COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 29, n° 1 (1974), p. 35-41

<http://www.numdam.org/item?id=CM_1974__29_1_35_0>

COMPOSITION MATHEMATICA, Vol. 29, Fasc. 1, 1974, pag. 35–42
Noordhoff International Publishing
Printed in the Netherlands

KUTTNER'S THEOREM FOR OPERATORS

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1.

In this note we extend Kuttner's theorem [1] to certain infinite matrices of bounded linear operators on a Banach space X into a Banach space Y. The method is to characterize the class $(w_p(X); c(Y))$ of matrices $A = (A_{nk})$ which are such that $\sum A_{nk} x_k$ converges for each n, and tends to a limit as $n \to \infty$, whenever $\bar{x} = (x_k) \in w_p(X)$. All sums without limits will be taken from $k = 1$ to $k = \infty$. It is shown in section 4 that, as in the classical case [3, Theorem M′], we have the inclusion $(w_p(X), c(Y) \subset (l_\infty(X), c(Y))$ when $0 < p < 1$.

2.

We now establish notation. Let X, Y be Banach spaces with (undifferentiated) norms $||x||$, $||y||$, and let $B(X, Y)$ be the Banach space of bounded linear operators on X into Y, with the usual operator norm. The continuous dual of Y is denoted by Y^*. If $T \in B(X, Y)$ we denote the adjoint of T by T^*, so that $(f, Tx) = (T^*f, x)$ for all $f \in Y^*$ and all $x \in X$, where as usual $(f, y) \equiv f(y)$ for $f \in Y^*$ and $y \in Y$.

We shall write

$$S^* = \{f \in Y^*: ||f|| \leqq 1\}$$

and make use of the fact that, by the Hahn-Banach theorem, for every $y \in Y$ there exists $f \in S^*$ such that $||y|| = f(y)$.

Throughout we shall suppose that $A_{nk} \in B(X, Y)$; $n, k = 1, 2, \cdots$.

$c(Y)$ denotes the space of convergent Y-valued sequences, $l_\infty(X)$ the space of bounded X-valued sequences, and $w_p(X)$ the space of strongly Cesàro summable sequences with values in X. For $0 < p < 1$, as in [3], we make $w_p(X)$ into a complete p-normed space with

$$||\bar{x}|| = \sup_r \frac{1}{2^r} \sum_r ||x_k||^p,$$

where $\sum_r$ denotes a sum over $2^r \leqq k < 2^{r+1}$. Following [3], w_p denotes $w_p(C)$, where C is the space of complex numbers.

By O we denote the zero element of $B(X, Y)$ and by T a fixed non-zero element of $B(X, Y)$. We shall use O and T in Theorem 1 below.

We regard to operators A_{nk}, a convergence statement such as $\sum A_{nk} \to T$ $(n \to \infty)$ refers to the topology of pointwise convergence, i.e. it means that $\sum A_{nk}x \to Tx$ $(n \to \infty)$ for each $x \in X$.

If $(B_k) = (B_1, B_2, \cdots)$ is an infinite sequence in $B(X, Y)$ we denote its group norm (see [2]) by

$$\|(B_k)\| = \sup \|\sum_{k=1}^{n} B_k x_k\|,$$

where the supremum is taken over all $n \geqq 1$ and all $\|x_k\| \leqq 1$. Also, we write R_{nm} for the mth "tail" of the nth row of the matrix A, i.e.

$$R_{nm} = (A_{nm}, A_{n,m+1}, A_{n,m+2}, \cdots). \tag{1}$$

3.

The following result, in the case $Y = X$ and T the identity operator, was proved by Robinson [5, Theorem IV]. The extension stated here is a trivial one.

THEOREM 1: *$A \in (c(X), c(Y))$ and $\lim \sum A_{nk} x_k = T(\lim x_n)$ if and only if*

$$A_{nk} \to O \qquad (n \to \infty, \text{ each } k), \tag{2}$$

$$\sum A_{nk} \to T \qquad (n \to \infty), \tag{3}$$

$$\sup_n \|R_{n1}\| < \infty. \tag{4}$$

We remark that (3) also involves the convergence of $\sum A_{nk}$ for each n.

To aid the determination of $(w_p(X), c(Y))$ we first prove

LEMMA 1: *Let $0 < p < 1$, and suppose $(B_k) \in B(X, Y)$. Then $\sum B_k x_k$ converges, whenever $\bar{x} \in w_p(X)$ if and only if*

$$M = \sup \sum_{r=0}^{\infty} 2^{r/p} \max_r \|B_k^* f\| < \infty, \tag{5}$$

where the supremum is taken over $f \in S^*$ and $\max_r$ *is over* $2^r \leqq k < 2^{r+1}$.

