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To Oscar Zariski in honor of his 75th birthday

THE TRANSVERSALITY OF A GENERAL TRANSLATE

Steven L. Kleiman

Introduction

A basic principle of classical Schubert calculus says that each solution to the general case of an enumerative problem appears with multiplicity one. The principle is a consequence of the following assertion: Consider two subvarieties of a Grassmann variety; if the intersection of the one and a general translate of the other is nonempty, the components each appear with multiplicity one. This assertion, made in characteristic zero, is listed as Fact (ii) on p. 338 of Hodge-Pedoe [4], but is inadequately justified by an analogy.

A simple, elementary proof is given below for the following transversality theorem: Assume the characteristic is zero, and consider two subvarieties, Y and Z, of a homogeneous space X. Then, Z intersects a general translate of Y properly, and transversally if Y and Z are smooth. In fact, Y and Z need not be subvarieties of X, just be over X (fibered product replaces intersection); this extension permits drawing Bertini's theorem and related results as corollaries. There are two basic principles involved in the proof. The first is that translating a map from a smooth variety into a homogeneous space yields a smooth parametrized family of maps. The second is that, if a parametrized family of maps is smooth, then the generic individual map is transversal to a given map from a smooth variety. (There are analogous principles with 'smooth' replaced 'Cohen-Macaulay', 'normal', or the like [cf. (7) below].) The second principle is used, more or less explicitly, in a proof of Bertini's theorem given in an unpublished preliminary version of part of the treatise. 'Elements de Geometrie Algebrique', by A. Grothendieck and J. Dieudonné. It is also essentially used in Zariski's proof [10] of Bertini's theorem. The formulation here is inspired by the remark after Lemma 4.6 in [1]; thanks are due to M. Golubitsky for pointing it out, for suggesting a proof, given in (3) below, of the first principle, and for remarking that the theorem carries over to the differentiable case.

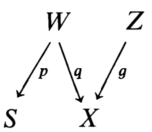
The situation is different in positive characteristic. For example [see (9) below], the Grassmann variety of 2-dimensional subspaces of a 4-

dimensional vector space contains a smooth subvariety that does not intersect a certain Schubert variety, nor any translate of it, transversally. However, the subvariety does not arise from an enumerative problem, and the principle of Schubert calculus, stated at the beginning, might still be valid in a useful form in positive characteristic.

In positive characteristic, the transversality result holds if the homogeneous space is a projective space. Indeed, the proof of Theorem I, p. 153, of Hodge-Pedoe [4] carries over. An updated version of it is presented in (10) below. This version underscores the action of the stability group of a point on its tangent space; this action is dramatically better in the case of a projective space than in the case of a more general Grassmann variety, contrary to the impression given in [4]. This version also yields the form of Bertini's theorem given in the unpublished part of the treatise by Grothendieck and Dieudonné cited above. It is easy, but instructive, to have examples showing Bertini's theorem needs more stringent hypotheses in positive characteristic. Zariski gives several examples in [10]; another example, one more in the spirit of this article, is given in (13) below.

Fix an algebraically closed ground field k.

1. Lemma: *Consider a diagram with integral algebraic schemes,*



(i) *Assume q is flat. Then, there exists a dense open subset U of S such that, for each point s in U, either the fibered product, $p^{-1}(s)\times_X Z$, is empty or it is equidimensional and its dimension is given by the formula,*

(1.1) $$\dim(p^{-1}(s)\times_X Z) = \dim(p^{-1}(s)) + \dim(Z) - \dim(X).$$

(ii) *Assume q is flat with (geometrically) regular fibers (in short, smooth). Assume Z is regular. Then, $p^{-1}(\sigma)\times_X Z$ is regular, where σ is the generic point of S, and, if the characteristic is zero, then $p^{-1}(s)\times_X Z$ is regular for each point s in an open dense subset of S.*

Proof:

