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P. L. CIJSOUW

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**TRANSCENDENCE MEASURES OF CERTAIN NUMBERS WHOSE
 TRANSCENDENCY WAS PROVED BY A. BAKER**

P. L. Cijsouw

1. Introduction

In the subsequent paper we continue the investigation of transcendence measures of certain transcendental numbers σ , i.e. positive lower bounds for $|P(\sigma)|$ in terms of the degree N and height H of P , where P is an arbitrary polynomial with integral coefficients. For more information about transcendence measures and the type of transcendence measures we will look for, see the earlier paper [4]; see also the authors thesis [3], which includes the results of the present paper.

Let $\alpha_1, \dots, \alpha_n$ be non-zero algebraic numbers such that, for any (fixed) values of the logarithms, $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over \mathcal{Q} . In this paper, transcendence measures are derived for numbers which can be written in one of the following ways:

- (i) $\beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n$ with $n \geq 2$, where β_1, \dots, β_n are algebraic numbers, not all zero
- (ii) $e^{\beta_0} \alpha_1^{\beta_1} \dots \alpha_n^{\beta_n}$ with $n \geq 1$ and

$$\alpha_i^{\beta_i} = e^{\beta_i \log \alpha_i}$$

where $\beta_0, \beta_1, \dots, \beta_n$ are algebraic numbers such that at least one of β_1, \dots, β_n is irrational. We prove the transcendence measure

$$\exp \{ -C_1 N^{n^2+n+\varepsilon} S(1 + \log S)^{n+1+\varepsilon} \}$$

for numbers of the form (1), and the transcendence measure

$$\exp \{ -C_2 N^{n^2+2n+2+\varepsilon} S^{n+1+\varepsilon} \}$$

for numbers of the form (2). Here $S = N + \log H$, ε is an arbitrary positive number and C_1 and C_2 are effectively computable numbers, depending on ε, n , the α 's and their logarithms and the β 's.

We remark, that these transcendence measures are the first explicit ones to be published for these numbers in which both the dependence on H and N is expressed. If one is interested merely in the height H , better results can be given. For numbers of the form (i), N. I. FEL'DMAN [6] proved the transcendence measure $\exp \{ -CS \}$, where C is a positive

number, depending on N , the α 's and their logarithms and the β 's. For numbers of the form (ii), a recent result of A. BAKER [2] implies the transcendence measure $\exp \{-C \log H \log \log H\}$ for $H \geq 4$, where C again is a positive number depending on N , the α 's and their logarithms and the β 's. An earlier result for the special case of numbers of the form $e^{\beta_0} \alpha_1^{\beta_1}$ can be found in [7].

The method of proof of our transcendence measures is A. BAKER's one with some improvements introduced by N. I. FEL'DMAN. In the proof we firstly derive a measure for the approximability of numbers of the types (i) and (ii); after that, the transcendence measures are given.

The transcendence of numbers of the form (i) follows immediately from e.g. Theorem 1 of A. BAKER's paper [1]. The transcendence of numbers of the form (ii) was proved by A. BAKER, distinguishing the cases $\beta_0 = 0$ and $\beta_0 \neq 0$; see the same paper.

2. Lemmas

We shall use the same notations (especially for the degree; height and size) as in [4]. For shortness, we use without reformulation the lemmas 3, 6, 7, 8 and 9 of [4]. Further, we need the following lemmas:

LEMMA 1: *Let $\alpha_1, \dots, \alpha_n$ be non-zero algebraic numbers such that, for any fixed values of the logarithms, $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over \mathcal{Q} . Let ε be positive and let d be a positive integer. Then there exists an effectively computable positive number*

$$\theta(\varepsilon, d) = \theta(\varepsilon, d, \log \alpha_1, \dots, \log \alpha_n)$$

such that

$$(1) \quad |\beta_0 + \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n| > \theta(\varepsilon, d) \exp \{-(\log h)^{1+\varepsilon}\}$$

for all algebraic numbers $\beta_0, \beta_1, \dots, \beta_n$, not all zero, of degrees at most d and of heights at most h .

PROOF: See Theorem 1 of [5].

LEMMA 2: *Let α and β be algebraic numbers, $\beta \neq 0$. Then*

$$(2) \quad s(\alpha\beta^{-1}) \leq 2d(\alpha)d(\beta) + d(\alpha)s(\beta) + d(\beta)s(\alpha).$$

PROOF: From Lemma 3 of [5] it follows that

$$h(\alpha\beta^{-1}) \leq 2^{d(\alpha)d(\beta)} \{h(\alpha)(d(\alpha)+1)\}^{d(\beta)} \{h(\beta)(d(\beta)+1)\}^{d(\alpha)}.$$

Using $d(\alpha)+1 < e^{d(\alpha)}$ and $d(\beta)+1 < e^{d(\beta)}$ we get

$$\begin{aligned} s(\alpha\beta^{-1}) &= d(\alpha\beta^{-1}) + \log h(\alpha\beta^{-1}) \\ &\leq d(\alpha)d(\beta)(1 + \log 2) + d(\beta)s(\alpha) + d(\alpha)s(\beta) \end{aligned}$$

from which (2) follows.

