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## A CONDITION EQUIVALENT TO COVERING DIMENSION FOR NORMAL SPACES

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In this paper a concept called *boundary covering dimension* is defined. Boundary covering dimension is proven to be equivalent to covering dimension for normal spaces. Also included is a definition of complete boundary covering dimension. Complete boundary covering dimension is proven to be equivalent to complete covering dimension for paracompact  $T_2$ -spaces (complete covering dimension is equivalent to covering dimension for paracompact  $T_2$ -spaces).

NOTATIONS: If  $X$  is a space and  $V \subset X$ , then  $B(V)$  denotes the boundary of  $V$ . If  $X$  is a space,  $M \subset X$ , and  $H \subset M$ , then  $B(M, H)$  denotes the boundary in the subspace  $M$  of  $H$ .

DEFINITIONS: The collection  $G$  of subsets of the space  $X$  is *discrete* means every point of  $X$  is contained in an open set that intersects at most one element of  $G$ .

*Covering dimension* is denoted by *dim*.  $\dim X \leq n$  means if  $G$  is a *finite* open cover of  $X$ , then there exists an open cover  $R$  of  $X$  such that  $R$  refines  $G$  and  $\text{ord } R \leq n + 1$ .

*Boundary covering dimension* is denoted by *bcd*. For  $n \geq 1$ ,  $\text{bcd } X \leq n$  means if  $H$  is a closed set,  $W$  is an open set,  $H \subset W$ , and  $G$  is a *finite* open cover of  $X$ , then there are an open set  $V$  and discrete collections  $G_1, G_2, \dots, G_n$  of closed sets such that  $H \subset V \subset W$ ,  $\bigcup_{j=1}^n G_j$  refines  $G$ , and  $B(V) = \bigcup (\bigcup_{j=1}^n G_j)$ . Now  $\text{bcd } X = n$  means  $\text{bcd } X \leq n$  and  $\text{bcd } X \not\leq n - 1$ .

*Complete covering dimension* is denoted by *complete dim*. Complete  $\dim X \leq n$  means if  $G$  is an open cover of  $X$ , then there exists an open cover  $R$  of  $X$  such that  $R$  refines  $G$  and  $\text{ord } R \leq n + 1$ .

*Complete boundary covering dimension* is denoted by *complete bcd*. For  $n \geq 1$ , complete  $\text{bcd } X \leq n$  means if  $H$  is a closed set,  $W$  is an open set,  $H \subset W$ , and  $G$  is an open cover of  $X$ , then there exist an open set  $V$  and discrete collections  $G_1, G_2, \dots, G_n$  of closed sets such that  $H \subset V \subset W$ ,  $\bigcup_{j=1}^n G_j$  refines  $G$ , and  $B(V) = \bigcup (\bigcup_{j=1}^n G_j)$ .

<sup>1</sup> The work for that paper was done while the author was on a Cottrell College Science Grant for Research Corporation.

REMARK: What is meant by  $\text{bcd } X \leq 0$ ? Let us note that the definition for  $\text{bcd}$  can be written another way.  $\text{bcd } X \leq n$  means if  $H$  is a closed set,  $W$  is an open set,  $H \subset W$ , and  $G$  is a finite open cover of  $X$ , then there are an open set  $V$  and a collection  $T$  of at most  $n$  elements such that  $H \subset V \subset W$ , each element of  $T$  is a discrete collection of closed sets,  $\bigcup T$  refines  $G$ , and  $B(V) = \bigcup (\bigcup T)$ . So when  $n = 0$ ,  $T = \phi$  and  $B(V) = \phi$ . Thus,  $\text{bcd } X \leq 0$  means  $\text{Ind } X \leq 0$ . In our proofs we will not be considering the case where  $n = 0$  since it will be evident what the proof would be for  $n = 0$ .