PROOF: First suppose that (5) holds. Write T_m for the mth tail of (B_k). Then for $s \geqq 0$ and $m \geqq 2^s$ we have

$$\begin{aligned} \|T_m\| \leqq \|T_{2^s}\| &= \sup \|\sum_{k=2^s}^{i} B_k x_k\| \\ &\leqq \sup \sum_{k=2^s}^{i} |(f, B_k x_k)|, \qquad \text{for some } f \in S^*, \end{aligned} \tag{6}$$

$$\leqq \sup \sum_{k=2^s}^{i} \|B_k^* f\| \, \|x_k\|$$

$$\leqq \sum_{r=s}^{\infty} \sum_{r} \|B_k^* f\|$$

$$\leqq M \cdot 2^{s/q}, \qquad \text{where } q = p/(p-1) < 0.$$

In (6) above $\|T_m\|$ denotes the group norm of the sequence T_m and the supremum is taken over all $i \geqq 2^s$ and all $\|x_k\| \leqq 1$. It now follows that (5) implies

$$(7) \qquad \|T_m\| \to 0 \qquad (m \to \infty).$$

From (7) we see immediately that $\sum B_k$ converges.

Now suppose that $\sum_r \|x_k - l\|^p = o(2^r)$ as $r \to \infty$. Since $\sum B_k l$ converges we have to show that $\sum B_k(x_k - l)$ converges. Take m and $s \geqq 0$ and let $n \geqq 2^m$. Then for some $f \in S^*$ we have

$$\|\sum_{k=n}^{n+s} B_k(x_k - l)\| \leqq \sum_{k=n}^{n+s} \|B_k^* f\| \, \|x_k - l\|$$

$$\leqq \sum_{r=m}^{\infty} \max_r \|B_k^* f\| \;\; \sum_r \|x_k - l\|$$

$$\leqq \sum_{r=m}^{\infty} 2^{r/p} \max_r \|B_k^* f\| \cdot (2^{-r} \sum_r \|x_k - l\|^p)^{1/p}.$$

By (5) it is now clear that $\sum B_k(x_k - l)$ converges. This proves the sufficiency.

Conversely let $\sum B_k x_k$ converge for all $\bar{x} \in w_p(X)$. Take $f \in Y^*$. Then $\sum (f, B_k x_k)$ converges for all $\bar{x} \in w_p(X)$. Now by definition of $\|B_k^* f\|$ we may choose z_k in the closed unit sphere in X such that $\|B_k^* f\| \leqq 2|(f, B_k z_k)|$. Let us take any complex sequence (a_k) such that $a_k \to o(w_p)$. Then $(a_k z_k) \in w_p(X)$; and so $\sum a_k(f, B_k z_k)$ converges whenever $a_k \to o(w_p)$. It follows from [3] that

$$\sum_{r=0}^{\infty} 2^{r/p} \max_r |(f, B_k z_k)| < \infty$$

and so

$$(8) \qquad \sum_{r=0}^{\infty} 2^{r/p} \max_r \|B_k{}^* f\| < \infty$$

for each $f \in Y^*$. Now let $q_n(f)$ be the nth partial sum of the series in (8). Then each q_n is a continuous seminorm on the Banach space Y^*, whence by a version of the Banach-Steinhaus theorem [4, corollary to Theorem

11, p, 114] the series in (8) is also a continuous seminorm on Y^*, so that (5) holds.

4.

We now present the main result.

THEOREM 2: *Let* $0 < p < 1$. *Then* $A \in (w_p(X), c(Y))$ *if and only if*:

$$\text{(9)} \qquad \text{There exists } \lim_n A_{nk} = A_k \text{ (each } k\text{)},$$

$$\text{(10)} \qquad M_1 = \sup \sum_{r=0}^{\infty} 2^{r/p} \max_r \|A_{nk}^* f\| < \infty,$$

$$\text{(11)} \qquad M_2 = \sup \sum_{r=0}^{\infty} 2^{r/p} \max_r \|(A_{nk}^* - A_k^*) f\| < \infty,$$

where in (10) *and* (11) *the supremum is taken over all* $n \geqq 1$ *and all* $f \in S^*$.