(i) Since W and X are integral and q is flat, the nonempty fibers of q are equidimensional with dimension, $\dim(W)-\dim(X)$, by (EGA IV_2, 6.1.1). Hence, clearly, the nonempty fibers of the projection,

$$pr_Z : W\times_X Z \to Z,$$

are also equidimensional with this dimension by (EGA IV_2, 4.2.8). On the other hand, pr_Z is flat since q is flat (*EGA* IV_2, 2.1.4), and so, by (EGA IV_2, 6.1.1), the fiber through a point b of $W \times_X Z$ has at b dimension, $\dim_b(W \times_Z Z) - \dim(Z)$, because Z is integral. Consequently, $W \times_X Z$ is equidimensional, and its dimension is given by the formula,

$$\dim (W \times_X Z) = \dim (W) + \dim (Z) - \dim (X).$$

There exists a dense open subset U_1 of S such that, for each s in U, either the fiber $p^{-1}(s)$ is empty or it is equidimensional and its dimension is given by the formula,

$$\dim (p^{-1}(s)) = \dim (W) - \dim (S),$$

(in fact, there is a dense open subset of S over which p is flat (EGA IV_2, 6.9.1), and it will do (EGA IV_2, 6.1.1)). Similarly, since $W \times_X Z$ is equidimensional, there exists a dense open subset U_2 of S such that, for each s in U_2, either the fiber over s of the morphism,

$$(p \circ pr_W) \colon (W \times_X Z) \to S,$$

is empty or it is equidimensional and its dimension is given by the formula,

$$\dim ((p \circ pr_X)^{-1}(s)) = \dim(W \times_X Z) - \dim (S).$$

Now, for each s in S, the associativity formula,

$$\operatorname{Spec} (k(s)) \times_S (W \times_X Z) = (\operatorname{Spec} (k(s)) \times_S W) \times_X Z,$$

may be rewritten in form,

(1.2) $$(p \circ pr_W)^{-1}(s) = p^{-1}(s) \times_X Z.$$

Consequently, for each s in $U = U_1 \cap U_2$, either $p^{-1}(s) \times_X Z$ is empty or it is equidimensional and its dimension is given by formula (1.1).

(ii) Since q is smooth, $pr_Z \colon W \times_X Z \to Z$ is smooth (EGA IV_4, 17.3.3 or EGA IV_2, 6.8.3). Hence, since Z is regular, $W \times_X Z$ is regular (EGA IV_2, 6.5.2). Therefore, the generic fiber, $(p \circ pr_W)^{-1}(\sigma)$, is regular because, at each point, its local ring is obviously equal to the local ring of $W \times_X Z$. If the characteristic is zero, then $(p \circ pr_X)^{-1}(\sigma)$ is geometrically regular (EGA IV_2, 6.7.4) and so $(p \circ pr_W)^{-1}(s)$ is (geometrically) regular for each s in an open dense subset of S (EGA IV_3, 9.9.4). In view of formula (1.2), the proof is now complete.

2. Theorem: *Let G be an integral algebraic group scheme, X an integral algebraic scheme with a transitive G-action. Let $f: Y \to X$ and $g: Z \to X$ be two maps of integral algebraic schemes. For each rational point s of G, let sY denote Y considered as an X-scheme via the map, $y \mapsto sf(y)$.*

(i) *Then, there exists a dense open subset U of G such that, for each rational point s in U, either the fibered product, $(sY) \times_X Z$, is empty or it is equidimensional and its dimension is given by the formula,*

$$\dim((sY) \times_X Z) = \dim(Y) + \dim(Z) - \dim(X).$$

(ii) *Assume the characteristic is zero, and Y and Z are regular. Then, there exists a dense open subset U of G such that, for each rational point s in U, the fibered product, $(sY) \times_X Z$, is regular.*

Proof: Consider the map,

$$q: G \times Y \to X, \qquad q(s, y) = sf(y).$$

By hypothesis, G and Y are integral; hence, $G \times Y$ is integral because the ground field is algebraically closed (EGA IV_2, 4.6.5). Therefore, q is flat over a dense open subset V of X (EGA IV_2, 6.9.1). Make G act on $G \times Y$ through the first factor; obviously, then q is a homogeneous map. By hypothesis, G acts transitively on X; hence, the translates, sV, as s runs through the rational points of G, form an open covering of X. Since q is homogeneous, each restriction, $q^{-1}(sV) \to sV$, is isomorphic to the restrictions, $q^{-1}(V) \to V$; hence, each is flat. Thus, q is flat.