LEMMA 3: *Let β be an algebraic number, and let k and ℓ be integers. Then*

$$(3) \quad h(k + \ell\beta) \leq h(\beta)|2k\ell|^{d(\beta)}.$$

PROOF: If $a_n z^n + \dots + a_1 z + a_0$ is the minimal polynomial of β , then

$$a_n(z-k)^n + \dots + a_1 \ell^{n-1}(z-k) + a_0 \ell^n$$

is a constant multiple of the minimal polynomial of $k + \ell\beta$. Thus, the coefficient of z^i ($i = 0, 1, \dots, n$) of this minimal polynomial is in absolute value at most

$$\left| a_n \binom{n}{i} (-k)^{n-i} + a_{n-1} \ell \binom{n-1}{i} (-k)^{n-1-i} + \dots + a_i \ell^{n-i} \right| \leq h(\beta)|k|^n |\ell|^n \left\{ \binom{n}{i} + \binom{n-1}{i} + \dots + 1 \right\}.$$

From the obvious inequality $\binom{m}{i} \leq 2^{m-1}$ for all positive integers m , it follows that

$$\binom{n}{i} + \binom{n-1}{i} + \dots + 1 \leq 2^n,$$

by which the proof is completed.

3. The case $\beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n$

First we give a measure for the approximability for numbers of the form (i), in the special case in which $\beta_n = -1$.

THEOREM 1: *Let, for $n \geq 2$, $\alpha_1, \dots, \alpha_n$ and $\gamma_1, \dots, \gamma_{n-1}$ be non-zero algebraic numbers such that, for any fixed values of the logarithms, $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over \mathcal{Q} . Let ε be a positive number. Then there exists an effectively computable number $S_1 = S_1(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \gamma_1, \dots, \gamma_{n-1})$ such that*

$$(4) \quad |\gamma_1 \log \alpha_1 + \dots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta| > \exp \{ -N^{n^2+n+\varepsilon} S(\log S)^{n+1+\varepsilon} \}$$

for all algebraic numbers η of degree N and size $S \geq S_1$.

PROOF: Put $\delta = (2n^3 + 4n^2 + 3n + 7)^{-1}\varepsilon$. For abbreviation, put

$$\sigma = \gamma_1 \log \alpha_1 + \dots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n$$

and

$$U = N^{n^2+n+(2n^3+4n^2+2n+1)\delta} S(\log S)^{n+1+(2n^2+3n+7)\delta}.$$

It is sufficient to prove that

$$|\sigma - \eta| > \exp \{-U\}$$

if $S \geq S_1$. In this proof we may restrict ourselves to the case in which δ is rather small. By c_1, c_2, \dots we denote positive numbers which depend only on $n, \log \alpha_1, \dots, \log \alpha_n, \gamma_1, \dots, \gamma_{n-1}$.

Suppose that

$$(5) \quad |\sigma - \eta| \leq \exp \{-U\}$$

for some algebraic number η of degree N and size S . We prove that this leads to a contradiction if S is sufficiently large.

Choose the following integers:

$$\begin{aligned} K &= [N^{n+(n^2-1)\delta}(\log S)^{1+(2n+1)\delta}], \\ M &= [N^{n+(n^2+n)\delta}S(\log S)^{1+(2n+3)\delta}], \\ C &= 2[\frac{1}{2} \exp \{N^{n+(n^2+n)\delta}S(\log S)^{2+(2n+4)\delta}\}], \\ T &= [N^{n+1+(n^2+n)\delta}(\log S)^{1+(2n+3)\delta}], \\ P &= [S(\log S)^{1+2\delta}], \\ R &= \left[\frac{n-1}{\delta} + 2n^2 + n \right], \\ T' &= [2^{-R}T] \text{ and} \\ P' &= [N^{n^2-1+(n^3-n)\delta}S(\log S)^{n+(2n^2+n+1)\delta}]. \end{aligned}$$

Put

$$F(z) = \sum_{k_1=0}^{K-1} \cdots \sum_{k_n=0}^{K-1} \sum_{m=0}^{M-1} \sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v z^m \times \exp \left\{ -k_n \eta z + \sum_{i=1}^{n-1} (k_i + k_n \gamma_i)(\log \alpha_i)z \right\},$$

where the numbers $C_{k_1 \dots k_n m v}$ are integers of absolute values at most C ; they will be specified later.

For $t = 0, 1, 2, \dots$ it is easily seen that

$$(6) \quad F^{(t)}(z) = \sum_{\tau_1 + \tau_2 + \dots + \tau_{n-1} = t} \frac{t!}{\tau_1! \tau_2! \cdots \tau_{n-1}!} \times (\log \alpha_1)^{\tau_1} \cdots (\log \alpha_{n-1})^{\tau_{n-1}} F_{\tau_1 \dots \tau_{n-1}}(z)$$

where

$$\begin{aligned} (7) \quad F_{\tau_1 \dots \tau_{n-1}}(z) &= \sum_{k_1} \cdots \sum_{k_n} \sum_m \sum_v C_{k_1 \dots k_n m v} \eta^v \\ &\times \sum_{\kappa=0}^m \binom{\tau}{\kappa} \frac{m!}{(m-\kappa)!} z^{m-\kappa} (-k_n)^{\tau-\kappa} \eta^{\tau-\kappa} \prod_{i=1}^{n-1} (k_i + k_n \gamma_i)^{\tau_i} \\ &\times \exp \left\{ \left(\sum_{i=1}^n k_i \log \alpha_i \right) z + k_n (\sigma - \eta) z \right\}. \end{aligned}$$

