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Introduction

This note concerns metric (measure theoretic) and topological properties of concrete dynamical systems on compact metric spaces with an invariant measure, as reflected in cocycle conditions. Cocycles arise specifically in group extension problems and problems concerned with velocity changes. These two classes of problem may be put together in terms of group extensions if the stringent condition that all flows under consideration must commute with the group, is relaxed. It has to be admitted, however, that the foundations of this point of view are yet to be laid. Instead of pursuing this line of thought, we display as our main purpose a connection between two examples; one of Kolmogorov's and one of Furstenberg s. A third example, related to one of Ellis , is displayed as an illustration of the cocycle technique.

1. Flows

Let T be a continuous flow of homeomorphisms on a compact metric space X i.e. for each $t \in R$, T_t is a homeomorphism of X, the map $X \times R \to X((x, t) \to T_t x)$ is continuous and $T_s \circ T_t = T_{s+t}$, $T_0 = I$. Let T preserve a normalised Borel measure m. If G is a locally compact abelian group acting continuously as m preserving homeomorphisms on X, such that $T_t gx = gT_t x$ for all $t \in R$, $g \in G$, $x \in X$ then we can construct other flows T^φ by $T_t^\varphi x = \varphi(x, t) T_t x$ where $\varphi: X \times R \to G$ is continuous as long as φ is a *cocycle* with values in G i.e.

$$(1.1) \qquad \varphi(x, s+t) = \varphi(T_t x, s)\varphi(x, t) \quad \text{for all } x \in X,\ s, t \in R$$

(The group operation in G is written multiplicatively) and φ is G invariant

$$(1.2) \qquad \varphi(gx, t) = \varphi(x, t) \quad \text{for } all\ g \in G,\ t \in R,\ x \in X.$$

The flow T^φ also preserves m and commutes with G.

The set of cocycles with values in G is an abelian group with pointwise multiplication denoted by $Z(T, G)$. Cocycles of the form $f(T_t x) f(x)^{-1}$ for continuous maps $f: X \to G$ are called *coboundaries* and they form a sub-group of $Z(T, G)$ denoted by $B(T, G)$. Subgroups $Z_0(T, G)$, $B_0(T, G)$ are defined by the condition (1.2) and correspond to the cocycles and coboundaries of an induced flow on X/G when this makes sense. In fact T can be thought of as a G-extension of its induced flow on X/G and along with it, under certain conditions, the flows T^φ exhaust all such extensions which commute with G. For compact G related problems have been studied in [1] [2] [3] [4] [5] and [6].

If a flow T is replaced by a single measure preserving homeomorphism S (so that the flow $\{S^n : n \in Z\}$ replaces $\{T_t : t \in R\}$), then the same considerations apply if (1.1) and (1.2) are replaced by

$$\varphi : X \to G \tag{1.1$'$}$$

$$\varphi(gx) = \varphi(x) \qquad \text{for all } g \in G,\ x \in X \tag{1.2$'$}$$

and a coboundary is defined as a function of the form $x \to f(Sx)f(x)^{-1}$, where $f : X \to G$ is continuous.

2. Change of velocity

We continue with a measure (m) preserving flow T on a compact metric space, but no auxiliary group G is involved. Let $k : X \to R$ be a strictly positive continuous function and for normalisation purposes assume that $\int_X k\,dm = 1$. Let $\bar{k}(x, t) = \int_0^t k(T_s x)ds$ so that

$$\bar{k}(x, s+t) = \bar{k}(T_t x, s) + \bar{k}(x, t). \tag{2.1}$$

$\bar{k}$ is a (positive) cocycle with values in R. It is easy to show there exists a unique continuous map $\bar{h} : X \times R \to R$ which is inverse to $\bar{k}$ in the sense that

$$\bar{k}(x, \bar{h}(x, t)) = t \qquad \text{for all } x \in X,\ t \in R.$$

If we define ${}^kT$ by ${}^kT_t(x) = T_{\bar{h}(x,t)}(x)$ then ${}^kT$ is a μ preserving flow where $\mu(B) = \int_B k\,dm$. The flows ${}^kT$ exhaust all flows which are obtained from T by changing velocity positively i.e. flows with the same orbits and orientation in which velocity is not changed abruptly as one traverses X.

