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Transcendence measures of certain numbers whose transcendency was proved by A. Baker


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TRANSCENDENCE MEASURES OF CERTAIN NUMBERS WHOSE TRANSCENDENCY WAS PROVED BY A. BAKER

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1. Introduction

In the subsequent paper we continue the investigation of transcendence measures of certain transcendental numbers $\sigma$, i.e. positive lower bounds for $|P(\sigma)|$ in terms of the degree $N$ and height $H$ of $P$, where $P$ is an arbitrary polynomial with integral coefficients. For more information about transcendence measures and the type of transcendence measures we will look for, see the earlier paper [4]; see also the authors thesis [3], which includes the results of the present paper.

Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers such that, for any (fixed) values of the logarithms, $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$. In this paper, transcendence measures are derived for numbers which can be written in one of the following ways:

(i) $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$ with $n \geq 2$, where $\beta_1, \ldots, \beta_n$ are algebraic numbers, not all zero

(ii) $\beta_0 \alpha_1^\beta_1 \cdots \alpha_n^\beta_n$ with $n \geq 1$ and

$$\alpha_i^{\beta_i} = \epsilon^{\beta_i \log \alpha_i}$$

where $\beta_0, \beta_1, \ldots, \beta_n$ are algebraic numbers such that at least one of $\beta_1, \ldots, \beta_n$ is irrational. We prove the transcendence measure

$$\exp \left\{ -C_1 N^{n^2 + n + \varepsilon} S (1 + \log S)^{n+1 + \varepsilon} \right\}$$

for numbers of the form (1), and the transcendence measure

$$\exp \left\{ -C_2 N^{n^2 + 2n + 2 + \varepsilon} S^{n+1 + \varepsilon} \right\}$$

for numbers of the form (2). Here $S = N + \log H$, $\varepsilon$ is an arbitrary positive number and $C_1$ and $C_2$ are effectively computable numbers, depending on $\varepsilon, n$, the $\alpha$’s and their logarithms and the $\beta$’s.

We remark, that these transcendence measures are the first explicit ones to be published for these numbers in which both the dependence on $H$ and $N$ is expressed. If one is interested merely in the height $H$, better results can be given. For numbers of the form (i), N. I. Fel’’dman [6] proved the transcendence measure $\exp \{ -CS \}$, where $C$ is a positive
number, depending on $N$, the $\alpha$'s and their logarithms and the $\beta$'s. For numbers of the form (ii), a recent result of A. Baker [2] implies the transcendence measure $\exp \{-C \log H \log \log H\}$ for $H \geq 4$, where $C$ again is a positive number depending on $N$, the $\alpha$'s and their logarithms and the $\beta$'s. An earlier result for the special case of numbers of the form $e^{\beta_0}x_1$ can be found in [7].

The method of proof of our transcendence measures is A. Baker's one with some improvements introduced by N. I. Fel'dman. In the proof we firstly derive a measure for the approximability of numbers of the types (i) and (ii); after that, the transcendence measures are given.

The transcendence of numbers of the form (i) follows immediately from e.g. Theorem 1 of A. Baker's paper [1]. The transcendence of numbers of the form (ii) was proved by A. Baker, distinguishing the cases $\beta_0 = 0$ and $\beta_0 \neq 0$; see the same paper.

2. Lemmas

We shall use the same notations (especially for the degree; height and size) as in [4]. For shortness, we use without reformulation the lemmas 3, 6, 7, 8 and 9 of [4]. Further, we need the following lemmas:

**Lemma 1:** Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers such that, for any fixed values of the logarithms, $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$. Let $\varepsilon$ be positive and let $d$ be a positive integer. Then there exists an effectively computable positive number

$$\theta(\varepsilon, d) = \theta(\varepsilon, d, \log \alpha_1, \ldots, \log \alpha_n)$$

such that

(1) $|\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n| > \theta(\varepsilon, d) \exp \{-(\log h)^{1+\varepsilon}\}$

for all algebraic numbers $\beta_0, \beta_1, \ldots, \beta_n$, not all zero, of degrees at most $d$ and of heights at most $h$.

**Proof:** See Theorem 1 of [5].

**Lemma 2:** Let $\alpha$ and $\beta$ be algebraic numbers, $\beta \neq 0$. Then

(2) $s(\alpha \beta^{-1}) \leq 2d(\alpha)d(\beta) + d(\alpha)s(\beta) + d(\beta)s(\alpha)$.

**Proof:** From Lemma 3 of [5] it follows that

$$h(\alpha \beta^{-1}) \leq 2^{d(\alpha)d(\beta)}(h(\alpha)(d(\alpha) + 1))^{d(\beta)}(h(\beta)(d(\beta) + 1))^{d(\alpha)}.$$ 

Using $d(\alpha) + 1 < e^{d(\alpha)}$ and $d(\beta) + 1 < e^{d(\beta)}$ we get

$$s(\alpha \beta^{-1}) = d(\alpha \beta^{-1}) + \log h(\alpha \beta^{-1})$$

$$\leq d(\alpha)d(\beta)(1 + \log 2) + d(\beta)s(\alpha) + d(\alpha)s(\beta)$$

from which (2) follows.
LEMMA 3: Let $\beta$ be an algebraic number, and let $k$ and $\ell$ be integers. Then

$$h(k+\ell \beta) \leq h(\beta)|2k\ell|^{d(\beta)}.$$  

PROOF: If $a_n z^n + \cdots + a_1 z + a_0$ is the minimal polynomial of $\beta$, then

$$a_n (z-k)^n + \cdots + a_1 \ell^{n-1} (z-k) + a_0 \ell^n$$

is a constant multiple of the minimal polynomial of $k+\ell \beta$. Thus, the coefficient of $z^i$ ($i = 0, 1, \cdots, n$) of this minimal polynomial is in absolute value at most

$$|a_n \binom{n}{i} (-k)^{n-i} + a_{n-1} \ell \binom{n-1}{i} (-k)^{n-1-i} + \cdots + a_1 \ell^{n-i}|$$

$$ \leq h(\beta)|k|^n |\ell|^n \left( \binom{n}{i} + \binom{n-1}{i} + \cdots + 1 \right).$$

From the obvious inequality $\binom{n}{i} \leq 2^{n-1}$ for all positive integers $m$, it follows that

$$\left( \binom{n}{i} + \binom{n-1}{i} + \cdots + 1 \right) \leq 2^n,$$

by which the proof is completed.