LEMMA 1: *If  $X$  is a topological space,  $\text{bcd } X \leq n$ , and  $M$  is a closed subset of  $X$ , then  $\text{bcd } M \leq n$ . (The proof is straight-forward and will not be given.)*

LEMMA 2: *If  $X$  is a topological space and  $\text{bcd } X \leq n$ , then if  $H$  is a closed set,  $W$  is an open set,  $H \subset W$ , and  $G$  is a finite open cover of  $X$ , then there are an open set  $V$  and finite discrete collections  $G_1, G_2, \dots, G_n$  of closed sets such that  $H \subset V \subset W$ ,  $\bigcup_{i=1}^n G_i$  refines  $G$ , and  $B(V) = \bigcup (\bigcup_{i=1}^n G_i)$ . (The proof is straight-forward and will not be given.)*

LEMMA 3: *If each of  $G_1, G_2, \dots, G_n$  is a finite open cover of the topological space  $X$ , then there is a finite open cover  $G$  of  $X$  such that for every  $i \in \{1, \dots, n\}$ ,  $G$  refines  $G_i$ .*

PROOF: For every  $p \in X$ , let  $T(p) = \{g | \exists i \in \{1, \dots, n\} \text{ such that } g \in G_i \text{ and } p \in g\}$ . Let  $G = \{\bigcap T(p) | p \in X\}$ .  $G$  is a finite open cover of  $X$  such that for every  $i \in \{1, \dots, n\}$ ,  $G$  refines  $G_i$ .

LEMMA 4: *If  $X$  is a paracompact  $T_2$ -space,  $M \subset X$ ,  $M$  is closed,  $n$  is a positive integer,  $G$  is a collection of open sets of  $X$  covering  $M$ , and no point of  $M$  belongs to  $n+1$  elements of  $G$ , then there exist discrete collections  $G_1, G_2, \dots, G_n$  of closed sets such that  $\bigcup_{j=1}^n G_j$  refines  $G$  and  $\bigcup (\bigcup_{j=1}^n G_j) = M$ .*

PROOF: Since every paracompact  $T_2$ -space is collectionwise normal, Theorem 2 of [1] can be applied to prove the Lemma.

LEMMA 5: *If  $X$  is a normal topological space,  $M \subset X$ ,  $M$  is closed,  $n$  is a positive integer,  $G$  is a finite collection of open sets of  $X$  covering  $M$ , and no point of  $M$  belongs to  $n+1$  elements of  $G$ , then there exist discrete collections  $G_1, G_2, \dots, G_n$  of closed sets such that  $\bigcup_{j=1}^n G_j$  refines  $G$  and  $\bigcup (\bigcup_{j=1}^n G_j) = M$ .*

PROOF: The proof is similar to the proofs of Theorem 1 and Theorem 2 of [1]. Only normality is needed instead of collectionwise normality since the open cover  $G$  is finite.

**THEOREM 1:** *If  $X$  is a normal topological space, then  $\text{bcd } X = \text{dim } X$ .*

**PROOF:**

*Part I: Show  $\text{dim } X \leq \text{bcd } X$ .* Assume  $n$  is a positive integer and  $\text{bcd } X \leq n$ . Assume  $G$  is a finite open cover of  $X$ . Let  $G = \{g_1, \dots, g_m\}$ . Let  $H_1 = g_1 - (\bigcup_{j=2}^m g_j) = X - \bigcup_{j=2}^m g_j$ . Now  $g_1$  is an open set containing the closed set  $H_1$ . Since  $\text{bcd } X \leq n$ , by Lemma 2, there exist an open set  $V_1$ , and finite discrete collections  $L_1, L_2, \dots, L_n$  of closed sets such that  $H_1 \subset V_1 \subset g_1$ ,  $\bigcup_{j=1}^n L_j$  refines  $G$ , and  $B(V_1) = \bigcup (\bigcup_{j=1}^n L_j)$ . For every  $j \in \{1, \dots, n\}$ , let  $S(1, j) = L_j$ . Let  $X_1 = X$ .

Assume  $k$  is a positive integer such that  $1 \leq k \leq m$  and for every  $i \in \{1, \dots, k\}$ ,

$$(a) X_i = X - \bigcup_{j=1}^{i-1} V_j = X_{i-1} - V_{i-1}$$

$$(b) H_i = X_i - \bigcup_{j=i+1}^m g_j$$

(c)  $H_i \subset V_i \subset g_i$ ,  $V_i \subset X_i$ ,  $V_i$  open in  $X_i$  (Hence  $X_i$  and  $H_i$  are closed in  $X$ )