Assume the characteristic is zero and Y is regular. Since G is also regular (G is reduced and homogeneous) and since the ground field is algebraically closed, $G \times Y$ is regular (EGA IV_2, 6.8.5). Hence, obviously, the generic fiber of q is regular; so it is geometrically regular because the characteristic is zero (EGA IV_2, 6.7.4). Therefore, the fibers of q over the points in a dense open subset of X are geometrically regular (EGA IV_3, 9.9.4). Since G acts transitively on X and q is a homogeneous map, it follows that every fiber of q is geometrically regular.

The assertions obviously now follow from (1) applied with G for S, with $G \times Y$ for W, with the projection from $G \times Y$ to G for p, and with q as above.

3. Remark: In (2), assume, for each rational point x of X, the differential of the map,

$$G \to X, \qquad s \mapsto sx,$$

is surjective at each rational point of G; this is always the case in characteristic zero, and commonly occurs in positive characteristic. Then, clearly, the differential of the map, $q: G \times Y \to X$, is also surjective at each rational point; hence, q is smooth (EGA IV_4, 17.11.1). Nevertheless, the hypothesis in (2, (ii)) that the characteristic be zero is needed to apply (1, (ii)) and cannot be eliminated.

4. COROLLARY: *Let G be an integral algebraic group scheme, X an integral algebraic scheme with a transitive G-action. Let Y and Z be integral subschemes of X.*

(i) *There exists a dense open subset U of G such that, for each rational point s in U, the translate, sY, and the subscheme Z intersect properly, that is, each component of their intersection has dimension,* $\dim(Y)+\dim(Z)-\dim(X)$.

(ii) *Assume the characteristic is zero and Y and Z are regular. Then, there exists a dense open subset U of G such that, for each rational point s in U, the subscheme, sY and Z, intersect transversally, that is, the intersection,* $(sY) \cap Z$, *is regular and has pure dimension,* $\dim(Y)+\dim(Z)-\dim(X)$.

PROOF: The assertions result from (2), applied with the inclusion maps of Y and Z into X for f and g.

5. COROLLARY (Bertini's theorem): *Assume the characteristic is zero. Then, a general member of a linear system on an integral algebraic scheme Z is regular off the base locus of the system and the singular locus of Z.*

PROOF: Discarding the base locus and the singular locus – both, closed subsets of Z – we may assume the system has no base points and Z is regular. Then, the system defines a map g of Z into a suitable projective space X in such a way that its members are exactly the scheme-theoretic inverse images of the hyperplanes. The group G of automorphisms of X (or, just as good, the corresponding general linear group) acts transitively on X and on the set of hyperplanes. Hence, the assertion results from (2), applied with the inclusion map of a fixed hyperplane Y into X for f.

6. REMARK: (5) generalizes easily. Let Z be an integral algebraic scheme, E a locally free sheaf with a finite rank r, and V a finite dimensional vector space of global sections of E generating E. Fix integers n and m satisfying the (natural) inequalities,

$$1 \leqq n \leqq \dim(V) \quad \text{and} \quad \max(0, r-\dim(V)+n) \leqq m \leqq \min(n, r).$$

Take n general, linearly independent elements of V, and consider the subscheme of Z of points t at which they span a subspace of $E \otimes k(t)$ with dimension at most m. Then, either this subscheme is empty or it has pure codimension, $(r-m)(n-m)$; moreover, if Z is regular and the characteristic is zero, then it is regular except exactly at the points t at which the

n sections span a subspace of $E \otimes k(t)$ with dimension at most $(m-1)$, (it is regular everywhere if $m = 0$).

Indeed, there is a natural map g of Z into the Grassmann variety X of r-dimensional quotients of V in such a way that the subscheme of Z in question is the scheme-theoretic inverse image of an analogous subscheme of X, which is a Schubert scheme (the one denoted in [6] by $\sigma_{(n-m)}(A)$ where A is the subspace of V generated by the n elements). It is known [6, 2.9, for example] that each Schubert scheme of this sort has pure codimension, $(r-m)(n-m)$, and that, in any characteristic, its singular locus is the Schubert scheme obtained by replacing m by $(m-1)$. Hence, the assertion results by (2).