Put

$$\begin{aligned} \Phi_{\tau\tau_1 \dots \tau_{n-1}}(z) &= \sum_{k_1} \dots \sum_{k_n} \sum_m \sum_\nu C_{k_1 \dots k_n m \nu} \eta^\nu \\ &\times \sum_{\kappa=0}^m \binom{\tau}{\kappa} \frac{m!}{(m-\kappa)!} z^{m-\kappa} (-k_n)^{\tau-\kappa} \eta^{\tau-\kappa} \prod_{i=1}^{n-1} (k_i + k_n \gamma_i)^{\tau_i} \\ &\times \exp \left\{ \left(\sum_{i=1}^n k_i \log \alpha_i \right) z \right\}. \end{aligned}$$

We estimate the difference

$$|F_{\tau\tau_1 \dots \tau_{n-1}}(p) - \Phi_{\tau\tau_1 \dots \tau_{n-1}}(p)|$$

as follows: by $|e^z - 1| \leq |z|e^{|z|}$, one has for $k_n = 0, 1, \dots, K-1$ and

$$p = 0, 1, \dots, [N^{n^2-1+(2n^3+3n^2+n)\delta} S(\log S)^{n+(2n^2+n+2)\delta}]$$

the inequality

$$|e^{k_n(\sigma-\eta)p} - 1| \leq \exp \left\{ -\frac{1}{2}U \right\}.$$

Hence,

$$(8) \quad |F_{\tau\tau_1 \dots \tau_{n-1}}(p) - \Phi_{\tau\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for $\tau, \tau_1, \dots, \tau_{n-1} = 0, 1, \dots, T-1$ and

$$p = 0, 1, \dots, [N^{n^2-1+(2n^3+3n^2+n)\delta} S(\log S)^{n+(2n^2+n+2)\delta}].$$

Let

$$P_{\tau\tau_1 \dots \tau_{n-1} p k_1 \dots k_n m} (z_0, z_1, \dots, z_{2n-1})$$

($\tau, \tau_1, \dots, \tau_{n-1} = 0, 1, \dots, T-1$; $p = 0, 1, \dots, P-1$; $k_1, \dots, k_n = 0, 1, \dots, K-1$; $m = 0, 1, \dots, M-1$) be the polynomials, chosen in the appropriate way, such that

$$\begin{aligned} \Phi_{\tau\tau_1 \dots \tau_{n-1}}(p) &= \sum_{k_1=0}^{K-1} \dots \sum_{k_n=0}^{K-1} \sum_{m=0}^{M-1} \sum_{\nu=0}^{N-1} C_{k_1 \dots k_n m \nu} \eta^\nu \\ &\times P_{\tau\tau_1 \dots \tau_{n-1} p k_1 \dots k_n m} (\eta, \alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_{n-1}). \end{aligned}$$

We apply Lemma 6 of [4] to these polynomials in the specified points. If r, s and B have the same meaning as in this lemma we have

$$\begin{aligned} r &= T^n P \leq N^{n^2+n+(n^3+n^2)\delta} S(\log S)^{n+1+(2n^2+3n+2)\delta}, \\ s &= K^n M \geq \frac{1}{2} N^{n^2+n+(n^3+n^2)\delta} S(\log S)^{n+1+(2n^2+3n+3)\delta}. \end{aligned}$$

Hence, $s \geq 4rd$ where $d = [Q(\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_{n-1}) : Q]$. Further,

$$B \leq \exp \left\{ c_1 N^{n+(n^2+n)\delta} S(\log S)^{2+(2n+3)\delta} \right\}.$$

Hence, the right hand side of the second condition in Lemma 6 of [4] is at most

$$\exp \{c_2 N^{n+1+(n^2+n)\delta} S(\log S)^{2+(2n+3)\delta}\} \leq C^N.$$

It follows that the numbers $C_{k_1 \dots k_{nmv}}$ can be chosen as integers, not all zero, of absolute values at most C , such that $\Phi_{\tau_1 \dots \tau_{n-1}}(p) = 0$ for $\tau, \tau_1, \dots, \tau_{n-1} = 0, 1, \dots, T-1$ and $p = 0, 1, \dots, P-1$. Doing so, we certainly have $\Phi_{\tau_1 \dots \tau_{n-1}}(p) = 0$ for $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T-1$ and $0 \leq p \leq P-1$. From (8) we now get

$$(9) \quad |F_{\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \{-\frac{1}{3}U\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T-1$ and $0 \leq p \leq P-1$.

Define T_r and P_r for $r = 0, 1, \dots, R$ by

$$T_r = [2^{-r}T]$$

and

$$P_r = [(N^{n+1} \log S)^{\delta} P].$$

Remark that

$$(10) \quad P_R \leq [N^{n^2-1+(2n^3+3n^2+n)\delta} S(\log S)^{n+(2n^2+n+2)\delta}].$$

LEMMA: For $r = 0, 1, \dots, R$ the inequality

$$(11) \quad |F_{\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \{-\frac{1}{3}U\}$$

holds for all non-negative integers $\tau, \tau_1, \dots, \tau_{n-1}$ and p with $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_r-1$ and $0 \leq p \leq P_r-1$.

