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METRICS IN LOCALLY COMPACT GROUPS

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According to a well-known theorem due to Birkhoff [1] and Kakutani [2], a topological group is metrizable if and only if it is first countable. (All topological groups are understood to be T_0.) In this case, the metric can be taken to be left invariant. If the group is also locally compact, then the spheres with sufficiently small radii are bounded (i.e., contained in compact sets). It is natural to ask: when does a group possess a metric such that *all* spheres of this metric are bounded? One sees immediately that such a group is second countable since it is necessarily the union of a countable family of compact sets (closed spheres), each of which is separable. It is the main purpose of this paper to prove a converse statement.

THEOREM: *A locally compact topological group G is metrizable with a left invariant metric all of whose spheres are bounded, if and only if G is second countable.*

Before giving the proof of this theorem we first prove two lemmas, the first of which yields a new proof of the Birkhoff-Kakutani theorem in the locally compact case. It is based on familiar properties of the Haar measure (c.f. [3].)

LEMMA 1: *Let G be a locally compact group with left Haar measure μ. If $\{V_n\}$ $n = 1, 2, \cdots$ is a decreasing countable (open) base at the identity e of G with V_1 bounded, then*

$$\rho(x, y) = \sup_n \mu(xV_n \,\Delta\, yV_n)$$

(where Δ is the symmetric difference) defines a left invariant metric in G which is compatible with the topology of G.

PROOF: It is clear that $\rho(x, y) = \rho(y, x)$ is well defined, is non-negative and finite, since the V_n are measurable and bounded and that $\rho(zx, zy) = \rho(x, y)$ for all x, y and z of G, since μ is left invariant. Moreover, since G is Hausdorff, if $x \neq y$, there exists an m such that the intersection $xV_m \cap yV_m$

is empty, and so $\rho(x, y) \geqq \mu(xV_m \Delta yV_m) = 2\mu(V_m) > 0$. Since for any n,

$$xV_n \Delta yV_n \subset (xV_n \Delta zV_n) \cup (zV_n \Delta yV_n),$$

we have also $\rho(x, y) \leqq \rho(x, z)+\rho(y, z)$ for all x, y and z of G. Thus ρ is a metric function in G. If the topology of G is discrete, then $\mu(\{e\}) > 0$ and there exists an m such that $V_m = \{e\}$. Hence for $x \neq y$, we have $\rho(x, y) \geqq \mu(xV_m \Delta yV_m) = \mu(\{x, y\}) = 2\mu(\{e\}) > 0$ and so the topology induced by ρ is also discrete. Suppose now that the topology of G is not discrete, then $\mu(V_n) \to 0$ as $n \to \infty$. If V is a neighborhood of e, there exists an m such that $V_m V_m^{-1} \subset V$, where as usual, $V_m^{-1} = \{x \in G: x^{-1} \in V_m\}$ and $V_m V_m^{-1}$ denotes the collection of all product pairs. We maintain that the ρ-sphere centered at e and of radius $\mu(V_m)$ is contained in V. Indeed, if $\rho(x, e) < \mu(V_m)$, then $\mu(xV_m \Delta V_m) < \mu(V_m)$. Hence $xV_m \cap V_m$ is not empty and so there exist $v_1,\ v_2 \in V_m$ such that $xv_1 = v_2$. Hence $x = v_2 v_1^{-1} \in V_m V_m^{-1} \subset V$. Now consider a ρ-sphere centered at e and of radius $r > 0$. There exists an m such that $\mu(V_n) < r/4$ for all $n \geqq m$. For each $k = 1, 2, \cdots, m-1$, the function $f_k(x) = \mu(xV_k \Delta V_k)$ is continuous in G [3]p. 266 and satisfies $f_k(e) = 0$. Hence there exists an l such that $\mu(xV_k \Delta V_k) < r$ for all $x \in V_l$ and $k = 1, 2, \cdots, m-1$. We maintain that V_l is contained in the ρ-sphere centered at e and of radius r. Indeed, if $x \in V_l$, then $\mu(xV_k \Delta V_k) < r$ for $k = 1, 2, \cdots, m-1$ and $\mu(xV_n \Delta V_n) \leqq 2\mu(V_n) < r/2$ for all $n \geqq m$. Thus $\rho(x, e) < r$. The last two results imply that the metric ρ is compatible with the topology of G and so the lemma is proved.