3. The case $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$

First we give a measure for the approximability for numbers of the form (i), in the special case in which $\beta_n = -1$.

THEOREM 1: Let, for $n \geq 2$, $\alpha_1, \cdots, \alpha_n$ and $\gamma_1, \cdots, \gamma_{n-1}$ be non-zero algebraic numbers such that, for any fixed values of the logarithms, $\log \alpha_1, \cdots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$. Let $\varepsilon$ be a positive number. Then there exists an effectively computable number $S_1 = S_1(\varepsilon, \log \alpha_1, \cdots, \log \alpha_n, \gamma_1, \cdots, \gamma_{n-1})$ such that

$$|\gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta|$$

$$ \geq \exp \left\{-N^{n+2} + S(\log S)^{n+1+\varepsilon}\right\}$$

for all algebraic numbers $\eta$ of degree $N$ and size $S \geq S_1$.

PROOF: Put $\delta = (2n^3 + 4n^2 + 3n + 7)^{-1} \varepsilon$. For abbreviation, put

$$\sigma = \gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n$$

and

$$U = N^{n^2 + n + (2n^3 + 4n^2 + 2n + 1)^{\delta}} S(\log S)^{n+1+(2n^3 + 3n + 7)^{\delta}}.$$
if $S \geq S_1$. In this proof we may restrict ourselves to the case in which
$\delta$ is rather small. By $c_1$, $c_2$, $\cdots$ we denote positive numbers which
depend only on $n$, $\log \alpha_1$, $\cdots$, $\log \alpha_n$, $\beta_1$, $\cdots$, $\gamma_{n-1}$.

Suppose that

$$|\sigma - \eta| > \exp \{ -U \}$$

(5) $$|\sigma - \eta| \leq \exp \{ -U \}$$

for some algebraic number $\eta$ of degree $N$ and size $S$. We prove that this
leads to a contradiction if $S$ is sufficiently large.

Choose the following integers:

$$K = \left[ N^{n^2-1} \log S \right]^{1/(2n+1)},$$

$$M = \left[ N^{n^2+n} \log S \right]^{1/(2n+3)},$$

$$C = 2 \left[ \frac{1}{2} \exp \left\{ N^{n^2+n} \log S \right\} \right],$$

$$T = \left[ N^{n^2+n} (\log S)^{1/(2n+3)} \right],$$

$$P = \left[ S (\log S)^{1+2\beta} \right],$$

$$R = \left[ \frac{n-1}{\delta} + 2n^2 + n \right],$$

$$T' = \left[ 2^{-R} T \right]$$

and

$$P' = \left[ N^{n^2-1} (n^3-n) S (\log S)^{n+2n+1} \right].$$

Put

$$F(z) = \sum_{k_1=0}^{K-1} \cdots \sum_{k_n=0}^{M-1} \sum_{m=0}^{N-1} \sum_{\nu=0}^{n-1} C_{k_1 \cdots k_n \nu} \eta^\nu z^m$$

$$\times \exp \{ -k_n \eta z + \sum_{i=1}^{n-1} (k_i + k_n \gamma_i) (\log \alpha_i) z \},$$

where the numbers $C_{k_1 \cdots k_n \nu}$ are integers of absolute values at most $C$;
they will be specified later.

For $t = 0, 1, 2, \cdots$ it is easily seen that

$$f^{(t)}(z) = \sum_{\tau + \tau_1 + \cdots + \tau_{n-1} = t} \frac{t!}{\tau! \tau_1! \cdots \tau_{n-1}!}$$

$$\times (\log \alpha_1)^{\tau_1} \cdots (\log \alpha_{n-1})^{\tau_{n-1}} F_{\tau_1 \cdots \tau_{n-1}}(z)$$

where

$$F_{\tau_1 \cdots \tau_{n-1}}(z) = \sum_{k_1} \cdots \sum_{k_n} \sum_{m} \sum_{\nu} C_{k_1 \cdots k_n \nu} \eta^\nu$$

$$\times \sum_{\kappa=0}^{m} \left( \frac{m!}{(m-\kappa)!} \right) z^{m-\kappa} (-k_n)^{\tau-\kappa} \eta^{\tau-\kappa}$$

$$\times \prod_{i=1}^{n-1} (k_i + k_n \gamma_i)^{\tau_i}$$

$$\times \exp \{ \sum_{i=1}^{n} k_i \log \alpha_i + k_n (\sigma - \eta) z \}.$$
Put
\[
\Phi_{\tau_1 \cdots \tau_{n-1}}(z) = \sum_{k_1} \cdots \sum_{k_n} \sum_{m} \sum_{v} C_{k_1 \cdots k_m v} \eta^v \times \frac{m!}{(m-\kappa)!} \frac{(\kappa)!}{m^{\kappa}} (-k_n)^{y-\kappa} \eta^{\kappa} \prod_{i=1}^{n-1} (k_i + k_n \gamma_i)^{\nu_i} \times \exp \left\{ \left( \sum_{i=1}^{n} k_i \log \alpha_i \right) z \right\}.
\]