Just as a variety of problems can be posed (and some solved) concerning G-extensions by considering G cocycles so a number of problems present themselves concerning changes of velocity. In this connection Chacon [7] showed (not using cocycles) that ergodic flows may be changed to weak-mixing flows by altering velocities. Humphreys proved a topological analogue to Chacon's theorem using cocycle techniques [8]. Actually early work in this area was begun by Hopf (who posed Chacon's problem)

p. 43 [9], and resembles analyses of certain wellknown examples of Kakutani and Von-Neumann. Maruyama [10] and Totoki [11] have also investigated velocity changes. The Chacon problem received further treatment by the author in [12].

As far as velocity changes are concerned this note investigates an example due to Kolmogorov [13]: There is an ergodic measure preserving flow on the torus which has discrete spectrum (metrically) but has no non-constant eigen-functions (i.e. is topologically weak-mixing.) The proof we shall present imitates Furstenberg's construction of a minimal homeomorphism of the torus which is not uniquely ergodic.

3. Furstenberg's example and discontinuous eigenfunctions

Let $X = \{(x, y): x, y \in C, |x| = |y| = 1\}$ be the two dimensional torus. Let $a = e^{2\pi i \alpha}$ where α is irrational. Let $S(x, y) = (ax, \varphi(x)y)$ where φ maps the circle to itself continuously.

S is *minimal* if and only if

$$f(ax) = \varphi(x)^k f(x), \tag{3.1}$$

has no continuous solution f except when $k = 0$ [2].

S is *uniquely ergodic* if and only if

$$f(ax) = \varphi(x)^k f(x) \quad \text{a.e.} \tag{3.2}$$

has no measurable solution f except when $k = 0$ [2], or, for that matter, if and only if S is ergodic with respect to Haar measure.

Furstenberg constructs an example in [2], where (3.2) has a solution for $k = 1$ but (3.1) has no continuous solution except when $k = 0$. In other words, the example is of a homeomorphism (actually C^∞) of X which is minimal but not ergodic (uniquely ergodic). Let S be Furstenberg's example, then it is easy to show that for almost all g in the circle $S_g(x, y) = (ax, g\varphi(x)y)$ is ergodic and uniquely ergodic, and using the metric conjugacy $(x, y) \to (x, \overline{f(x)}y)$, we see that S_g is metrically isomorphic to $(x, y) \to (ax, gy)$ which is ergodic with discrete spectrum. However the eigenfunction of Sg,

$$F(x, y) = \overline{f(x)}y$$

$$(FS_g(x, y) = \overline{f(ax)}g\varphi(x)y = gF(x, y) \text{ a.e.})$$

corresponding to the eigenvalue g cannot be made continuous, for otherwise (3.1) would have a continuous solution. In fact metrically the group of eigenvalues $= \{\alpha^m g^n : m, n \in Z\}$ whereas topologically, the group of eigenvalues $= \{\alpha^m : m \in Z\}$. Kolmogorov's example, which we will analyse, reduces the group (topologically) even further to the trivial group. Before investigating this, however, we shall consider an example of a minimal homeomorphism on the Klein bottle.

4. Klein bottle X′

(c.f. Ellis [3] for the first example of a minimal homeomorphism on X'.)

Let $S(x, y) = (ax, \varphi(x)y)$ be a homeomorphism of the torus X, where as usual $a = e^{2\pi i\alpha}$ with α irrational. Consider the Z_2 action on X given by $1:(x, y) \to (x, y)$, $-1:(x, y) \to (-x, y^{-1})$. Then $X' = X/Z_2$ is the Klein bottle, and Z_2 commutes with S if and only if $\varphi(-x) = \varphi(x)^{-1}$. Let $\varphi(x) = e^{2\pi i r(x)}$ where r is a continuous map of the circle to the reals with absolutely convergent Fourier series $r(x) = \sum a_n x^n$ $(= -r(-x) = \sum -a_n x^n(-1)^n$ if, as we suppose $a_{2n} = 0$, for all $n \in Z)$. S will induce a minimal homeomorphism S' on X' if S is minimal. The problem then is to *find a real valued absolutely convergent Fourier series*

$$r(x) = \sum_{n=1}^{\infty} (a_n x^{2n+1} + \bar{a}_n x^{-2n-1})$$

such that $S(x, y) = (ax, \varphi(x)y)$, $\varphi(x) = e^{2\pi i r(x)}$ is minimal. In other words we require (3.1) to have no continuous solution except when $k = 0$. By considering the absolute value in equation (3.1) it is clear that solutions have constant absolute value so there is no loss in generality in restricting our attention to functions f such that $|f| = 1$. If f is a continuous function such that $|f| = 1$ then $f(x) = x^l e^{2\pi . ih(x)}$ where l is an integer and $h: X \to R$ is continuous. For (3.1) to have the solution f we must have