It is clear that the above metric ρ will not yield bounded spheres for all radii unless, perhaps, G is compact, since G is necessarily covered by each (open) sphere of radius greater than $2\mu(V_1)$. In the proof of the next lemma we make use of a new construction reminiscent of that used by Birkhoff and Kakutani, but designed to modify the metric ρ for the larger radii so that all the modified spheres become bounded.

LEMMA 2: *Let G be a locally compact, second countable group. Then there exists a family of open subsets U_r (r ranging over all positive numbers) satisfying the following conditions: For all positive r and s,* (1) *U_r is bounded,* (2) *$U_r = U_r^{-1}$ and* (3) *$U_r U_s \subset U_{r+s}$.* (4) *The family $\{U_r\}$ for $r > 0$ is a base at e.* (5) *$\bigcup_{r>0} U_r = G$.*

PROOF: Since G is first countable, by Lemma 1 there exists a left invariant, compatible metric ρ. Since G is locally compact, we may assume that the (open) spheres

$$S_r = \{x \in G: \rho(x, e) < r\}$$

are bounded for $0 < r \leqq 2$. For each r satisfying $0 < r < 2$ we define $U_r = S_r$. We observe that U_r is open and bounded, $U_r = U_r^{-1}$ and

$U_r U_s \subset U_{r+s}$ for $0 < r < 2$ and $0 < s < 2$ with $r+s < 2$. (For the last property note that $\rho(xy, e) \leqq \rho(xy, x)+\rho(x, e) = \rho(y, e)+\rho(x, e)$.) Moreover, the family $\{U_r\}$ for $0 < r < 2$ is a base at e. Since G is locally compact and second countable, there exists a countable open base $\{W_{2^n}\}$, $n = 1, 2, \cdots$ for the topology of G in which each W_{2^n} is bounded. We may assume further that $W_{2^n} = W_{2^n}^{-1}$ for $n = 1, 2, \cdots$. We then define $U_2 = S_2 \cup W_2$ and observe that the above stated properties hold for $0 < r \leqq 2$ and $0 < s \leqq 2$ with $r+s \leqq 2$ and in addition that we have $W_2 \subset U_2$. We proceed by induction and so assume that we have defined the family of open bounded sets U_r for $0 < r \leqq 2^n$ such that $U_r = U_r^{-1}$ and $U_r U_s \subset U_{r+s}$ for $0 < r \leqq 2^n$ also we may assume that the family $\{U_r\}$ is a base at e and that $W_{2^m} \subset U_{2^m}$ for $m = 1, 2, \cdots, n$. Let $2^n < r < 2^{n+1}$. Then we define

$$U_r = \bigcup U_{t_1} U_{t_2} \cdots U_{t_m}$$

where the union ranges over all finite subsets of positive numbers $\{t_1, t_2, \cdots, t_m\}$ satisfying $0 < t_i \leqq 2^n$ and $\sum_{i=1}^m t_i = r$. We observe that for any r satisfying $2^n < r < 2^{n+1}$ and any finite set of positive numbers t_i satisfying $r = \sum_{i=1}^m t_i$ and $0 < t_i \leqq 2^n$ there must exist integers k and l such that $1 \leqq k \leqq l < m$ and

$$\sum_{i=1}^{k} t_i \leqq 2^n, \qquad \sum_{i=k+1}^{l} t_i \leqq 2^n \quad \text{and} \quad \sum_{i=l+1}^{m} t_i \leqq 2^n.$$

It follows that $U_{t_1} \cdots U_{t_m} \subset U_{2^n} U_{2^n} U_{2^n}$, and hence that $U_r \subset U_{2^n} U_{2^n} U_{2^n}$ for $0 < r < 2^{n+1}$.