(d)  $\forall j \in \{1, \dots, n\}$ ,  $S(i, j)$  is a finite discrete collection of closed sets and  $S(i, j)$  refines  $G$ , and

$$(e) \bigcup_{j=1}^n (\bigcup_{i=1}^k S(i, j)) = \bigcup_{j=1}^n B(X_j, V_j).$$

Now let  $X_{k+1} = X - \bigcup_{j=1}^k V_j = X_k - V_k$  and let  $H_{k+1} = X_{k+1} - \bigcup_{j=k+2}^m g_j$

Now  $H_{k+1} \subset g_{k+1}$ . For every  $j \in \{1, \dots, n\}$ , let  $E_j = \{e(j, w) | w \in S(k, j)\}$  and  $F_j = \{f(j, w) | w \in S(k, j)\}$  be finite discrete collections of open sets such that  $F_j$  refines  $G$ ,  $\forall w \in S(k, j)$   $w \subset e(j, w) \subset \overline{e(j, w)} \subset f(j, w)$  and  $f(j, w)$  intersects only one element of  $S(k, j)$ , and let  $T_j = \{f(j, w) \cap X_{k+1} | w \in S(k, j)\} \cup \{g - (\bigcup E_j) \cap X_{k+1} | g \in G\}$ . By Lemma 3, there is a finite cover  $T$  of  $X_{k+1}$  such that each element of  $T$  is open in  $X_{k+1}$ , and for every  $j \in \{1, \dots, n\}$ ,  $T$  refines  $T_j$ . By Lemma 1,  $\text{bcd } X_{k+1} \leq n$  so by Lemma 2 there exist a set  $V_{k+1}$ , open in  $X_{k+1}$ , and finite discrete collections  $G_1, G_2, \dots, G_n$  of closed sets such that  $H_{k+1} \subset V_{k+1} \subset g_{k+1}$ ,  $\bigcup_{j=1}^n G_j$  refines  $T$ , and  $B(X_{k+1}, V_{k+1}) = \bigcup (\bigcup_{j=1}^n G_j)$ .  $\forall j \in \{1, \dots, n\}$ ,  $\forall w \in S(k, j)$ , let  $b(j, w) = \{w\} \cup \{h | h \in G_j \text{ and } h \subset f(j, w)\}$ .  $\forall j \in \{1, \dots, n\}$ , let  $M_j = \{h | h \in G_j \text{ and } \forall w \in S(k, j), h \not\subset b(j, w)\}$  and let  $S(k+1, j) = \{\bigcup b(j, w) | w \in S(k, j)\} \cup M_j$ .  $\forall j \in \{1, \dots, n\}$ ,  $S(k+1, j)$  is a finite collection of closed sets and  $S(k+1, j)$  refines  $G$ .

Assume  $j \in \{1, \dots, n\}$ . It will now be shown that  $S(k+1, j)$  is discrete. Since  $S(k+1, j)$  is finite, we need only to show that no two elements of  $S(k+1, j)$  intersect. It should be clear that no two elements of  $M_j$  intersect and no two elements of  $\{\bigcup b(j, w) | w \in S(k, j)\}$  intersect. Assume  $\exists w_0 \in S(k, j)$  and  $h_0 \in M_j$  such that  $\bigcup b(j, w_0)$  intersects  $h_0$ .

*Case 1:*  $\exists h_1 \in G_j$  such that  $h_1 \subset f(j, w_0)$  and  $h_1$  intersects  $h_0$ . Since no two elements of  $G_j$  intersect,  $h_0 = h_1$ .  $\forall w \in S(k, j)$ ,  $h_0 \notin b(j, w)$  since

$h_0 \in M_j$ . But  $h_0$ , which is  $h_1$ , is an element of  $b(j, w_0)$ . Contradiction.

*Case 2:  $h_0$  intersects  $w_0$ .* Since  $G_j$  refines  $T$  which refines  $T_j$ , there is an element  $g_0$  of  $T_j$  such that  $h_0 \subset g_0$ . Thus  $g_0$  intersects  $w_0$ , and  $w_0 \subset \bigcup E_j$ . No element of  $\{[g - (\bigcup E_j)] \cap X_{k+1} \mid g \in G\}$  intersects  $\bigcup E_j$  so  $g_0 \in \{f(j, w) \cap X_{k+1} \mid w \in S(k, j)\}$ . Thus  $\exists w_1 \in S(k, j)$  such that  $g_0 = f(j, w_1) \cap X_{k+1}$ . This means  $h_0 \subset f(j, w_1)$ . So  $h_0 \in b(j, w_1)$ . Since  $h_0 \in M_j$ , we know that  $\forall w \in S(k, j)$ ,  $h_0 \notin b(j, w)$ . This means  $h_0 \notin b(j, w_1)$ , but  $h_0 \in b(j, w_1)$ . Contradiction. Therefore, no two elements of  $S(k+1, j)$  intersect.