We estimate the difference
\[
|F_{\tau_1 \cdots \tau_{n-1}}(p) - \Phi_{\tau_1 \cdots \tau_{n-1}}(p)|
\]
as follows: by \(|e^z - 1| \leq |z|e^{|z|}\), one has for \(k_n = 0, 1, \cdots, K-1\) and
\[
p = 0, 1, \cdots, [N^{n-1} + (2n^3 + 3n^2 + n)\delta] (\log S)^{n-1} + (2n^2 + n + 2)\delta]
\]
the inequality
\[
|e^{\delta(p-n)} - e^{\delta(p-1)}| \leq \exp \left\{ -\frac{1}{2} U \right\}.
\]

Hence,
\[
|F_{\tau_1 \cdots \tau_{n-1}}(p) - \Phi_{\tau_1 \cdots \tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{2} U \right\}
\]
for \(\tau, \tau_1, \cdots, \tau_{n-1} = 0, 1, \cdots, T-1\) and
\[
p = 0, 1, \cdots, [N^{n-1} + (2n^3 + 3n^2 + n)\delta] (\log S)^{n-1} + (2n^2 + n + 2)\delta].
\]

Let
\[
P_{\tau_1 \cdots \tau_{n-1} p k_1 \cdots k_m}(z_0, z_1, \cdots, z_{2n-1})
\]
\((\tau, \tau_1, \cdots, \tau_{n-1} = 0, 1, \cdots, T-1; p = 0, 1, \cdots, P-1; k_1, \cdots, k_n = 0, 1, \cdots, K-1; m = 0, 1, \cdots, M-1)\) be the polynomials, chosen in the appropriate way, such that
\[
\Phi_{\tau_1 \cdots \tau_{n-1}}(p) = \sum_{k_1} \cdots \sum_{k_n} \sum_{m} \sum_{v} C_{k_1 \cdots k_m v} \eta^v \times P_{\tau_1 \cdots \tau_{n-1} p k_1 \cdots k_m}(\eta, \alpha_1, \cdots, \alpha_n, \gamma_1, \cdots, \gamma_{n-1}).
\]

We apply Lemma 6 of [4] to these polynomials in the specified points. If \(r, s\) and \(B\) have the same meaning as in this lemma we have
\[
r = T^n p \leq N^{n^2 + n + (n^3 + n^2)\delta} (\log S)^{n+1} + (2n^2 + 3n + 2)\delta,
\]
\[
s = K^n M \leq \frac{1}{2} N^{n^2 + n + (n^3 + n^2)\delta} (\log S)^{n+1} + (2n^2 + 3n + 3)\delta.
\]

Hence, \(s \geq 4rd\) where \(d = [Q(\alpha_1, \cdots, \alpha_n, \gamma_1, \cdots, \gamma_{n-1}) : Q]\). Further,
\[
B \leq \exp \left\{ c_1 N^{n^2 + n + (n^3 + n^2)\delta} (\log S)^{2 + (n^2 + 3)\delta} \right\}.\]
Hence, the right hand side of the second condition in Lemma 6 of [4] is at most
\[ \exp \{ c_2 N^{n+1} + (n^2+n)\delta S(\log S)^{2+(2n+3)\delta} \} \leq C^n. \]

It follows that the numbers \( C_{k_1\cdots k_{mv}} \) can be chosen as integers, not all zero, of absolute values at most \( C \), such that \( \Phi_{\tau_1\cdots\tau_{n-1}}(p) = 0 \) for \( \tau, \tau_1, \ldots, \tau_{n-1} = 0, 1, \ldots, T-1 \) and \( p = 0, 1, \ldots, P-1 \). Doing so, we certainly have \( \Phi_{\tau_1\cdots\tau_{n-1}}(p) = 0 \) for \( 0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T-1 \) and \( 0 \leq p \leq P-1 \). From (8) we now get
\[ |F_{\tau_1\cdots\tau_{n-1}}(p)| \leq \exp \{ -\frac{1}{2} U \} \]
for \( 0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T-1 \) and \( 0 \leq p \leq P-1 \).

Define \( T_r \) and \( P_r \) for \( r = 0, 1, \ldots, R \) by
\[ T_r = [2^{-r}T] \]
and
\[ P_r = [(N^{n+1} \log S)^{\delta r}] \]

Remark that
\[ (10) \quad P_R \leq [N^{n^2-1} + (2n^3+3n^2+n)\delta S(\log S)^{n+(2n^2+n+2)\delta}] \]

**Lemma:** For \( r = 0, 1, \ldots, R \) the inequality
\[ (11) \quad |F_{\tau_1\cdots\tau_{n-1}}(p)| \leq \exp \{ -\frac{1}{2} U \} \]
holds for all non-negative integers \( \tau, \tau_1, \ldots, \tau_{n-1} \) and \( p \) with
\( 0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T_r - 1 \) and \( 0 \leq p \leq P_r - 1 \).

**Proof:** We proceed by induction on \( r \). For \( r = 0 \) the statement is proved in (9). Let \( r \) be an integer with \( 0 \leq r \leq R-1 \) for which
\[ (12) \quad |F_{\tau_1\cdots\tau_{n-1}}(p)| \leq \exp \{ -\frac{1}{2} U \} \]
for \( 0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T_r - 1 \) and \( 0 \leq p \leq P_r - 1 \). Since
\[ F_{\tau_1\cdots\tau_{n-1}}(z) = \sum_{k_1} \cdots \sum_{k_{mv}} \sum_{r} C_{k_1\cdots k_{mv}} \eta^{-\sum_{i=1}^{n-1} (\log \alpha_i)^{-r}} \times (z^m e^{-k_n z})^{(r)} \prod_{i=1}^{n-1} (\exp \{(k_i + k_n \gamma_i)(\log \alpha_i)z\})^{(n_i)} \]
we have for \( t = 0, 1, 2, \cdots \)
\[ F_{\tau_1\cdots\tau_{n-1}}^{(t)}(z) = \sum_{\mu_1 + \cdots + \mu_{n-1} = t} \frac{t!}{\mu_1! \cdots \mu_{n-1}!} \times \prod_{i=1}^{n-1} (\log \alpha_i)^{\mu_i} \times F_{\tau + \mu_1, \tau_1 + \mu_1, \cdots, \tau_{n-1} + \mu_{n-1}}(z). \]
Together with (12) we obtain

$$|F_{\tau_1 \cdots \tau_{n-1}}(p)| \leq \exp\{-\frac{1}{4}U\}$$

for

$$0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} = T_{r+1} - 1, \quad 0 \leq t \leq T_{r+1} - 1 \quad \text{and} \quad 0 \leq p \leq P_r - 1.$$