PROOF: Consider the necessity. That (9) is necessary is trivial. Let us show that (11) is necessary – the necessity of (10) may be shown by similar reasoning. Define $T_n(x) = \sum A_{nk} x_k$, the series converging for each n and all $\bar{x} \in w_p(X)$. By the Banach-Steinhaus theorem each T_n is a continuous linear operator on $w_p(X)$ into Y, whence for each triple (m, n, h) of positive integers the set

$$E(m, n, h) = \{\bar{x} \in w_p(X) \colon \|(T_n - T_m)(x)\| \leqq h\}$$

is closed. Hence $E(h)$, the intersection over all (m, n), is closed, and $w_p(X)$ is the union of the $E(h)$. Since $w_p(X)$ is of the second category a standard type of argument yields the existence of an absolute constant H such that $\sup_{m,n} \|(T_n - T_m)(x)\| \leqq H \|\bar{x}\|^{1/p}$ for all $\bar{x} \in w_p(X)$.

Now let s be a positive integer, θ the zero of X and consider only those sequences $\bar{x}$ such that $x_k = \theta$ for $k \geqq 2^{s+1}$. Also, write $B_{nk} = A_{nk} - A_k$. Then, using (9), but letting $m \to \infty$, we obtain

$$\Big\| \sum_{k=1}^{2^{s+1}-1} B_{nk} x_k \Big\| \leqq H \|\bar{x}\|^{1/p},$$

so that, for every $f \in Y^*$,

$$\text{(12)} \qquad \Big| \sum_{k=1}^{2^{s+1}-1} (f, B_{nk} x_k) \Big| \leqq \|f\| \cdot H \cdot \|\bar{x}\|^{1/p}.$$

Next we determine $\|z_{nk}\| \leqq 1$ such that $\|B_{nk}^* f\| \leqq 2|(f, B_{nk} z_{nk})|$. Then by suitable choice of a complex sequence (a_k), with w_p norm equal 1 (see [4, p. 173]) we have $(a_k z_{nk}) \in w_p(X)$, so from (12) we get

$$\sum_{r=0}^{s} 2^{r/p} \max_r |(f, B_{nk} z_{nk})| \leqq \|f\| \cdot H,$$

whence

$$\sup_n \sum_{r=0}^{\infty} 2^{r/p} \max_r \|B_{nk}^* f\| \leqq 2\|f\| \cdot H$$

for every $f \in Y^*$, which implies (11).

Conversely, let (9), (10) and (11) hold. Let R_{nm} be given by (1) and write R_m for the mth tail of the sequence (A_k). By the argument used to prove that (5) implied (7), we see that (11) implies

$$\sup_n \|R_{nm} - R_m\| \to 0 \qquad (m \to \infty). \tag{13}$$

Also, (10) implies

$$\|R_{nm}\| \to 0 \qquad (\mathrm{m} \to \infty, \text{ each } n). \tag{14}$$

Now let $x_k \to l(w_p(X))$. Then by (10) and the argument of the sufficiency part of Lemma 1,

$$\|\sum_{k=m}^{m+s} A_{nk}(x_k - l)\| \leqq \varepsilon M_1, \tag{15}$$

for each $\varepsilon > 0$, for all (n, s) and all sufficiently large m. Letting $n \to \infty$ in (15) we get

$$\|\sum_{k=m}^{m+s} A_k(x_k - l)\| \leqq \varepsilon M_1. \tag{16}$$

Hence by (16),

$$\|\sum_{k=m}^{m+s} A_k x_k\| \leqq \varepsilon M_1 + \|\sum_{k=m}^{m+s} (A_k - A_{1k})l\| + \|\sum_{k=m}^{m+s} A_{1k} l\|$$

$$\leqq \varepsilon M_1 + \|R_{1m} - R_m\| \cdot \|l\| + \|R_{1m}\| \cdot \|l\|,$$

so by (13) and (14) we see that $\sum A_k x_k$ converges for each $\bar{x} \in w_p(X)$.

Is is now a simple matter to show that (9), (13) and (14) are sufficient for A to be in $(l_\infty(X), c(Y))$, and that for such A,

$$\lim_n \sum \mathrm{A}_{nk} \mathrm{x}_k = \sum \mathrm{A}_k \mathrm{x}_k \tag{17}$$

for each $\bar{x} \in l_\infty(X)$. We remark that (9), (13) and (14) are also necessary for

A to be in $(l_\infty(X), c(Y))$, but the proof is not completely trivial, and we do not require the necessity for our present purpose.