PROOF: We proceed by induction on r . For $r = 0$ the statement is proved in (9). Let r be an integer with $0 \leq r \leq R-1$ for which

$$(12) \quad |F_{\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \{-\frac{1}{3}U\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_r-1$ and $0 \leq p \leq P_r-1$. Since

$$F_{\tau_1 \dots \tau_{n-1}}(z) = \sum_{k_1} \dots \sum_{k_n} \sum_m \sum_v C_{k_1 \dots k_{nmv}} \eta^v \prod_{i=1}^{n-1} (\log \alpha_i)^{-\tau_i} \\ \times (z^m e^{-k_n z})^{(\tau)} \prod_{i=1}^{n-1} (\exp \{(k_i + k_n \gamma_i)(\log \alpha_i)z\})^{(\tau_i)}$$

we have for $t = 0, 1, 2, \dots$

$$F_{\tau_1 \dots \tau_{n-1}}^{(t)}(z) = \sum_{\mu + \mu_1 + \dots + \mu_{n-1} = t} \frac{t!}{\mu! \mu_1! \dots \mu_{n-1}!} \\ \times \prod_{i=1}^{n-1} (\log \alpha_i)^{\mu_i} \times F_{\tau + \mu, \tau_1 + \mu_1, \dots, \tau_{n-1} + \mu_{n-1}}(z).$$

Together with (12) we obtain

$$(13) \quad |F_{\tau\tau_1 \dots \tau_{n-1}}^{(t)}(p)| \leq \exp \left\{ -\frac{1}{4}U \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$, $0 \leq t \leq T_{r+1} - 1$ and $0 \leq p \leq P_r - 1$.

From (7) we know that

$$(14) \quad \max_{|z| \leq 6P_{r+1}} |F_{\tau\tau_1 \dots \tau_{n-1}}(z)| \leq \exp \left\{ c_3 N^{n+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+4)\delta} \right\}$$

for $\tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$. We apply Lemma 7 of [4] to $F_{\tau\tau_1 \dots \tau_{n-1}}$ with $R = P_{r+1}$, $A = 6$, $T = T_{r+1}$ and $P = P_r$. From (13), (14) and the inequality

$$N^{n+1} \log S \leq \exp \left\{ N^\delta (\log S)^\delta \right\}$$

we then obtain

$$\max_{|z| \leq P_{r+1}} |F_{\tau\tau_1 \dots \tau_{n-1}}(z)| \leq \exp \left\{ -2^{-(r+3)} N^{n+1+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+5)\delta} \right\}.$$

In particular,

$$|F_{\tau\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \left\{ -2^{-(r+3)} N^{n+1+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+5)\delta} \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$ and $0 \leq p \leq P_{r+1} - 1$.

From (8) and (10) it follows that

$$(15) \quad |\Phi_{\tau\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \left\{ -2^{-(r+4)} N^{n+1+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+5)\delta} \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$ and $0 \leq p \leq P_{r+1} - 1$. But for these values of τ , $\tau_1, \dots, \tau_{n-1}$ and p , we can consider $\Phi_{\tau\tau_1 \dots \tau_{n-1}}(p)$ as a polynomial in η , $\alpha_1, \dots, \alpha_n$, $\gamma_1, \dots, \gamma_{n-1}$, of degree less than $T_{r+1} + N$ in η , KP_{r+1} in $\alpha_1, \dots, \alpha_n$ and T_{r+1} in $\gamma_1, \dots, \gamma_{n-1}$. If B denotes the sum of the absolute values of the coefficients, then we have

$$B \leq \exp \left\{ 2N^{n+(n^2+n)\delta} S(\log S)^{2+(2n+4)\delta} \right\}.$$

According to Lemma 3 of [4] we thus have either $\Phi_{\tau\tau_1 \dots \tau_{n-1}}(p) = 0$ or

$$(16) \quad |\Phi_{\tau\tau_1 \dots \tau_{n-1}}(p)| \geq \exp \left\{ -c_4 N^{n+1+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+4)\delta} \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$ and $0 \leq p \leq P_{r+1} - 1$. Hence, $\Phi_{\tau\tau_1 \dots \tau_{n-1}}(p) = 0$ for these τ , $\tau_1, \dots, \tau_{n-1}$ and p . Again from (8) and (10) we obtain

$$|F_{\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$ and $0 \leq p \leq P_{r+1} - 1$. The lemma has been proved.

From (11) with $r = R$ we get

$$(17) \quad |F_{\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_R - 1$ and $0 \leq p \leq P_R - 1$. From their definitions we have $T_R = T'$. Since

$$R \geq \frac{n-1}{\delta} + 2n^2 + n - 1$$

we see that

$$P_R \geq \frac{1}{2} N^{n^2-1+(2n^3+3n^2-1)\delta} S(\log S)^{n+(2n^2+n+1)\delta} \geq P'.$$

Hence,

$$(18) \quad |F_{\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T' - 1$ and $0 \leq p \leq P' - 1$. From (6) it now follows that

$$(19) \quad |F^{(t)}(p)| \leq \exp \left\{ -\frac{1}{4}U \right\}$$

for $t = 0, 1, \dots, T' - 1$ and $p = 0, 1, \dots, P' - 1$.