From (7) we know that

$$\max_{|z| \leq R} |F_{\tau_1 \cdots \tau_{n-1}}(z)| \leq \exp\{c_3 N^{n+a_2+n+r+r}\delta S(\log S)^2 + (2n+r+4)^6\}$$

for

$$\tau + \tau_1 + \cdots + \tau_{n-1} \leq T_{r+1} - 1.$$ 

We apply Lemma 7 of \cite{4} to \(F_{\tau_1 \cdots \tau_{n-1}}\) with \(R = P_{r+1}, \ A = 6, \ T = T_{r+1} \) and \(P = P_r\). From (13), (14) and the inequality

$$N^{n+1} \log S \leq \exp\{N^\delta(\log S)^\delta\}$$

we then obtain

$$\max_{|z| \leq P_{r+1}} |F_{\tau_1 \cdots \tau_{n-1}}(z)| \leq \exp\{-2^{-r+3}(n+1+a_2+n+r+r)\delta S(\log S)^2 + (2n+r+5)^6\}.$$ 

In particular,

$$\max_{|z| \leq P_{r+1}} |F_{\tau_1 \cdots \tau_{n-1}}(p)| \leq \exp\{-2^{-r+3}(n+1+a_2+n+r+r)\delta S(\log S)^2 + (2n+r+5)^6\}$$

for

$$0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T_{r+1} - 1 \quad \text{and} \quad 0 \leq p \leq P_{r+1} - 1.$$

From (8) and (10) it follows that

$$|\Phi_{\tau_1 \cdots \tau_{n-1}}(p)| \leq \exp\{-2^{-r+4}(n+1+a_2+n+r+r)\delta S(\log S)^2 + (2n+r+5)^6\}$$

for

$$0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T_{r+1} - 1 \quad \text{and} \quad 0 \leq p \leq P_{r+1} - 1.$$ 

But for these values of \(\tau, \tau_1, \cdots, \tau_{n-1}\) and \(p\), we can consider \(\Phi_{\tau_1 \cdots \tau_{n-1}}(p)\) as a polynomial in \(\eta, \alpha_1, \cdots, \alpha_n, \gamma_1, \cdots, \gamma_{n-1}\), of degree less than \(T_{r+1} + N\) in \(\eta, KP_{r+1}\) in \(\alpha_1, \cdots, \alpha_n\) and \(T_{r+1}\) in \(\gamma_1, \cdots, \gamma_{n-1}\). If \(B\) denotes the sum of the absolute values of the coefficients, then we have

$$B \leq \exp\{2N^{n+(n+1)+a_2+n+r+r}\delta S(\log S)^2 + (2n+r+4)^6\}.$$ 

According to Lemma 3 of \cite{4} we thus have either \(\Phi_{\tau_1 \cdots \tau_{n-1}}(p) = 0\) or

$$|\Phi_{\tau_1 \cdots \tau_{n-1}}(p)| \geq \exp\{-c_4 N^{n+1+a_2+n+r+r}\delta S(\log S)^2 + (2n+r+4)^6\}$$

for

$$0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T_{r+1} - 1 \quad \text{and} \quad 0 \leq p \leq P_{r+1} - 1.$$ 

Hence, \(\Phi_{\tau_1 \cdots \tau_{n-1}}(p) = 0\) for these \(\tau, \tau_1, \cdots, \tau_{n-1}\) and \(p\). Again from (8) and (10) we obtain
\[ |F_{\tau_1 \cdots \tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\} \]

for \( 0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T_r + 1 \) and \( 0 \leq p \leq P_r + 1 \). The lemma has been proved.

From (11) with \( r = R \) we get

\[ |F_{\tau_1 \cdots \tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\} \]

for \( 0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T_R - 1 \) and \( 0 \leq p \leq P_R - 1 \). From their definitions we have \( T_R = T' \). Since

\[ R \geq n - 1 \delta + 2n^2 + n - 1 \]

we see that

\[ P_R \geq \frac{1}{2} N^{n^2 - 1 + 2n^3 + 3n^2 - 1} S (\log S)^{n + (2n^2 + n + 1)\delta} \geq P'. \]

Hence,

\[ |F_{t_1, \cdots, t_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\} \]

for \( 0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T' - 1 \) and \( 0 \leq p \leq P' - 1 \). From (6) it now follows that

\[ |F^{(t)}(p)| \leq \exp \left\{ -\frac{1}{4}U \right\} \]

for \( t = 0, 1, \cdots, T' - 1 \) and \( p = 0, 1, \cdots, P' - 1 \).

We apply Lemma 8 of [4] to \( F \) with \( K \) replaced by \( K^\alpha \). Let \( \Omega \) and \( \omega \) have the same meaning as in this lemma. Since the exponents of \( F \) have the form

\[ k_1 \log \alpha_1 + \cdots + k_n \log \alpha_n + k_n(\sigma - \eta), \]

we see that

\[ \Omega \leq c_5 N^{n + (n^2 - 1)\delta} (\log S)^{1 + (2n + 1)\delta}. \]

With this, the condition

\[ T'P' \geq 2K^\alpha M + 13\Omega P' \]

is easily checked. Further, we know from Lemma 1, applied with \( \varepsilon = \delta \), that

\[ |k_1 \log \alpha_1 + \cdots + k_n \log \alpha_n| \geq \exp \left\{ -(\log K)^{1 + 2\delta} \right\} \]

\[ \geq \exp \left\{ -N^\delta (\log S)^\delta \right\} \]

for all integers \( k_1, \cdots, k_n \), not all zero, with \( |k_1| \leq K - 1, \cdots, |k_n| \leq K - 1 \). From (21) and (5) it follows that

\[ \omega \geq \exp \left\{ -N^\delta (\log S)^\delta \right\} - \exp \left\{ -\frac{1}{2}U \right\} \geq \exp \left\{ -2N^\delta (\log S)^\delta \right\}. \]
From (20) we have

\[ \Omega \leq \exp \{ N^3 \log S \} \].