Finally, let us write

$$\sum A_{nk}x_k = \sum A_k x_k + \sum (A_{nk}-A_k)(x_k - l) + \sum (A_{nk}-A_k)l.$$

Then (9) and (13) imply $\sum(A_{nk}-A_k)(x_k-l) \to \theta$ $(n \to \infty)$, and (17) implies $\sum(A_{nk}-A_k)l \to \theta$ $(n \to \infty)$. Hence

$$\lim_n \sum A_{nk}x_k = \sum A_k x_k \tag{18}$$

for every $\bar{x} \in w_p(X)$. This proves the theorem.

Next we give the generalization of Kuttner's theorem.

THEOREM 3: *Let $0 < p < 1$ and let T be a fixed non-zero element of $B(X, Y)$. Suppose that $A = (A_{nk})$ is as in Theorem 1. Then there is a sequence in $w_p(X)$ which is not summable A.*

PROOF: Suppose, if possible, that $A \in (w_p(X), c(Y))$. Take $x \in X$ such that $T(x) \neq \theta$. Then by (18) we have $\lim_n \sum A_{nk}x$. But (3) implies $\lim_n \sum A_{nk}x = T(x)$ and (2) implies $\sum A_k x = \theta$, whence $T(x) = \theta$, contrary to the choice of x.

5.

We now briefly consider the case $1 \leqq p < \infty$. The norm

$$\|\bar{x}\| = \sup_r \left(\frac{1}{2^r}\sum_r \|x_k\|^p\right)^{1/p}$$

makes $w_p(X)$ into a Banach space.

The analogue of Lemma 1 is:

LEMMA 2: *Let $1 \leqq p < \infty$ and $(B_k) \in B(X, Y)$. Then $\sum B_k x_k$ converges, whenever $\bar{x} \in w_p(X)$ if and only if*

$$\sum B_k \text{ converges,} \tag{19}$$

$$\sup \sum_{r=0}^{\infty} 2^{r/p}\left(\sum_r \|B_k^* f\|^q\right)^{1/q} < \infty, \tag{20}$$

where $1/p + 1/q = 1$, the supremum is taken over $f \in S^*$ and $\sum_r$ denotes a sum over $2^r \leqq k < 2^{r+1}$. The case $p = 1$ of (20) is interpreted as (5) with $p = 1$.

We remark that (19) and (20) are independent. For example, in the space of complex numbers, if $b_k = (-1)^k/k$ then (19) holds but

$$\sum_{r=0}^{\infty} 2^{r/p}\left(\sum |b_k|^q\right)^{1/q} = \infty,$$

so that (20) fails. On the other hand, let $X = Y = \{x \in w_p : x_k \to 0(w_p)\}$. Then by [3, p. 290], $f \in X^*$ if and only if $f(x) = \sum a_k x_k$ where a is such that

$$(21) \qquad \|f\| = \sum_{r=0}^{\infty} 2^{r/p} (\sum_r |a_k|^q)^{1/q} < \infty.$$

Let us define $B_k : X \to X$ by $B_k x = (0, 0, \cdots, x_1, 0, 0, \cdots)$ with x_1 in the kth place. Then $(B_k^* f, x) = (f, B_k x) = a_k x_1$, so that $\|B_k^* f\| = |a_k|$. Hence (21) implies

$$\sum_{r=0}^{\infty} 2^{r/p} (\sum \|B_k^* f\|^q)^{1/q} < \infty$$

for each $f \in X^*$. Thus, by the argument immediately following (8) we see that (20) holds. Now take $x = (1, 0, 0, \cdots)$ and write $y(n) = \sum_{k=1}^{n} B_k x$. Then $z \equiv y(2^{r+1}-1) - y(2^r - 1)$ is a sequence such that $z_k = 1$ for $2^r \leqq k < 2^{r+1}$ and $z_k = 0$ otherwise. Hence $\|z\| = 1$, so that $\sum B_k x$ diverges, which means that (19) fails.

Finally, using arguments similar to those in the proof of Theorem 2 we can establish

THEOREM 4: *Let $1 \leqq p < \infty$. Then $A \in (w_p(X), c(Y))$ if and only if: There exists*

$$(22) \qquad \lim_n A_{nk} \text{ (each } k) \text{ and } \lim_n \sum A_{nk},$$

$$(23) \qquad \sup \sum_{r=0}^{\infty} 2^{r/p} (\sum_r \|A_{nk}^* f\|^q)^{1/q} < \infty,$$

$$(24) \qquad \sup \sum_{r=0}^{\infty} 2^{r/p} (\sum_r \|(A_{nk}^* - A_k^*)\|^q)^{1/q} < \infty,$$

the suprema being over all $n \geqq 1$ and all $f \in S^*$.

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(Oblatum 12–IX–1973)

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