Taking $r = 1$ and $m = 0$ yields (5) because the members of a linear system without base points are the schemes of zeros of the elements of a finite dimensional vector space of global sections of an invertible sheaf.

7. Remark: It is easy to prove similarly the analogous forms of (1, (ii)), of (2, (ii)), of (4, (ii)), of (5) and a strengthened, analogous form of (6) with the adjective 'regular' replaced throughout by the adjective 'Cohen-Macaulay' (resp. 'normal', 'reduced', 'locally integral' etc.); in the new from of (6), the subscheme of Z can be proved Cohen-Macaulay, etc., everywhere because the corresponding Schubert variety is so [3, 5, 7, 8, or 9]. In the forms with 'Cohen-Macaulay', the characteristic can be arbitrary.

8. Corollary: *Let G be an integral algebraic group scheme, X an integral (regular) algebraic scheme with a transitive G-action. Let Y and Z be integral subschemes of X. Assume either the characteristic is zero or the induced action of G on the projective tangent variety PTX $(= \mathbb{P}(\Omega^1_X))$ is transitive. Then, there exists a dense open subset U of G such that, for each rational point s in U, the intersection, $(sY) \cap Z$, is proper and each of its components appears with multiplicity one, that is, there exists a dense open subset of $(sY) \cap Z$ that is regular and has pure dimension,* $\dim(Y)+\dim(Z)-\dim(X)$.

Proof: Let Y' (resp. Z') denote the open set of regular points of Y (resp. of Z). Clearly, (4, (i)) implies that, for each rational point s in a dense open subset U' of G, the following four intersections are proper:

$$(sY) \cap Z, \qquad (sY') \cap Z', \qquad (sY) \cap (Z-Z'), \qquad (sY-sY') \cap Z.$$

Hence, by dimension considerations, $(sY') \cap Z'$ is a dense (open) subset of $(sY) \cap Z$ for each such s; moreover, it is equidimensional and its dimension is given by the formula,

$$\dim((sY') \cap Z') = \dim(Y)+\dim(Z)-\dim(X).$$

Assume the characteristic is zero. Then, (4, (ii)) implies that, for each rational point s in a dense open subset U of G contained in U', the dense open subset $(sY') \cap Z'$ of $(sY) \cap Z$ is regular and has pure dimension, $\dim(Y) + \dim(Z) - \dim(X)$, as desired.

Assume the induced G-action on PTX is transitive. Then, (4, (i)) implies that, for each rational point s in a dense open subset U of G contained in U', the subschemes $s(PTY')$ and PTZ' of PTX intersect properly. Fix such an s, and consider the projection,

$$p\colon s(PTY') \cap PTZ' \to sY' \cap Z'$$

Since p is surjective and the two intersections are proper, there is a dense open subset V of $sY' \cap Z'$ over which the fibers of p have dimension, $\dim(Y) + \dim(Z) - \dim(X) - 1$. Hence, at each rational point of V, the tangent spaces to sY' and Z' meet in a vector space with dimension, $\dim(Y) + \dim(Z) - \dim(X)$. Therefore, V is a dense open subset of $(sY) \cap Z$ that is regular and has pure dimension, $\dim(Y) + \dim(Z) - \dim(X)$, as desired.

9. EXAMPLE (showing some auxiliary hypothesis is necessary in (8)): Let X be the Grassmann variety of 2-dimensional subspaces of a 4-dimensional vector space V. Fix an element f of V, and let $Y = Y(f)$ denote the Schubert scheme parametrizing the subspaces containing f. Locally, X can be coordinatized as the affine space of 2×2-matrices (t_{ij}) via a choice of basis, (e_1, e_2, e_3, e_4), of V and a decomposition of the basis into two subsets of two elements, say, (e_1, e_2) and (e_3, e_4), [6, 1.6, for example]. In this coordinate patch, Y is defined by the following two linear equations [6, 2.3, for example]:

$$F_1 \equiv s_1 + s_3 t_{11} + s_4 t_{12} = 0$$

$$F_2 \equiv s_2 + s_3 t_{21} + s_4 t_{22} = 0,$$

where the s_i are determined by the relation,

$$f = s_1 e_1 + \cdots + s_4 e_4.$$

In particular, Y is obviously smooth.