From lemma 8 of [4], with (19), (22) and (23) we obtain

\[
\begin{align*}
\sum_{v=0}^{N-1} C_{k_1 \ldots k_{nv}} \eta^v & \leq \exp \left\{ c_6 N^{n^2+n+(n^2+n^2+1)\delta} (\log S)^{n+1+(2n^2+3n+4)\delta} - \frac{1}{2} U \right\} \\
& \leq \exp \left\{ -\frac{1}{2} U \right\}
\end{align*}
\]

for \( k_1, \cdots, k_n = 0, 1, \cdots, K-1 \) and \( m = 0, 1, \cdots, M-1 \).

But according to Lemma 3 of [4] we have either

\[
\sum_{v=0}^{N-1} C_{k_1 \ldots k_{nv}} \eta^v = 0
\]

or

\[
\sum_{v=0}^{N-1} C_{k_1 \ldots k_{nv}} \eta^v \geq \exp \left\{ -2N^{n+1+(n^2+n)\delta} (\log S)^{2+(2n^2+4)\delta} \right\}
\]

for the same values of \( k_1, \cdots, k_n \) and \( m \). Hence,

\[
\sum_{v=0}^{N-1} C_{k_1 \ldots k_{nv}} \eta^v = 0 \quad \text{for} \quad k_1, \cdots, k_n = 0, 1, \cdots, K-1
\]

and \( m = 0, 1, \cdots, M-1 \). Since \( \eta \) has the degree \( N \), it follows that all integers \( C_{k_1 \ldots k_{nv}} \) are zero, in contradiction to their choice. This contradiction proves Theorem 1.

We have the following

**COROLLARY:** Under the conditions of Theorem 1, there exists an effectively computable, number \( C_3 = C_3 (\epsilon, \log \alpha_1, \cdots, \log \alpha_n, \gamma_1, \cdots, \gamma_{n-1}) > 0 \) such that

\[
|\gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta| > \exp \left\{ -C_3 N^{n^2+n+\epsilon} (1+\log S)^{n+1+\epsilon} \right\}
\]

for all algebraic numbers \( \eta \) of degree at most \( N \) and size at most \( S \).

**PROOF:** There are only finitely many algebraic numbers \( \eta \) of size \( s(\eta) < S_1 \). Choose \( C_3 \geq 1 \) such that (26) holds for these finitely many numbers.

**THEOREM 2:** Let, for \( n \geq 2, \alpha_1, \cdots, \alpha_n \) and \( \beta_1, \cdots, \beta_n \) be non-zero algebraic numbers such that, for any fixed values of the logarithms, \( \log \alpha_1, \cdots, \log \alpha_n \) are linearly independent over \( \mathbb{Q} \). Let \( \epsilon \) be a positive number. Then there exists an effectively computable positive number \( C_4 = C_4 (\epsilon, \log \alpha_1, \cdots, \log \alpha_n, \beta_1, \cdots, \beta_n) \) such that
for all algebraic numbers $\xi$ of degree $N$ and size $S$.

**Proof:** Put $\gamma_i = -\beta_i/\beta_n$ ($i = 1, \cdots, n-1$) and $\eta = -\xi/\beta_n$. Then

$$(-1/\beta_n)(\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi)$$

$$= \gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta.$$  

We have $d(\eta) \leq c_7 N$ with $c_7 = d(\beta_n)$ and, by Lemma 2, $s(\eta) \leq c_8 S$ with $c_8 = 3d(\beta_n) + s(\beta_n)$. From (26) we now obtain

$$|\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi|$$

$$= |\beta_n| \gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta| >$$

$$|\beta_n| \exp \{-C_3(c_1 N)^{n^2+n+\varepsilon} c_2 S(1 + \log S)^{n+1+\varepsilon}\}$$

$$\geq \exp \{-C_4 N^{n^2+n+\varepsilon} S(1 + \log S)^{n+1+\varepsilon}\}$$

for some effectively computable positive number $C_4$.

**Theorem 3:** Under the conditions of Theorem 2, there exists an effectively computable number $C_1 = C_1(\varepsilon, \log \alpha_1, \cdots, \log \alpha_n, \beta_1, \cdots, \beta_n) > 0$, such that

$$\exp \{-C_1 N^{n^2+n+\varepsilon} S(1 + \log S)^{n+1+\varepsilon}\}$$

is a transcendence measure of $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$.

**Proof:** Apply Lemma 9 of [4] to the result of Theorem 2 and put $C_1 = 6C_4(1 + \log 2)^{n+1+\varepsilon}$.

**4. The case $\varepsilon \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$**

**Theorem 4:** Let $n$ be a positive integer. Let $\alpha_0$ be algebraic and let $\alpha_1, \cdots, \alpha_n$ be non-zero algebraic numbers such that, for any fixed values of the logarithms, $\log \alpha_1, \cdots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$. Let $\beta_1, \cdots, \beta_n$ be algebraic numbers, not all rational. Put

$$\alpha_i^{\beta_i} = e^{\beta_i \log \alpha_i} \text{ for } i = 1, \cdots, n.$$  

Let $\varepsilon$ be a positive number. Then there exists an effectively computable number $S_2 = S_2(\varepsilon, \log \alpha_1, \cdots, \log \alpha_n, \beta_0, \beta_1, \cdots, \beta_n) > 0$ such that

$$|e^{\varepsilon \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}} - \xi| > \exp \{-N^{n^2+2n+2+\varepsilon} S^{n+1+\varepsilon}\}$$

for all algebraic $\xi$ of degree $N$ and size $S \geq S_2$.