Since U_{2^n} is bounded, $U_{2^n} U_{2^n} U_{2^n}$ is bounded and hence U_r is bounded whenever $0 < r < 2^{n+1}$. Since $(U_{t_1} \cdots U_{t_m})^{-1} = U_{t_m}^{-1} \cdots U_{t_1}^{-1}$ we have $U_r = U_r^{-1}$ whenever $0 < r < 2^{n+1}$. Also since $(U_{t_1} \cdots U_{t_m})(U_{\tau_1} \cdots U_{\tau_j}) = U_{t_1} \cdots U_{t_m} U_{\tau_1} \cdots U_{\tau_j}$ and $(t_1 + \cdots + t_m)+(\tau_1 + \cdots + \tau_j) = t_1 + \cdots t_m + \tau_1 + \cdots + \tau_j$ we have $U_r U_s \subset U_{r+s}$ whenever $0 < r$, $0 < s$, $r+s < 2^{n+1}$. Suppose now $0 < r$, $0 < s$ and $r+s = 2^{n+1}$. If $r = s = 2^n$ then $U_r U_s = U_{2^n} U_{2^n}$. If $r > 2^n$ then $s < 2^n$. Since $r < 2^{n+1}$ we have

$$U_r \subset U_{2^n} U_{2^n} U_{2^n}.$$

Hence

$$U_r U_s \subset (U_{2^n} U_{2^n} U_{2^n}) U_{2^n}.$$

In any event we have

$$U_r U_s \subset U_{2^n} U_{2^n} U_{2^n} U_{2^n}.$$

We then define $U_{2^{n+1}} = U_{2^n} U_{2^n} U_{2^n} U_{2^n} \cup W_{2^{n+1}}$ and readily verify that: U_r is open and bounded and $U_r = U_r^{-1}$ for $0 < r \leqq 2^{n+1}$, $U_r U_s \subset U_{r+s}$

whenever $0 < r$, $0 < s$ and $r+s \leqq 2^{n+1}$, the family $\{U_r\}$ for $0 < r \leqq 2^{n+1}$ is a base at e, and $W_{2^m} \subset U_{2^m}$ for $m = 1, 2, \cdots, n+1$. Thus by induction the family $\{U_r\}$ for all $r > 0$ is defined and satisfies the five conditions of the lemma.

Finally, to prove the theorem we need only construct the desired metric for a locally compact, second countable group G. Let $\{U_r\}$, $r > 0$ be the family of open bounded sets constructed in Lemma 2. For each pair of elements x and y of G we define

$$d(x, y) = \inf \{r : y^{-1}x \in U_r\}$$

and maintain that d satisfies the conditions of the theorem. It is clear from condition (5) that for any pair x and y of G, $y^{-1}x \in U_r$ for some r and that $d(x, y) \geqq 0$. If $y^{-1}x \neq e$, then by condition (4) there exists an $r_0 > 0$ such that $y^{-1}x \notin U_{r_0}$. Since condition (3) implies that $U_{r_0} \subset U_r$ for $0 < r_0 \leqq r$, we have $d(x, y) \geqq r_0$. On the other hand, e belongs to each U_r and so $d(x, x) = 0$. Since $U_r = U_r^{-1}$, we have $y^{-1}x \in U_r$ if and only if $x^{-1}y \in U_r$ and thus $d(x, y) = d(y, x)$. We note further that for $x, y, z \in G$, we have

$$d(zx, zy) = \inf \{r : (zy)^{-1}zx \in U_r\} = \inf \{r : y^{-1}x \in U_r\} = d(x, y).$$

Hence d is a left invariant metric. Now for each $r > 0$, the d-sphere centered at e and of radius r is contained in U_r while this same d-sphere contains $U_{r'}$ for any $0 < r' < r$. Thus by condition (4) the metric d is compatible with the topology of G and by condition (1) all d-spheres are bounded. This completes the proof of the theorem.

Remark: It is known [4] p. 79 that a topological group G has a compatible, two-sided invariant metric d if and only if G has a countable open base $\{V_n\}$, $n = 1, 2, \cdots$ at the identity e of G which satisfies $xV_nx^{-1} = V_n$ for $x \in G$ and $n = 1, 2, \cdots$. If the d-spheres are also to be bounded it is necessary that G be second countable and have the property that $\bigcup_{x \in G} xKx^{-1}$ be bounded whenever K is bounded. All the above constructions can be carried out to show that these last two properties of G are also sufficient to obtain a compatible, two-sided invariant metric in G all of whose spheres are bounded.