Assume the characteristic is $p > 0$. Fis the coordinate patch U, and define a subscheme Z of U by the following two equations:

$$G_1 \equiv t_{11} + t_{21} + t_{11}^p = 0$$

$$G_2 \equiv t_{12} + t_{22} + t_{11}^p = 0.$$

The differentials of these equations are

$$dG_1 = dt_{11}+dt_{21}$$

$$dG_2 = dt_{12}+dt_{22}.$$

Since they are linearly independent, Z is smooth.

Assume s_4 and (s_3+s_4) are nonzero. Then, solving the equations defining Y and Z simultaneously shows that Y and Z have exactly one point in common. Now, we obviously have the relation,

$$s_3 dG_1+s_4 dG_2 = dF_1+dF_2.$$

Therefore, Y and Z do not intersect transversally.

The general linear group G of V acts transitively on X, and clearly we have the formula,

$$sY(f) = Y(sf), \qquad \text{for each rational point } s \text{ of } G.$$

Consequently, for each rational point s in a dense open subset of G, the regular subschemes, $sY(f)$ and Z, of X do not intersect transversally, in fact, their intersection consists of a single, non-reduced point.

10. Theorem: *Let G be an integral algebraic group scheme, X an integral algebraic scheme with a transitive G-action. Assume that, for each rational point x of X, the induced homomorphism,*

(10.1) $$G_x \to Gl(T_x(X)),$$

from the stability group of x into the general linear group of the tangent space of X at x, is surjective. Let $f: Y \to X$ and $g: Z \to X$ be two maps of integral algebraic schemes. Assume f and g are unramified (for example, embeddings). Then, there exists a dense open subset U of G such that, for each rational point s of U, the fibered product, $(sY)\times_X Z$, where sY denotes Y considered as an X-scheme via the map, $y \mapsto sf(y)$, is empty or regular with pure dimension, $\dim(Y)+\dim(Z)-\dim(X)$.

Proof: Since f is unramified, the map $f^*\Omega^1_X \to \Omega^1_Y$ is surjective (EGA IV_4, 17.4.2, c)). So, there is a canonical closed embedding of $PTY = \mathbb{P}(\Omega^1_Y)$ in $(PTX)\times_X Y = \mathbb{P}(f^*\Omega^1_Y)$; it corresponds to a (linear) map, $PTf: PTY \to PTX$, lifting f. Similarly, since g is unramified, there an analogous map, $PTg: PTZ \to PTX$.

The projections, $PTY \to Y$, etc., induce a map,

$$h: (G\times(PTY))\times_{PTX} PTZ \to (G\times Y)\times_X Z,$$

where $G\times PTY$ is considered as a PTX-scheme via the map, $(s, t) \mapsto s(PTf(t))$, the action of G on PTX being the one lifting that of G on X, and where GXY is considered as an X-scheme via the map, $(s, y) \mapsto sf(y)$.

Since the projections are proper, h is proper. Hence, the following set is closed:

$$V = \{v \in (G \times Y) \times_X Z \,|\dim(h^{-1}(v)) \geqq \dim(Y) + \dim(Z) - \dim(X) + 1\}.$$

Let s, x, y, z be rational points of G, X, Y, Z, and assume the relation, $x = sf(y) = g(z)$, holds. Since f and g are unramified, the projective tangent spaces, $PT_y(Y)$ and $PT_z(Z)$, can be considered as (linear) subspaces of $PT_x(X)$ via PTf and PTg. It is easy to see that the point, $v = (s, y, z)$, of $(G \times Y) \times_X Z$ lies in V if and only if the following inequality holds:

(10.2) $\dim((sPT_y(Y)) \cap PT_z(Z)) \geqq \dim(Y) + \dim(Z) - \dim(X) + 1.$

In general, let A and B be subspaces of a vector space C, let a, b and c denote the respective dimensions, and let p be an integer satisfying the condition,

$$\max(0, a+b-c) \leqq p \leqq \min(a, b).$$

It is well-known that the closed set,

$$\{m \in Gl(C) | \dim((mA) \cap B) \geqq p\},$$

has codimension, $p(c-a-b+p)$. (The set maps onto the Schubert variety parametrizing the a-dimensional subspaces of C that meet B in a space of dimension at least p.)

