leqq \frac{\max(1, K|\log \alpha|)}{\min(1, |\log \alpha|)} \leqq K \frac{\max(1, |\log \alpha|)}{\min(1, |\log \alpha|)}.$$

Using this inequality and (46) we obtain from (21)

(47) $$\left|\sum_{\nu=0}^{N-1} C_{km\nu} \xi^\nu\right| \leqq \exp\{-10^9(s(\alpha))^7 N^2 S\}$$

for $k = 0, 1, \cdots, K-1$ and $m = 0, 1, \cdots, M-1$.

From Lemma 3 it now follows that

$$\sum_{\nu=0}^{N-1} C_{km\nu} \xi^\nu = 0$$

for all $k$ and $m$. This implies that all integers $C_{km\nu}$ are zero. By the contradiction to the choice of these integers, Theorem 4 has been proved.

We proceed to prove Theorem 2. From (39) it follows that there exists an effectively computable number $C_9 = C_9(\log \alpha) > 0$ such that

$$|\log \alpha - \xi| > \exp\{-C_9 N^2 S\} \tag{48}$$

for all algebraic numbers $\xi$ of degree $N$ and size $S$ with $N \geqq (\frac{1}{2}S)^{\frac{1}{4}}$, since there are only finitely many algebraic numbers of size smaller than $S_2$. Taking $C_{10} = \max(2C_8, C_9)$ we see from (38) and (48) that

$$|\log \alpha - \xi| > \exp\{-C_{10} N^2 S(1+\log N)^2\}$$

for all algebraic numbers $\xi$ of degree $N$ and size $S$. As before, the application of Lemma 9 completes the proof.

## REFERENCES


[1] P. Bundschuh: Zu einem Transzendenzsatz der Herren Baron und Braune, To appear in *J.f.d. reine und angew. Math.*

[2] P. L. Cijsouw: *Transcendence measures.* (Thesis) University of Amsterdam, 1972.

[3] N. I. Fel'dman: Approximation of certain transcendental numbers. I. The approximation of logarithms of algebraic numbers. (Russian) *Izv. Akad. Nauk SSSR, ser. mat., 15* (1951) 53–74. English translation: *Amer. Math. Soc. Transl., II ser., 59* (1966) 224–245.

[4] N. I. Fel'dman: Approximation of the logarithms of algebraic numbers by algebraic numbers. (Russian) *Izv. Akad. Nauk SSSR, ser. mat., 24* (1960) 475–492. English translation: *Amer. Math. Soc. Transl., II ser., 58* (1966) 125–142.

[5] N. I. Fel'dman: On the problem of the measure of transcendence of e. (Russian) *Uspekhi Mat. Nauk 18* (3) (1963) 207–213.

[6] N. I. Fel'dman: Estimate for a linear form of logarithms of algebraic numbers. (Russian) *Mat. Sbornik (N.S.) 76 (118)*(1968) 304–319. English translation: *Math. USSR Sbornik 5* (1968) 291–307.

[7] A. I. Galočkin: On the diophantine approximation of values of the exponential function and of solutions of certain transcendental equations. (Russian) *Vestnik Moskov. Univ., Ser. I, 27* (3) (1972) 16–23.

[8] A. O. Gel'fond: Transcendental and algebraic numbers. (translated from the first Russian edition) Dover Publ., New York 1960.

[9] F. Lindemann, Ueber die Zahl $\pi$. *Math. Ann. 20* (1882) 213–225.

[10] K. Mahler: An application of Jensen's formula to polynomials. *Mathematika 7* (1960) 98–100.

[11] T. Schneider: Einführung in die Transzendenten Zahlen. Springer Verlag, Berlin 1957.

[12] R. Tijdeman: On the algebraic independence of certain numbers. *Nederl. Akad. Wet. Proc., ser A, 74* (1971) 146–162 = *Indag. Math. 33* (1971) 146–162.

[13] R. Tijdeman: An auxiliary result in the theory of transcendental numbers. *J. Number Th. 5* (1973) 80–94.

[14] M. Waldschmidt: Un premier cours sur les nombres transcendants. To appear in the Lecture Notes in Mathematics, Springer Verlag, Berlin.


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