It follows that  $\bigcup (\bigcup_{j=1}^n S(k+1, j)) = \bigcup_{j=1}^{k+1} B(X_j, V_j)$ . We have now completed our inductive definition. Thus each of  $S(m, 1), S(m, 2), \dots, S(m, n)$  is a finite discrete collection of closed sets that refines  $G$ .  $\forall j \in \{1, \dots, n\}$ , let  $Z_j$  be a finite discrete collection of open sets such that  $S(m, j)$  refines  $Z_j$  and  $Z_j$  refines  $G$ .  $\forall i \in \{1, \dots, m\}$ , let  $V'_i = V_i - [\bigcup (\bigcup_{j=1}^n Z_j)]$ . Now  $\{V'_1, V'_2, \dots, V'_m\}$  is a finite collection of mutually exclusive closed sets such that  $\forall i \in \{1, \dots, m\}$ ,  $V'_i \subset g_i$ . Let  $Z_{n+1} = \{a_1, \dots, a_m\}$  be a finite discrete collection of open sets such that  $\forall i \in \{1, \dots, m\}$ ,  $V'_i \subset a_i \subset g_i$ . Let  $Z = \bigcup_{j=1}^{n+1} Z_j$ .  $Z$  is an open cover of  $X$  such that  $Z$  refines  $G$  and  $\text{ord } Z \leq n+1$ . Thus  $\dim X \leq n$ .

*Part II: Show  $\text{bcd } X \leq \dim X$ .* Assume  $n$  is a positive integer and  $\dim X \leq n$ . Assume  $H$  is a closed set,  $W$  is an open set  $H \subset W$ , and  $G$  is a finite open cover of  $X$ . Let  $F$  be a finite open cover of  $X$  such that  $F$  refines  $G$  and every element of  $F$  that intersects  $H$  is a subset of  $W$ . Let  $T = \{t_i \mid i = 1, \dots, k\}$  be a finite open cover of  $X$  such that  $T$  refines  $F$ ,  $\text{ord } T \leq n+1$ , and if  $i \neq j$ , then  $t_i \neq t_j$ . Let  $R = \{r_i \mid i = 1, \dots, k\}$  be an open cover of  $X$  such that  $\forall i \in \{1, \dots, k\}$ ,  $r_i \subset t_i$ . Let  $V = \bigcup \{r_i \mid i \in \{1, \dots, k\} \text{ and } r_i \text{ intersects } H\}$ . Assume  $p \in B(V)$  and  $n+1$  elements of  $R$  contain  $p$ . There exist positive integers  $j_1 < j_2 < \dots < j_{n+1} \leq k$  such that  $\forall i \in \{1, \dots, n+1\}$ ,  $p \in r_{j_i}$ . Since  $R$  is finite,  $\exists j_{n+2} \in \{1, \dots, k\}$  such that  $p \in B(r_{j_{n+2}})$ .  $\forall i \in \{1, \dots, n+2\}$ ,  $p \in t_{j_i}$  since  $r_{j_i} \subset t_{j_i}$ . Thus,  $n+2$  elements of  $T$  contain  $p$ , which is a contradiction. Therefore no point of  $B(V)$  is contained by  $n+1$  elements of  $R$ . By Lemma 5, there exist discrete collections  $G_1, G_2, \dots, G_n$  of closed sets such that  $\bigcup_{j=1}^n G_j$  refines  $G$  and  $B(V) = \bigcup (\bigcup_{j=1}^n G_j)$ . So  $\text{bcd } X \leq n$ .