We apply Lemma 8 of [4] to F with K replaced by K^n . Let Ω and ω have the same meaning as in this lemma. Since the exponents of F have the form

$$k_1 \log \alpha_1 + \dots + k_n \log \alpha_n + k_n(\sigma - \eta),$$

we see that

$$(20) \quad \Omega \leq c_5 N^{n+(n^2-1)\delta} (\log S)^{1+(2n+1)\delta}.$$

With this, the condition

$$T'P' \geq 2K^n M + 13\Omega P'$$

is easily checked. Further, we know from Lemma 1, applied with $\varepsilon = \delta$, that

$$(21) \quad |k_1 \log \alpha_1 + \dots + k_n \log \alpha_n| \geq \exp \left\{ -(\log K)^{1+2\delta} \right\} \\ \geq \exp \left\{ -N^\delta (\log S)^\delta \right\}$$

for all integers k_1, \dots, k_n , not all zero, with $|k_1| \leq K-1, \dots, |k_n| \leq K-1$. From (21) and (5) it follows that

$$(22) \quad \omega \geq \exp \left\{ -N^\delta (\log S)^\delta \right\} - \exp \left\{ -\frac{1}{2}U \right\} \geq \exp \left\{ -2N^\delta (\log S)^\delta \right\}.$$

From (20) we have

$$(23) \quad \Omega \leq \exp \{N^\delta (\log S)^\delta\}.$$

From lemma 8 of [4], with (19), (22) and (23) we obtain

$$(24) \quad \begin{aligned} & \left| \sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v \right| \\ & \leq \exp \{c_6 N^{n^2+n+(n^3+n^2+1)\delta} S (\log S)^{n+1+(2n^2+3n+4)\delta} - \frac{1}{4}U\} \\ & \leq \exp \{-\frac{1}{5}U\} \end{aligned}$$

for $k_1, \dots, k_n = 0, 1, \dots, K-1$ and $m = 0, 1, \dots, M-1$.

But according to Lemma 3 of [4] we have either

$$\sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v = 0$$

or

$$(25) \quad \left| \sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v \right| \geq \exp \{-2N^{n+1+(n^2+n)\delta} S (\log S)^{2+(2n+4)\delta}\}$$

for the same values of k_1, \dots, k_n and m . Hence,

$$\sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v = 0 \text{ for } k_1, \dots, k_n = 0, 1, \dots, K-1$$

and $m = 0, 1, \dots, M-1$. Since η has the degree N , it follows that all integers $C_{k_1 \dots k_n m v}$ are zero, in contradiction to their choice. This contradiction proves Theorem 1.

We have the following

COROLLARY: *Under the conditions of Theorem 1, there exists an effectively computable, number $C_3 = C_3(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \gamma_1, \dots, \gamma_{n-1}) > 0$ such that*

$$(26) \quad \begin{aligned} & |\gamma_1 \log \alpha_1 + \dots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta| \\ & > \exp \{-C_3 N^{n^2+n+\varepsilon} S (1 + \log S)^{n+1+\varepsilon}\} \end{aligned}$$

for all algebraic numbers η of degree at most N and size at most S .

PROOF: There are only finitely many algebraic numbers η of size $s(\eta) < S_1$. Choose $C_3 \geq 1$ such that (26) holds for these finitely many numbers.

THEOREM 2: *Let, for $n \geq 2$, $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n be non-zero algebraic numbers such that, for any fixed values of the logarithms, $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over \mathcal{Q} . Let ε be a positive number. Then there exists an effectively computable positive number $C_4 = C_4(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \beta_1, \dots, \beta_n)$ such that*

$$(27) \quad |\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi| > \exp \{-C_4 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon}\}$$

for all algebraic numbers ξ of degree N and size S .

PROOF: Put $\gamma_i = -\beta_i/\beta_n$ ($i = 1, \dots, n-1$) and $\eta = -\xi/\beta_n$. Then

$$\begin{aligned} &(-1/\beta_n)(\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi) \\ &= \gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta. \end{aligned}$$

We have $d(\eta) \leq c_7 N$ with $c_7 = d(\beta_n)$ and, by Lemma 2, $s(\eta) \leq c_8 S$ with $c_8 = 3d(\beta_n) + s(\beta_n)$. From (26) we now obtain

$$\begin{aligned} &|\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi| \\ &= |\beta_n| |\gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta| \\ &> |\beta_n| \exp \{-C_3(c_1 N)^{n^2+n+\varepsilon} c_2 S(1+\log c_2 S)^{n+1+\varepsilon}\} \\ &\geq \exp \{-C_4 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon}\} \end{aligned}$$

for some effectively computable positive number C_4 .

THEOREM 3: Under the conditions of Theorem 2, there exists an effectively computable number $C_1 = C_1(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \beta_1, \dots, \beta_n) > 0$, such that

$$\exp \{-C_1 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon}\}$$

is a transcendence measure of $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$.

PROOF: Apply Lemma 9 of [4] to the result of Theorem 2 and put $C_1 = 6C_4(1+\log 2)^{n+1+\varepsilon}$.

4. The case $e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$

THEOREM 4: Let n be a positive integer. Let β_0 be algebraic and let $\alpha_1, \dots, \alpha_n$ be non-zero algebraic numbers such that, for any fixed values of the logarithms, $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over \mathcal{Q} . Let β_1, \dots, β_n be algebraic numbers, not all rational. Put

$$\alpha_i^{\beta_i} = e^{\beta_i \log \alpha_i} \text{ for } i = 1, \dots, n.$$

Let ε be a positive number. Then there exists an effectively computable number $S_2 = S_2(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \beta_0, \beta_1, \dots, \beta_n) > 0$ such that

$$(28) \quad |e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n} - \xi| > \exp \{-N^{n^2+2n+2+\varepsilon} S^{n+1+\varepsilon}\}$$

for all algebraic ξ of degree N and size $S \geq S_2$.