**Proof:** Put $\delta = (2n^3 + 8n^2 + 10n + 4)^{-1}\varepsilon$. For the sake of brevity, put
\[ \sigma = e^{\beta_0 \beta_1 \cdots \beta_n} \]

and

\[ U = N^{n^2 + 1 + (2n^3 + 8n^2 + 10n + 4)\delta S^{n + 1 + (2n^2 + 7n + 11)\delta}}. \]

It is sufficient to prove that

\[ |\sigma - \xi| > \exp \{ -U \} \]

if \( S \geq S_2 \); in this proof we may assume that \( \delta \) is rather small. By \( c_1, c_2, \cdots \) we shall denote positive numbers which depend on \( n, \log \alpha_1, \cdots, \log \alpha_n, \beta_0, \beta_1, \cdots, \beta_n \) only.

Suppose that

\[ (29) \quad |\sigma - \xi| \leq \exp \{ -U \} \]

for some algebraic number \( \xi \) of degree \( N \) and size \( S \). We prove that this is impossible if \( S \) is sufficiently large.

Choose the following integers:

\[
\begin{align*}
K &= \left[ N^{n^2 + n} S^{1 + (2n + 3)\delta} \right], \\
L &= \left[ N^{n^2 + 1 + (n^2 + n)\delta} S^{(2n + 3)\delta} \right], \\
M &= \left[ N^{n^2 + 1 + (n^2 + 2n + 1)\delta} S^{1 + (2n + 5)\delta} \right], \\
C &= 2^{\left[ \frac{1}{2} \exp \left\{ N^{n^2 + 1 + (n^2 + 2n + 1)\delta} S^{1 + (2n + 6)\delta} \right\} \right]}, \\
T &= \left[ N^{n^2 + 1 + (n^2 + 2n + 1)\delta} S^{1 + (2n + 5)\delta} \right] (= M), \\
P &= \left[ NS^{2\delta} \right], \\
R &= \left[ \frac{n}{\delta} + 2n^2 + 5n + 3 \right], \\
T' &= \left[ 2^{-RT} \right], \text{ and} \\
P' &= \left[ \frac{1}{2} N^{n^2 + n + 1 + (n^2 + 2n + n)\delta} S^{n + (2n^2 + 5n + 4)\delta} \right].
\end{align*}
\]

Put

\[
F(z) = \sum_{k_1=0}^{K-1} \cdots \sum_{k_n=0}^{K-1} \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} C_{k_1 \cdots k_n l m v} z^v z^m \\
\times \exp \left\{ \ell \beta_0 z + \sum_{i=1}^{n} (k_i + \ell \beta_i)(\log \alpha_i)z \right\},
\]

where the numbers \( C_{k_1 \cdots k_n l m v} \) are integers of absolute values at most \( C \); they will be specified later.

For \( t = 0, 1, 2, \cdots \) we have

\[ (30) \quad F^{(t)}(z) = \sum_{\tau + \tau_1 + \cdots + \tau_n = t} \frac{t!}{\tau! \tau_1! \cdots \tau_n!} \prod_{i=1}^{n} (\log \alpha_i)^{\nu_i} F_{\tau_1 \cdots \tau_n}(z) \]

where
Define \( \Phi_{\tau_1 \cdots \tau_n} \) by

\[
\Phi_{\tau_1 \cdots \tau_n}(z) = \sum_{k_1} \cdots \sum_{k_n} \sum_{l} \sum_{m} \sum_{v} C_{k_1 \cdots k_n l m v} \xi^v \sum_{k} \frac{m!}{(m-k)!} \\
\times z^{m-k} \ell_{-k} \beta_0^{r-k} \prod_{i=1}^{n} (k_i + \ell \beta_i) t_i \exp \left\{ \sum_{i=1}^{n} k_i (\log \alpha_i) z \right\} \xi^z.
\]

For \( \ell = 0, 1, \cdots, L-1 \) and

\[
p = 0, 1, \cdots, \left[ Nn^2 + n + 1 + (2n^3 + 7n^2 + 8n + 3) \delta \right] S n + (2n^2 + 5n + 5) \delta
\]

one has

\[
|\sigma^p - \xi^p| \leq lp |\sigma - \xi| (|\sigma| + 1)^p \leq \exp \left\{ -\frac{1}{2} U \right\}.
\]

Hence,

\[
(F_{\tau_1 \cdots \tau_n}(p) - \Phi_{\tau_1 \cdots \tau_n}(p)) \leq \exp \left\{ -\frac{1}{2} U \right\}
\]

for \( \tau, \tau_1, \cdots, \tau_n = 0, 1, \cdots, T-1 \) and

\[
p = 0, 1, \cdots, \left[ Nn^2 + n + 1 + (2n^3 + 7n^2 + 8n + 3) \delta \right] S n + (2n^2 + 5n + 5) \delta
\]

We apply Lemma 6 of [4] to the polynomials

\[
P_{\tau_1 \cdots \tau_n p k_1 \cdots k_n l m}(r_1, \tau_1, \cdots, \tau_n = 0, 1, \cdots, T-1 \}; p = 0, 1, \cdots, P-1; \]

\[
k_1, \cdots, k_n = 0, 1, \cdots, K-1; \ell = 0, 1, \cdots, L-1 \text{ and } m = 0, 1, \cdots, M-1, \]

chosen in the appropriate way such that

\[
\Phi_{\tau_1 \cdots \tau_n}(p) = \sum_{k_1} \cdots \sum_{k_n} \sum_{l} \sum_{m} \sum_{v} C_{k_1 \cdots k_n l m v} \xi^v \\
\times P_{\tau_1 \cdots \tau_n p k_1 \cdots k_n l m}(\xi, \alpha_1, \cdots, \alpha_n, \beta_0, \beta_1, \cdots, \beta_n).
\]