Example 1: Let G be the group of all matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ for $a > 0$ and b real, together with the Euclidean topology of the half plane. This is a well-known example of a non-Abelian, second countable, locally compact group. For each $r > 0$ let

$$U_r = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : \begin{matrix} e^{-r} < a < e^r \\ |b| < \min(are^r, re^r) \end{matrix} \right\}.$$

It is clear that this family $\{U_r\}$ for $r > 0$ satisfies conditions (1), (4) and

(5) of Lemma 2. We shall verify (2) and (3) as well. Indeed if $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \varepsilon U_r$ and $\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1}$, then $e^{-r} < a < e^r$, $|b| < \min(are^r, re^r)$ and $ac = 1$, $ad + b = 0$. Hence $e^{-r} < c < e^r$, and

$$|d| = c|b| \leqq c \min(are^r, re^r) = \min(re^r, cre^r).$$

Thus $\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \in U_r$ also and so $U_r^{-1} \subset U_r$.By a symmetrical argument, $U_r \subset U_r^{-1}$ and hence condition (2) is satisfied. Now let $r > 0$ and $s > 0$. Then $U_r U_s$ is the set of all $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a = xz$, $b = xw + y$ and x, y, z, w vary according to the inequalities

$$e^{-r} < x < e^r, \qquad |y| < \min(xre^r, re^r)$$

and

$$e^{-s} < z < e^s, \qquad |w| < \min(zse^s, se^s).$$

By a straightforward and elementary computation, it follows that

$$e^{-r-s} < a < e^{r+s}, \qquad |b| < \min[a(r+s)e^{r+s}, (r+s)e^{r+s}]$$

and hence, $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in U_{r+s}$. Thus we see that the family $\{U_r\}$ for $r > 0$ satisfies all the conditions of Lemma 2 and that

$$d\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}\right) = \inf\{r : |\log a - \log c| < r, |b-d| < re^r \min(a, c)\}$$

defines a compatible, left invariant metric d in G all of whose spheres are bounded.

In Example 1, it is easy to see that the function defined by $f(r) = \mu(S_r)$, where S_r is the open d-sphere centered at e and of radius r, is continuous. In the general case, the regularity of the Haar measure implies only that f is lower semi-continuous. Similarly, the function defined by $g(r) = \mu(T_r)$, where T_r is the closed d-sphere centered at e and of radius r, is upper semi-continuous and f and/or g is continuous at r_0 if and only if $f(r_0) = g(r_0)$. However, continuity need not be present, as seen in the following example.

EXAMPLE 2: Let G_n, $n = 1, 2, \cdots$ be a sequence of finite groups considered as discrete compact groups. Then the product $G = \prod_{n=1}^{\infty} G_n$ is a compact group, by the Tychonov theorem. Let $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ belong to G. For $x = y$ we define $d(x, y) = 0$ and for $x \neq y$ we define $d(x, y) = 1/N(x, y)$, where $N(x, y)$ is the least integer r such that $x_n \neq y_n$. It is not difficult to verify that d is a two-sided invariant metric function. In particular,

$$d(x^{-1}y, z^{-1}w) \leqq d(x^{-1}y, x^{-1}w) + d(x^{-1}w, z^{-1}w) = d(y, w) + d(x, z),$$

and hence the group operations in G are continuous in the topology induced by d. This is clearly the product topology of G and since G is compact, all d-spheres are bounded. In this case, $S_r = S_{1/n}$ for $1/n+1 < r \leqq 1/n$, while $S_{1/n+1}$ is closed as well as open. Hence $S_{1/n} - S_{1/n+1}$ is a nonempty open Borel set and so

$$f(1/n) - f(1/n+1) = \mu(S_{1/n} - S_{1/n+1}) > 0$$

and f is discontinuous at $r = 1/n$, $n = 1, 2, \cdots$.

These results are contained in the author's 1951 Ph.D. dissertation written at the University of Notre Dame under the direction of Professor Ky Fan. Other results were published earlier in [5].

REFERENCES

[1] G. Birkhoff: A note on topological groups, *Compositio Math.*, *3* (1936) 427–430.
[2] S. Kakutani: Über die Metrisation der topologischen Gruppen, *Proc. Imp. Acad. Tokyo*, *12* (1936) 82–84.
[3] P. Halmos: *Measure Theory*, D. Van Nostrand, New York, 1950.
[4] E. Hewitt and K. Ross: *Abstract Harmonic Analysis*, Springer-Verlag, Berlin, 1963.
[5] R. Struble: Almost periodic functions on locally compact groups, *Proc. Nat. Acad. Sci.*, U.S.A., *39* (1953) 122–126.

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