PROOF: Put $\delta = (2n^3 + 8n^2 + 10n + 4)^{-1} \varepsilon$. For the sake of brevity, put

$$\sigma = e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$$

and

$$U = N^{n^2+2n+2+(2n^3+8n^2+10n+4)\delta} S^{n+1+(2n^2+7n+11)\delta}$$

It is sufficient to prove that

$$|\sigma - \xi| > \exp \{-U\}$$

if $S \geq S_2$; in this proof we may assume that δ is rather small. By c_1, c_2, \dots we shall denote positive numbers which depend on $n, \log \alpha_1, \dots, \log \alpha_n, \beta_0, \beta_1, \dots, \beta_n$ only.

Suppose that

$$(29) \quad |\sigma - \xi| \leq \exp \{-U\}$$

for some algebraic number ξ of degree N and size S . We prove that this is impossible if S is sufficiently large.

Choose the following integers:

$$\begin{aligned} K &= [N^{n+(n^2+n)\delta} S^{1+(2n+3)\delta}], \\ L &= [N^{n+1+(n^2+n)\delta} S^{(2n+3)\delta}], \\ M &= [N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+5)\delta}], \\ C &= 2[\frac{1}{2} \exp \{N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+6)\delta}\}], \\ T &= [N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+5)\delta}] (= M), \\ P &= [NS^{2\delta}], \\ R &= \left[\frac{n}{\delta} + 2n^2 + 5n + 3 \right], \\ T' &= [2^{-RT}], \text{ and} \\ P' &= [\frac{1}{2} N^{n^2+n+1+(n^3+2n^2+n)\delta} S^{n+(2n^2+5n+4)\delta}]. \end{aligned}$$

Put

$$\begin{aligned} F(z) &= \sum_{k_1=0}^{K-1} \cdots \sum_{k_n=0}^{K-1} \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} \sum_{v=0}^{N-1} C_{k_1 \dots k_n l m v} \xi^v z^m \\ &\quad \times \exp \{ \ell \beta_0 z + \sum_{i=1}^n (k_i + \ell \beta_i) (\log \alpha_i) z \}, \end{aligned}$$

where the numbers $C_{k_1 \dots k_n l m v}$ are integers of absolute values at most C ; they will be specified later.

For $t = 0, 1, 2, \dots$ we have

$$(30) \quad F^{(t)}(z) = \sum_{\tau_1 + \dots + \tau_n = t} \frac{t!}{\tau_1! \cdots \tau_n!} \prod_{i=1}^n (\log \alpha_i)^{\tau_i} F_{\tau_1 \dots \tau_n}(z)$$

where

$$(31) \quad F_{\tau_1 \dots \tau_n}(z) = \sum_{k_1} \dots \sum_{k_n} \sum_l \sum_m \sum_v C_{k_1 \dots k_n l m v} \zeta^v \sum_{\kappa=0}^m \binom{\tau}{\kappa} \frac{m!}{(m-\kappa)!} \\ \times z^{m-\kappa} \ell^{\tau-\kappa} \beta_0^{\tau-\kappa} \prod_{i=1}^n (k_i + \ell \beta_i)^{\tau_i} \exp \left\{ \sum_{i=1}^n k_i (\log \alpha_i) z \right\} \sigma^{Lz}.$$

Define $\Phi_{\tau_1 \dots \tau_n}$ by

$$\Phi_{\tau_1 \dots \tau_n}(z) = \sum_{k_1} \dots \sum_{k_n} \sum_l \sum_m \sum_v C_{k_1 \dots k_n l m v} \zeta^v \sum_{\kappa=0}^m \binom{\tau}{\kappa} \frac{m!}{(m-\kappa)!} \\ \times z^{m-\kappa} \ell^{\tau-\kappa} \beta_0^{\tau-\kappa} \prod_{i=1}^n (k_i + \ell \beta_i)^{\tau_i} \exp \left\{ \sum_{i=1}^n k_i (\log \alpha_i) z \right\} \xi^{Lz}.$$

For $\ell = 0, 1, \dots, L-1$ and

$$p = 0, 1, \dots, [N^{n^2+n+1+(2n^3+7n^2+8n+3)\delta} S^{n+(2n^2+5n+5)\delta}]$$

one has

$$|\sigma^{lp} - \xi^{lp}| \leq lp |\sigma - \xi| (|\sigma| + 1)^{lp} \leq \exp \left\{ -\frac{1}{2} U \right\}.$$

Hence,

$$(32) \quad |F_{\tau_1 \dots \tau_n}(p) - \Phi_{\tau_1 \dots \tau_n}(p)| \leq \exp \left\{ -\frac{1}{3} U \right\}$$

for $\tau, \tau_1, \dots, \tau_n = 0, 1, \dots, T-1$ and

$$p = 0, 1, \dots, [N^{n^2+n+1+(2n^3+7n^2+8n+3)\delta} S^{n+(2n^2+5n+5)\delta}].$$

We apply Lemma 6 of [4] to the polynomials

$$P_{\tau_1 \dots \tau_n p k_1 \dots k_n l m}(\tau, \tau_1, \dots, \tau_n = 0, 1, \dots, T-1; p = 0, 1, \dots, P-1; \\ k_1, \dots, k_n = 0, 1, \dots, K-1; \ell = 0, 1, \dots, L-1 \text{ and } m = 0, 1, \dots, M-1),$$

chosen in the appropriate way such that

$$\Phi_{\tau_1 \dots \tau_n}(p) = \sum_{k_1} \dots \sum_{k_n} \sum_l \sum_m \sum_v C_{k_1 \dots k_n l m v} \zeta^v \\ \times P_{\tau_1 \dots \tau_n p k_1 \dots k_n l m}(\xi, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n).$$