$$e^{2\pi i l\alpha} x^l e^{2\pi i h(ax)} = e^{2\pi i k r(x)} x^l e^{2\pi i h(x)}$$

i.e. $l\alpha + h(ax) = kr(x) + h(x) + b$ where b is an integer. Integration shows that

$$h(ax) = kr(x) + h(x), \tag{4.1}$$

so that dividing by k, we need only find a real valued absolutely convergent Fourier series

$$r(x) = \sum_{n=1}^{\infty} (a_n x^{2n+1} + \overline{a_n} x^{-2n-1})$$

such that (4.1) has no continuous solution h, if $k \neq 0$. In fact if (4.1) has a continuous solution h with Fourier series $\sum_{n \in Z} b_n x^n$, then again $b_{2n} = 0$ and $(a^{2n+1} - 1)b_{2n+1} = a_n$. Hence if we let $a_n = 0$ for $n \in N$ and $a_n = (a^{2n+1} - 1)$ for $n \notin N$ where $\sum_{n \notin N} |a^{2n+1} - 1| < \infty$, then $b_{2n+1} = 1$ for $n \notin N$ and $\sum_{n \in N} b_n x^n$ is *not* the Fourier series of a continuous function.

This completes the construction. Notice that we have not shown that S is uniquely ergodic. In fact there are minimal non uniquely ergodic homeomorphisms of a Klein bottle using Furstenberg's construction.

5. Kolmogorov's example [13]

The considerations of this section are closely related to Arnold [14], Kowada [15]. We consider the flow T_t on X (the torus) given by $T_t(x, y) = (e^{2\pi it\alpha}x, e^{2\pi it}y)$ where α is an irrational to be chosen later. We wish to find a change of velocity flow ${}^kT$ (preserving the measure kdm) which has discrete spectrum but no continuous eigenfunctions other than constants. An eigenfunction f of ${}^kT$ satisfies

$$f({}^kT_t) = e^{2\pi i\beta t}f \qquad \text{i.e.}$$
$$f(T_t) = e^{2\mathcal{O}i\beta\bar{k}(z,t)}f,\ z = (x, y).$$

If f is continuous then $f(x, y) = x^m y^n e^{2\pi i\varphi(x,y)}$ so that

$$e^{2\pi it(\alpha m+n)}x^m y^n e^{2\pi i\varphi T_t(x,y)} = e^{2\pi i(\beta\bar{k}(z,t)+\varphi(xy))}x^m y^n \qquad \text{i.e.}$$

$$\varphi T_t(x, y) - \varphi(x, y) = \beta\bar{k}(z, t) - (\alpha m+n)t.$$

and integration yields

$$\varphi T_t(x, y) - \varphi(x, y) = \beta(\bar{k}(z, t) - t) = \beta\int_0^t k_0(T_s)\,\mathrm{d}s$$

$$\alpha m + n = \beta,$$

where $k_0 = k-1$, $\overline{k_0} = \bar{k}(x, t) - t$. We shall arrange for the equation

(5.1) $$\varphi T_t - \varphi = \bar{k}_0$$

to have no continuous solution, whereas the equation

(5.2) $$\varphi T_t - \varphi = \bar{k}_0 \qquad \text{a.e.}$$

for each $t \in R$ will have a solution in L^2.

In this case, as we have seen, the flow ${}^kT$ will have no continuous eigenfunctions other than constants. On the other hand the solution to (5.2) allows us to establish a metric conjugacy between T and ${}^kT$ showing that ${}^kT$ has discrete spectrum (since T has discrete spectrum.)

If k_0 is a sufficiently small real valued continuous function on X with $\int k_0\,\mathrm{d}m = 0$ then $k = 1+k_0$ is a strictly positive continuous function allowing us to construct the velocity change ${}^kT$. Evidently

$$\bar{k}_0(z, t) = \int_0^t k_0(T_s z)\,\mathrm{d}s$$

and if φ is a solution of (5.2) then the diagram

$$\begin{tikzcd} z \arrow[r] \arrow[d] & T_{\varphi(z)}(z) \arrow[d] \\ {}^kT_t(z) \arrow[r] \arrow[d, equal] & T_{\varphi({}^kT_t z)} \circ T_{\bar{h}(z,t)}(x) = T_t \circ T_{\varphi(z)}(z) \\ T_{\bar{h}(z,t)}(z) & \end{tikzcd}$$

commutes a.e. since, substituting $\bar{k}(x, t)$ for t, we have $\varphi(T_t z)+t = \bar{k}(z, t)+\varphi(z)$ a.e. or, in other words, $\varphi T_t - \varphi = \bar{k}_0(z, t)$ a.e. It is not difficult to show that $z \to T_{\varphi(z)}(z)$ is a measurable measure preserving ($k\,\mathrm{d}m$ to $\mathrm{d}m$) invertible (mod 0) transformation of X to itself; thus defining a conjugacy between ${}^kT$ and T.