If \( r, s \) and \( B \) denote the same numbers as in Lemma 6 of [4], we have

\[
r = T^{n+1} p \leq N^{n^2 + 2n + 2 + (n^3 + 3n^2 + 3n + 1) \delta} S n + 1 + (2n^2 + 7n + 7) \delta,
\]

\[
s = K^n LM \geq \frac{1}{2} N^{n^2 + 2n + 2 + (n^3 + 3n^2 + 3n + 1) \delta} S n + 1 + (2n^2 + 7n + 8) \delta
\]

and

\[
B \leq \exp \left\{ c_1 N^{n+1} + (n^2 + 2n + 1) \delta S^{n+1} (2n+5) \log S \right\}.
\]

From these inequalities it is easy to check the conditions of this lemma. Hence, we can fix the numbers \( C_{k_1 \cdots k_n l m} \) as integers, not all zero, of
absolute values at most $C$, such that $\Phi_{\tau_1 \ldots \tau_n} = 0$ for $\tau, \tau_1, \ldots, \tau_n = 0, 1, \ldots, T-1$ and $p = 0, 1, \ldots, P-1$. With (32) this implies

(33) \[ |F_{\tau_1 \ldots \tau_n}(p)| \leq \exp \{-\frac{1}{3}U\} \]

for $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T-1$ and $0 \leq p \leq P-1$.

Define $T_r$ and $P_r$ for $r = 0, 1, \ldots, R$ by

\[ T_r = [2^{-r}T] \]

and

\[ P_r = [(N^{n+1}S)^{\delta}P]. \]

Observe that

(34) \[ P_R \leq N^{n^2+n+1+(2n^2+7n^2+8n+3)\delta}S^{n+(2n^2+5n+5)\delta}. \]

**Lemma:** For $r = 0, 1, \ldots, R$ the inequality

(35) \[ |F_{\tau_1 \ldots \tau_n}(p)| \leq \exp \{-\frac{1}{3}U\} \]

holds for all non-negative integers $\tau, \tau_1, \ldots, \tau_n$ and $p$ with $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_r-1$ and $0 \leq p \leq P_r-1$.

**Proof:** We use induction on $r$. For $r = 0$ the inequality has already been proved in (33). Let $r$ be an integer with $0 \leq r \leq R-1$ for which

(36) \[ |F_{\tau_1 \ldots \tau_n}(p)| \leq \exp \{-\frac{1}{3}U\} \]

for $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_r-1$ and $0 \leq p \leq P_r-1$. Since

\[ F_{\tau_1 \ldots \tau_n}(p) = \sum_{k_1} \cdots \sum_{k_n} \sum_{l_1} \sum_{l_n} C_{k_1} \cdots k_n l m \xi^y \]

\[ \times \prod_{i=1}^{n} (\log \alpha_i)^{-\tau_i} \times (z^{m} e^{\beta_0 z^{(t)}}) \times \prod_{i=1}^{n} (e^{(k_1+1)\beta_0 (\log \alpha_i(z))^{(t)}}) \]

it follows that for $t = 0, 1, 2, \ldots$

\[ F_{\tau_1 \ldots \tau_n}(z) = \sum_{\mu \mu_1 + \cdots + \mu_n = t} \frac{t!}{\mu! \mu_1 ! \cdots \mu_n !} \]

\[ \times \prod_{i=1}^{n} (\log \alpha_i)^{\mu_1} F_{\tau + \mu_1 + \tau_1 + \mu_1, \ldots, \tau_n + \mu_n}(z). \]

Hence, (36) implies

(37) \[ |F_{\tau_1 \ldots \tau_n}(p)| \leq \exp \{-\frac{1}{3}U\} \]

for $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_{r+1}-1$, $0 \leq t \leq T_{r+1}-1$ and $0 \leq p \leq P_r-1$.

For the same values of $\tau, \tau_1, \ldots, \tau_n$ we obtain from (31)
We apply Lemma 7 of [4] to $F_{z_1 \cdots z_n}$ with $R = P_{r+1}$, $A = 6$, $T = T_{r+1}$ and $P = P_r$. From (37), (38) and (34) we then obtain
\[
\max_{|z| \leq 6P_{r+1}} |F_{z_1 \cdots z_n}(z)| \leq \exp \{ c_2^{n+1 + (n^2 + nr + 2n + r + 1)S^1 + (2n + r + 6)S^2} \}.
\]
Consequently,
\[
|F_{z_1 \cdots z_n}(p)| \leq \exp \{ -2^-(r+3)N^{n+2 + (n^2 + nr + 2n + r + 1)S^1 + (2n + r + 7)S^2} \}
\]
for $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_{r+1} - 1$ and $0 \leq p \leq P_{r+1} - 1$. From (32) and (34), it follows that
\[
|\Phi_{z_1 \cdots z_n}(p)| \leq \exp \{ -2^-(r+4)N^{n+2 + (n^2 + nr + 2n + r + 1)S^1 + (2n + r + 7)S^2} \}
\]
for $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_{r+1} - 1$ and $0 \leq p \leq P_{r+1} - 1$.