If r, s and B denote the same numbers as in Lemma 6 of [4], we have

$$r = T^{n+1}P \leq N^{n^2+2n+2+(n^3+3n^2+3n+1)\delta} S^{n+1+(2n^2+7n+7)\delta}, \\ s = K^N L M \geq \frac{1}{2} N^{n^2+2n+2+(n^3+3n^2+3n+1)\delta} S^{n+1+(2n^2+7n+8)\delta}$$

and

$$B \leq \exp \{ c_1 N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+5)\delta} \log S \}.$$

From these inequalities it is easy to check the conditions of this lemma. Hence, we can fix the numbers $C_{k_1 \dots k_n l m v}$ as integers, not all zero, of

absolute values at most C , such that $\Phi_{\tau\tau_1 \dots \tau_n p} = 0$ for $\tau, \tau_1, \dots, \tau_n = 0, 1, \dots, T-1$ and $p = 0, 1, \dots, P-1$. With (32) this implies

$$(33) \quad |F_{\tau\tau_1 \dots \tau_n}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T-1$ and $0 \leq p \leq P-1$.

Define T_r and P_r for $r = 0, 1, \dots, R$ by

$$T_r = [2^{-r}T]$$

and

$$P_r = [(N^{n+1}S)^{\delta}P].$$

Observe that

$$(34) \quad P_R \leq N^{n^2+n+1+(2n^3+7n^2+8n+3)\delta} S^{n+(2n^2+5n+5)\delta}.$$

LEMMA: For $r = 0, 1, \dots, R$ the inequality

$$(35) \quad |F_{\tau\tau_1 \dots \tau_n}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

holds for all non-negative integers $\tau, \tau_1, \dots, \tau_n$ and p with $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_r - 1$ and $0 \leq p \leq P_r - 1$.

PROOF: We use induction on r . For $r = 0$ the inequality has already been proved in (33). Let r be an integer with $0 \leq r \leq R-1$ for which

$$(36) \quad |F_{\tau\tau_1 \dots \tau_n}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_r - 1$ and $0 \leq p \leq P_r - 1$. Since

$$F_{\tau\tau_1 \dots \tau_n}(p) = \sum_{k_1} \dots \sum_{k_n} \sum_l \sum_m \sum_v C_{k_1 \dots k_n l m v} \xi^v \\ \times \prod_{i=1}^n (\log \alpha_i)^{-\tau_i} \times (z^m e^{l\beta_0 z})^{(\tau)} \times \prod_{i=1}^n (e^{(k_i + l\beta_i)(\log \alpha_i \tau_i)})^{(\tau_i)}$$

it follows that for $t = 0, 1, 2, \dots$

$$F_{\tau\tau_1 \dots \tau_n}^{(t)}(z) = \sum_{\mu + \mu_1 + \dots + \mu_n = t} \frac{t!}{\mu! \mu_1! \dots \mu_n!} \\ \times \prod_{i=1}^n (\log \alpha_i)^{\mu_i} F_{\tau + \mu, \tau_1 + \mu_1, \dots, \tau_n + \mu_n}(z).$$

Hence, (36) implies

$$(37) \quad |F_{\tau\tau_1 \dots \tau_n}^{(t)}(p)| \leq \exp \left\{ -\frac{1}{4}U \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_{r+1} - 1$, $0 \leq t \leq T_{r+1} - 1$ and $0 \leq p \leq P_r - 1$.

For the same values of $\tau, \tau_1, \dots, \tau_n$ we obtain from (31)

$$(38) \quad \max_{|z| \leq 6P_{r+1}} |F_{\tau_1 \dots \tau_n}(z)| \leq \exp \{c_2 N^{n+1+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+6)\delta}\}.$$

We apply Lemma 7 of [4] to $F_{\tau_1 \dots \tau_n}$ with $R = P_{r+1}$, $A = 6$, $T = T_{r+1}$ and $P = P_r$. From (37), (38) and (34) we then obtain

$$\max_{|z| \leq P_{r+1}} |F_{\tau_1 \dots \tau_n}(z)| \leq \exp \{-2^{-(r+3)} N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+7)\delta}\}.$$

Consequently,

$$|F_{\tau_1 \dots \tau_n}(p)| \leq \exp \{-2^{-(r+3)} N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+7)\delta}\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_{r+1} - 1$ and $0 \leq p \leq P_{r+1} - 1$. From (32) and (34), it follows that

$$(39) \quad |\Phi_{\tau_1 \dots \tau_n}(p)| \leq \exp \{-2^{-(r+4)} N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+7)\delta}\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_{r+1} - 1$ and $0 \leq p \leq P_{r+1} - 1$.

However, for these values of $\tau, \tau_1, \dots, \tau_n$ and p , we can consider $\Phi_{\tau_1 \dots \tau_n}(p)$ as a polynomial in $\xi, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n$, of degree less than $LP_{r+1} + N$ in ξ, KP_{r+1} in $\alpha_1, \dots, \alpha_n$ and T_{r+1} in $\beta_0, \beta_1, \dots, \beta_n$. The sum of the absolute values of its coefficients is at most

$$\exp \{2N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+6)\delta}\}.$$

According to Lemma 3 of [4] we have either $\Phi_{\tau_1 \dots \tau_n}(p) = 0$ or

$$(40) \quad |\Phi_{\tau_1 \dots \tau_n}(p)| \geq \exp \{-c_3 N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+6)\delta}\}.$$

Hence,

$$\Phi_{\tau_1 \dots \tau_n}(p) = 0 \text{ for } 0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_{r+1} - 1 \text{ and } 0 \leq p \leq P_{r+1} - 1.$$

From (32) and (34) we see

$$|F_{\tau_1 \dots \tau_n}(p)| \leq \exp \{-\frac{1}{3}U\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_{r+1} - 1$ and $0 \leq p \leq P_{r+1} - 1$, which proves the lemma.