Our task then is to find a continuous real function k_0 with $\int k_0\,\mathrm{d}m = 0$ such that (5.2) has a solution whilst (5.1) has no continuous solution. (Making k_0 small is no problem, since we may simply multiply it by a small number.)

Let

$$k_0(x, y) = \sum_{r=1}^{\infty} a(r)(x^{m_r}y^{-n_r} + x^{-m_r}y^{n_r})$$

be an absolutely convergent Fourier series where the positive integers m_r, n_r and the real coefficients $a(r)$ are to be determined.

$$\begin{aligned} \bar{k}_0(z, y, t) &= \int_0^t k_0\, T_s(x, y)\,\mathrm{d}t \\ &= \sum_{r=1}^{\infty} \frac{a(r)}{2\pi i} \frac{(e^{2\pi i(m_r\alpha - n_r)t} - 1)}{m_r\alpha - n_r} x^{m_r}y^{-n_r} + \bar{S} \end{aligned}$$

where S is the first part of this expression, so that if $\varphi \in L^2(X)$ has Fourier series

$$\varphi = \sum_{r=0}^{\infty} (b(r)x^{m_r}y^{-n_r} + \overline{b(r)}x^{-m_r}y^{n_r}) \tag{5.3}$$

and satisfies (5.2) then

$$b(r)(e^{2\pi i(m_r\alpha - n_r)t} - 1) = \frac{a(r)}{2\pi i} \frac{(e^{2\pi i(m_r\alpha - n_r)t} - 1)}{(m_r\alpha - n_r)}$$

$$\text{i.e.} \qquad b(r) = \frac{a(r)}{2\pi i(m_r\alpha - n_r)} \qquad r = 1, 2, \cdots$$

Let $a(r) = (m_r\alpha - n_r)/r$ so that $b_r = 1/2\pi ir$ $(r \neq 0)$. In this case we see that (5.3) is the Fourier series of an L^2 function. Let $\alpha = \sum_{j=1}^{\infty} 1/2^{v_j}$ where $v_{n+1} - v_n = \theta_n$ then

$$2^{v_n}\alpha - [2^{v_n}\alpha] = \sum_{j=n+1}^{\infty} \frac{2^{v_n}}{2^{v_j}} \leqq 2 \cdot \frac{2^{v_n}}{2^{v_{n+1}}} = \frac{2}{2^{\theta_n}}\,.$$

Let $m_r = 2^{v_r}$, $n_r = [2^{v_r}\alpha]$ and note that $m_r \equiv 0 \bmod 4$, $n_r \equiv 1 \bmod 4$ (if $\theta_r \geqq 2$). In this case (5.3) is not the Fourier series of a continuous function since (putting $x = Ri = y$)

$$\lim_{\substack{R \to 1 \\ R \ 1}} \sum_{r=1}^{\infty} \left(\frac{1}{2\pi iri} R^{m_r - n_r} - \frac{1}{2\pi ir} (1/R)^{m_r - n_r} \right) = -\frac{1}{\pi} \sum_{r=1} \frac{1}{r} = -\infty$$

(Abel convergence test c.f. [16].)

The function k_0 will be determined once we have specified the positive sequence θ_n. The Fourier coefficient of the term $x^{m_r}y^{-n_r}$ is

$$a(r) = \frac{m_r\alpha - n_r}{r} \quad \text{and} \quad |m_r\alpha - n_r| \leqq \frac{2}{2^{\theta_r}}\,.$$

By choosing θ_r judiciously one can make k_0 a C^∞ function. In other words C^∞ flows on the torus exist with (metric) discrete spectrum and with no non-constant eigenfunctions.

6. Conclusion

Each of the problems considered here amounts to a cocycle - coboundary problem. In Section 3 Furstenberg required a continuous cocycle which was the coboundary of a measurable but not continuous function. In Section 4 we required a special continuous cocycle which was not a coboundary. In Section 5 we required a continuous cocycle which was the coboundary of a measurable but not continuous function.

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