However, for these values of $\tau$, $\tau_1$, $\cdots$, $\tau_n$ and $p$, we can consider $\Phi_{z_1 \cdots z_n}(p)$ as a polynomial in $\zeta$, $\alpha_1$, $\cdots$, $\alpha_n$, $\beta_0$, $\beta_1$, $\cdots$, $\beta_n$, of degree less than $L P_{r+1} + N$ in $\zeta$, $KP_{r+1}$ in $\alpha_1$, $\cdots$, $\alpha_n$ and $T_{r+1}$ in $\beta_0$, $\beta_1$, $\cdots$, $\beta_n$. The sum of the absolute values of its coefficients is at most
\[
\exp \{ 2N^{n+1 + (n^2 + 2n + 1)S^1 + (2n + 6)S^2} \}.
\]
According to Lemma 3 of [4] we have either $\Phi_{z_1 \cdots z_n}(p) = 0$ or
\[
|\Phi_{z_1 \cdots z_n}(p)| \geq \exp \{ -c_3^{N^{n+2 + (n^2 + nr + 2n + r + 1)S^1 + (2n + r + 6)S^2}} \}.
\]
Hence,
\[
\Phi_{z_1 \cdots z_n}(p) = 0 \quad \text{for } 0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_{r+1} - 1 \quad \text{and} \quad 0 \leq p \leq P_{r+1} - 1.
\]
From (32) and (34) we see
\[
|F_{z_1 \cdots z_n}(p)| \leq \exp \{ -\frac{1}{3}U \}
\]
for $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_{r+1} - 1$ and $0 \leq p \leq P_{r+1} - 1$, which proves the lemma.

From (35) with $r = R$ we get
\[
|F_{z_1 \cdots z_n}(p)| \leq \exp \{ -\frac{1}{3}U \}
\]
for $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_R - 1$ and $0 \leq p \leq P_R - 1$. We have $T_R = T'$. From $R \geq n/\delta + 2n^2 + 5n + 2$ we see...
Thus,

\[ |F_{\tau_1, \ldots, \tau_n}(p)| \leq \exp\left\{-\frac{1}{3} U\right\} \]

for \( 0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T' - 1 \) and \( 0 \leq p \leq P' - 1 \). From (30) we now obtain

\[ |F^{(i)}(p)| \leq \exp\left\{-\frac{4}{3} U\right\} \]

for \( t = 0, 1, \cdots, T' - 1 \) and \( p = 0, 1, \cdots, P' - 1 \).

The exponents of \( F \) have the form

\[ \ell \beta_0 + (k_1 + \ell \beta_1) \log \alpha_1 + \cdots + (k_n + \ell \beta_n) \log \alpha_n. \]

Let \( \Omega \) and \( \omega \) have the same meaning as in Lemma 8 of [4]. Then

\[ \Omega \leq c_4 N^{n+(n^2+n)^\delta} S^{1+(2n+3)^\delta}, \]

from which the condition

\[ T'P' \geq 2KL^M + 13\Omega P' \]

follows by direct computation.

The difference of two exponents of \( F \) is of the form

\[ \ell \beta_0 + (k_1 + \ell \beta_1) \log \alpha_1 + \cdots + (k_n + \ell \beta_n) \log \alpha_n \]

with integral \( k_1, \cdots, k_n, \ell, \) not all zero, and \( |k_i| \leq K - 1 \) for \( i = 1, \cdots, n \) and \( |\ell| \leq L - 1 \). Moreover, at least one of the numbers \( k_i + \ell \beta_i \) \((i = 1, \cdots, n)\) is non-zero, since \( \beta_1, \cdots, \beta_n \) are not all rational. The degrees of \( \ell \beta_0, k_1 + \ell \beta_1, \cdots, k_n + \ell \beta_n \) are constants. We estimate their heights by means of Lemma 3; we then see that these heights do not exceed

\[ c_5(2KL)^{c_6} \leq S^{c_7} \]

in which \( c_5 \) and \( c_6 \) are upper bounds for the heights and degrees resp. of \( \beta_0, \beta_1, \cdots, \beta_n \). From Lemma 1 with \( \varepsilon = \delta \) it follows that

\[ |\ell \beta_0 + (k_1 + \ell \beta_1) \log \alpha_1 + \cdots + (k_n + \ell \beta_n) \log \alpha_n| > \exp\left\{- (\log S)^{1+2\delta}\right\}. \]

Hence, the exponents of \( F \) are distinct and

\[ \omega > \exp\left\{- (\log S)^{1+2\delta}\right\} > \exp\left\{- S^{\varepsilon}\right\}. \]

From Lemma 8 of [4], using (41), (42) and (43) we obtain the inequality

\[ \sum_{v=0}^{N-1} C_{k_1, \cdots, k_m} \xi^v \leq \exp\left\{-\frac{1}{3} U\right\} \]

for \( k_1, \cdots, k_n = 0, 1, \cdots, K - 1; \ell = 0, 1, \cdots, L - 1 \) and \( m = 0, 1, \cdots, M - 1 \).
According to Lemma 3 of [4] we have either

$$\sum_{v=0}^{N-1} C_{k_1 \cdots k_n \ell m v} \xi^v = 0 \text{ or}$$

$$|\sum_{v=0}^{N-1} C_{k_1 \cdots k_n \ell m v} \xi^v| > \exp\{-2N^{n+2+(n^2+2n+1)\delta S t + (2n+6)\delta}\}$$

for the same values of $k_1, \cdots, k_n, \ell$ and $m$. It follows that

$$\sum_{v=0}^{N-1} C_{k_1 \cdots k_n \ell m v} \xi^v = 0$$

for all of these values. Since $\xi$ has the degree $N$, this implies that all integers $C_{k_1 \cdots k_n \ell m v}$ are zero, in contradiction to their choice. The theorem has been proved.

Using the fact, that there are only finitely many algebraic numbers $\xi$ of size $S < S_2$, and using Lemma 9 of [4], one immediately obtains the following theorem:

**Theorem 5**: Under the conditions of Theorem 4, there exists an effectively computable, number $C_2 = C_2(e, \log \alpha_1, \cdots, \log \alpha_n, \beta_0, \beta_1, \cdots, \beta_n) > 0$ such that

$$\exp\{-C_2 N^{n^2+2n+2+\varepsilon S n+1+\varepsilon}\}$$

is a transcendence measure of $e^{\beta_0 \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}}$.

**References**