Consider the natural map, $r \colon V \to Y \times Z$. Fix a rational point (y, z) of $Y \times Z$, and let s be a rational point of G. Obviously, the point (s, y, z) of $(G \times Y) \times_X Z$ lies in the fiber $r^{-1}(y, z)$ if and only if it lies in V, so if and only if the inequality (10.2) holds. Hence, since the map (10.1) is surjective, the general fact just recalled yields the formula,

$$\operatorname{codim}(r^{-1}(y, z), G_X) = (\dim(Y) + \dim(Z) - \dim(X) + 1) \cdot 1$$

Therefore, there is an inequality,

(10.3) $$\dim(V) \leqq \dim(G) - 1.$$

(In fact, equality holds because, as is easy to see, r is surjective.)

In view of (10.3), the natural map from V into G is not surjective. Hence, the complement in G of the closure of the image ov V is a dense open set U. Fix a rational point s of U. Let y and z be rational points of Y and Z, and assume the relation, $sf(y) = g(z)$, holds. Then, clearly, the following equality holds:

$$\dim((sPT_y(Y)) \cap PT_z(Z)) = \dim(Y) + \dim(Z) - \dim(X).$$

It follows that $(sY) \times_X Z$ is regular with dimension, $\dim(Y) + \dim(Z) -$

$\dim(X)$, at (y, z); this is a local matter, which is easily verified using the completions, for an unramified morphism is analytically a closed embedding (EGA IV_4, 17.4.4, f″)). Thus, $(sY)\times_X Z$ is empty or regular with dimension, $\dim(Y)+\dim(Z)-\dim(X)$.

11. COROLLARY: *Let X be projective n-space, G the corresponding general linear group. Let $f: Y \to X$ and $g: Z \to X$ be two unramified maps of integral algebraic schemes. Then, there exists a dense open subset U of G such that, for each rational point s in U, the product, $(sY)\times_X Z$, where sY denotes Y considered as an X-scheme via the map, $y \mapsto sf(y)$, is empty or regular with dimension,* $\dim(Y)+\dim(Z)-n$.

PROOF: Let x be a rational point of X. Coordinatize X so that x becomes $(1, 0, \cdots, 0)$ and, using the dual numbers, identify the tangent space $T_x(X)$ with the set of $(n+1)$-typles $(1, b_1\varepsilon, \cdots, b_n\varepsilon)$ where the b_i are arbitrary scalars and ε^2 is equal to zero. Then, a matrix m in the general linear group, $Gl(T_x(X))$, is obviously the image of the matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & m\end{smallmatrix}\right)$ in the stability subgroup G_x of x. Thus, the homomorphism (10.1) is surjective, and the assertion follows from (10).

12. COROLLARY (Bertini's theorem in arbitrary characteristic): *A general member of a linear system without base points on a regular, integral algebraic scheme Z is regular if the system separates infinitely near points, that is, if the system defines an unramified map g of Z into a suitable projective space X.*

PROOF: The assertion results from (11), applied with the inclusion map of a fixed hyperplane Y into X for f.

13. EXAMPLE (showing some auxiliary hypothesis is necessary in Bertini's theorem in characteristic $p > 0$): Let Z be the affine plane of variables x, y. The equations,

$$u = y^p + x^p y, \qquad v = x^{p+1},$$

define a finite, surjective map g from Z onto the affine (u, v)-plane. (It is ramified along the y-axis). The corresponding linear system on Z consists of the curves defined by the equations,

$$F_{a,b,c} \equiv a(y^p + x^p y) + bx^{p+1} + c = 0,$$

where a, b, c are arbitrary scalars. Obviously, we have the formula,

$$\mathrm{d}F_{a,b,c} = ax^p \mathrm{d}y + bx^p \mathrm{d}x.$$

So, if a is nonzero, the curve, $F_{a,b,c} = 0$, has a unique singular point at $x = 0$, $y = (c/a)^{1/p}$.

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