**THEOREM 2:** *If  $X$  is a paracompact  $T_2$ -space, then  $\text{bcd } X = \dim X = \text{complete bcd } X = \text{complete dim } X$ .*

**PROOF:** Assume  $X$  is a paracompact  $T_2$ -space. Theorem II.6 page 22 of [2] makes it clear  $\dim X = \text{complete dim } X$ , Theorem 1 gives us  $\text{bcd } X = \dim X$ . It is trivial that  $\text{bcd } X \leq \text{complete bcd } X$ . It will now be shown that  $\text{complete bcd } X \leq \text{bcd } X$ . Assume  $n$  is positive integer and  $\text{bcd } X \leq n$ . Thus  $\dim X \leq n$ , and hence  $\text{complete dim } X \leq n$ .

Assume  $H$  is a closed set,  $W$  is an open set,  $H \subset W$ , and  $G$  is an open cover of  $X$ . Let  $F$  be an open cover of  $X$  such that  $F$  refines  $G$  and every element of  $F$  that intersects  $H$  is a subset of  $W$ . Let  $T = \{t_b | b \in B\}$  be a locally finite open cover of  $X$  such that  $T$  refines  $F$ ,  $\text{ord } T \leq n+1$ , and if  $b_1, b_2 \in B$  and  $b_1 \neq b_2$  then  $t_{b_1} \neq t_{b_2}$  (Theorem 3 of [1] assures the existence of such a  $T$ ). Let  $R = \{r_b | b \in B\}$  be an open cover of  $X$  such that  $\forall b \in B, \overline{r_b} \subset t_b$ . Let  $V = \bigcup \{r_b | b \in B \text{ and } r_b \text{ intersects } H\}$ . Assume  $p \in B(V)$  and  $n+1$  elements of  $R$  contain  $p$ . There exist  $n+1$  elements  $b_1, b_2, \dots, b_{n+1}$  of  $B$  such that  $\forall i \in \{1, \dots, n+1\}, p \in r_{b_i}$ . Since  $R$  is locally finite, there exists  $b_{n+2} \in B$  such that  $p \in B(r_{b_{n+2}})$ .  $\forall i \in \{1, \dots, n+2\}, p \in t_{b_i}$ , since  $\overline{r_{b_i}} \subset t_{b_i}$ . Thus  $n+2$  elements of  $R$  contain  $p$ , which is a contradiction. Therefore, no point of  $B(V)$  is contained by  $n+1$  elements of  $R$ . By Lemma 4, there exist discrete collections  $G_1, G_2, \dots, G_n$  of closed sets such that  $\bigcup_{j=1}^n G_j$  refines  $G$  and  $B(V) = \bigcup (\bigcup_{j=1}^n G_j)$ . Thus  $\text{complete bcd } X \leq n$ . Therefore  $\text{bcd } X = \text{dim } X = \text{complete bcd } X = \text{complete dim } X$ .

**COROLLARY:** *Assume  $X$  is a normal topological space. Then  $\text{dim } X \leq n$  if and only if for all mutually exclusive closed sets  $H$  and  $K$ , for every finite (the word 'finite' can be deleted for  $X$  a paracompact  $T_2$ -space) open cover  $G$  of  $X$ , there exist mutually exclusive open sets  $D_H$  and  $D_K$  and a collection  $T$  of at most  $n$  elements such that  $H \subset D_H, K \subset D_K$ , every element of  $T$  is a discrete collection of closed sets,  $\bigcup T$  refines  $G$ , and  $X - (D_H \cup D_K) = \bigcup (\bigcup T)$ .*

**PROOF:** The proof follows from Theorem 1 (If  $X$  is  $T_2$ -paracompact and the open cover  $G$  is not necessarily finite, then the proof follows from Theorem 2).

**REMARK:** Note the similarity between the above Corollary and the following familiar theorem on large inductive dimension (denoted  $\text{Ind}$ ): For  $X$  normal,  $\text{Ind } X \leq n$  if and only if for all mutually exclusive open sets  $H$  and  $K$ , there exist mutually exclusive open sets  $D_H$  and  $D_K$  and a closed set  $T$  such that  $H \subset D_H, K \subset D_K, \text{Ind } T \leq n-1$ , and  $X - (D_H \cup D_K) = T$ . The similarity of the Corollary and this theorem on  $\text{Ind}$  enable one to pattern some  $\text{dim}$  proofs after some  $\text{Ind}$  proofs.

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