From (35) with $r = R$ we get

$$|F_{\tau_1 \dots \tau_n}(p)| \leq \exp \{-\frac{1}{3}U\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_R - 1$ and $0 \leq p \leq P_R - 1$. We have $T_R = T'$. From $R \geq n/\delta + 2n^2 + 5n + 2$ we see

$$P_R \geq \left[\frac{1}{2} N^{n^2+n+1+(2n^3+7n^2+7n+2)\delta} S^{n+(2n^2+5n+4)\delta} \right] \geq P'$$

Thus,

$$|F_{\tau_1 \dots \tau_n}(p)| \leq \exp \left\{ -\frac{1}{3} U \right\}$$

for $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T' - 1$ and $0 \leq p \leq P' - 1$. From (30) we now obtain

$$(41) \quad |F^{(t)}(p)| \leq \exp \left\{ -\frac{1}{4} U \right\}$$

for $t = 0, 1, \dots, T' - 1$ and $p = 0, 1, \dots, P' - 1$.

The exponents of F have the form

$$\ell \beta_0 + (k_1 + \ell \beta_1) \log \alpha_1 + \dots + (k_n + \ell \beta_n) \log \alpha_n.$$

Let Ω and ω have the same meaning as in Lemma 8 of [4]. Then

$$(42) \quad \Omega \leq c_4 N^{n+(n^2+n)\delta} S^{1+(2n+3)\delta},$$

from which the condition

$$T'P' \geq 2K^rLM + 13\Omega P'$$

follows by direct computation.

The difference of two exponents of F is of the form

$$\ell \beta_0 + (k_1 + \ell \beta_1) \log \alpha_1 + \dots + (k_n + \ell \beta_n) \log \alpha_n$$

with integral k_1, \dots, k_n, ℓ , not all zero, and $|k_i| \leq K - 1$ for $i = 1, \dots, n$ and $|\ell| \leq L - 1$. Moreover, at least one of the numbers $k_i + \ell \beta_i$ ($i = 1, \dots, n$) is non-zero, since β_1, \dots, β_n are not all rational. The degrees of $\ell \beta_0, k_1 + \ell \beta_1, \dots, k_n + \ell \beta_n$ are constants. We estimate their heights by means of Lemma 3; we then see that these heights do not exceed

$$c_5(2KL)^{c_6} \leq S^{c_7}$$

in which c_5 and c_6 are upper bounds for the heights and degrees resp. of $\beta_0, \beta_1, \dots, \beta_n$. From Lemma 1 with $\varepsilon = \delta$ it follows that

$$|\ell \beta_0 + (k_1 + \ell \beta_1) \log \alpha_1 + \dots + (k_n + \ell \beta_n) \log \alpha_n| > \exp \left\{ -(\log S)^{1+2\delta} \right\}.$$

Hence, the exponents of F are distinct and

$$(43) \quad \omega > \exp \left\{ -(\log S)^{1+2\delta} \right\} > \exp \left\{ -S^\delta \right\}.$$

From Lemma 8 of [4], using (41), (42) and (43) we obtain the inequality

$$(44) \quad \left| \sum_{v=0}^{N-1} C_{k_1 \dots k_n \ell m v} \xi^v \right| \leq \exp \left\{ -\frac{1}{5} U \right\}$$

for $k_1, \dots, k_n = 0, 1, \dots, K - 1$; $\ell = 0, 1, \dots, L - 1$ and $m = 0, 1, \dots, M - 1$.

According to Lemma 3 of [4] we have either

$$(45) \quad \sum_{v=0}^{N-1} C_{k_1 \dots k_n l m v} \xi^v = 0 \text{ or} \\ \left| \sum_{v=0}^{N-1} C_{k_1 \dots k_n l m v} \xi^v \right| > \exp \{ -2N^{n+2+(n^2+2n+1)\delta} S^{1+(2n+6)\delta} \}$$

for the same values of k_1, \dots, k_n, l and m . It follows that

$$\sum_{v=0}^{N-1} C_{k_1 \dots k_n l m v} \xi^v = 0$$

for all of these values. Since ξ has the degree N , this implies that all integers $C_{k_1 \dots k_n l m v}$ are zero, in contradiction to their choice. The theorem has been proved.

Using the fact, that there are only finitely many algebraic numbers ξ of size $S < S_2$, and using Lemma 9 of [4], one immediately obtains the following theorem:

THEOREM 5: *Under the conditions of Theorem 4, there exists an effectively computable, number $C_2 = C_2(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \beta_0, \beta_1, \dots, \beta_n) > 0$ such that*

$$\exp \{ -C_2 N^{n^2+2n+2+\varepsilon} S^{n+1+\varepsilon} \}$$

is a transcendence measure of $e^{\beta_0} \alpha_1^{\beta_1} \dots \alpha_n^{\beta_n}$.

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Instituut voor Propedeutische Wiskunde
Universiteit van Amsterdam
Roetersstraat 15
Amsterdam.