COMPOSITIO MATHEMATICA

I. J. MADDOX

Kuttner's theorem for operators

Compositio Mathematica, tome 29, nº 1 (1974), p. 35-41

<http://www.numdam.org/item?id=CM_1974__29_1_35_0>

COMPOSITION MATHEMATICA, Vol. 29, Fasc. 1, 1974, pag. 35–42
Noordhoff International Publishing
Printed in the Netherlands

KUTTNER'S THEOREM FOR OPERATORS

I. J. Maddox

1.

In this note we extend Kuttner's theorem [1] to certain infinite matrices of bounded linear operators on a Banach space X into a Banach space Y. The method is to characterize the class $(w_p(X); c(Y))$ of matrices $A = (A_{nk})$ which are such that $\sum A_{nk} x_k$ converges for each n, and tends to a limit as $n \to \infty$, whenever $\bar{x} = (x_k) \in w_p(X)$. All sums without limits will be taken from $k = 1$ to $k = \infty$. It is shown in section 4 that, as in the classical case [3, Theorem M′], we have the inclusion $(w_p(X), c(Y) \subset (l_\infty(X), c(Y))$ when $0 < p < 1$.

2.

We now establish notation. Let X, Y be Banach spaces with (undifferentiated) norms $||x||$, $||y||$, and let $B(X, Y)$ be the Banach space of bounded linear operators on X into Y, with the usual operator norm. The continuous dual of Y is denoted by Y^*. If $T \in B(X, Y)$ we denote the adjoint of T by T^*, so that $(f, Tx) = (T^*f, x)$ for all $f \in Y^*$ and all $x \in X$, where as usual $(f, y) \equiv f(y)$ for $f \in Y^*$ and $y \in Y$.

We shall write

$$S^* = \{f \in Y^*: ||f|| \leqq 1\}$$

and make use of the fact that, by the Hahn-Banach theorem, for every $y \in Y$ there exists $f \in S^*$ such that $||y|| = f(y)$.

Throughout we shall suppose that $A_{nk} \in B(X, Y)$; $n, k = 1, 2, \cdots$.

$c(Y)$ denotes the space of convergent Y-valued sequences, $l_\infty(X)$ the space of bounded X-valued sequences, and $w_p(X)$ the space of strongly Cesàro summable sequences with values in X. For $0 < p < 1$, as in [3], we make $w_p(X)$ into a complete p-normed space with

$$||\bar{x}|| = \sup_r \frac{1}{2^r} \sum_r ||x_k||^p,$$

where $\sum_r$ denotes a sum over $2^r \leqq k < 2^{r+1}$. Following [3], w_p denotes $w_p(C)$, where C is the space of complex numbers.

By O we denote the zero element of $B(X, Y)$ and by T a fixed non-zero element of $B(X, Y)$. We shall use O and T in Theorem 1 below.

We regard to operators A_{nk}, a convergence statement such as $\sum A_{nk} \to$ T $(n \to \infty)$ refers to the topology of pointwise convergence, i.e. it means that $\sum A_{nk} x \to Tx$ $(n \to \infty)$ for each $x \in X$.

If $(B_k) = (B_1, B_2, \cdots)$ is an infinite sequence in $B(X, Y)$ we denote its group norm (see [2]) by

$$\|(B_k)\| = \sup \|\sum_{k=1}^{n} B_k x_k\|,$$

where the supremum is taken over all $n \geqq 1$ and all $\|x_k\| \leqq 1$. Also, we write R_{nm} for the mth "tail" of the nth row of the matrix A, i.e.

$$R_{nm} = (A_{nm}, A_{n,m+1}, A_{n,m+2}, \cdots). \tag{1}$$

3.

The following result, in the case $Y = X$ and T the identity operator, was proved by Robinson [5, Theorem IV]. The extension stated here is a trivial one.

THEOREM 1: *$A \in (c(X), c(Y))$ and* $\lim \sum A_{nk} x_k = T(\lim x_n)$ *if and only if*

$$A_{nk} \to O \qquad (n \to \infty, \text{ each } k), \tag{2}$$

$$\sum A_{nk} \to T \qquad (n \to \infty), \tag{3}$$

$$\sup_n \|R_{n1}\| < \infty. \tag{4}$$

We remark that (3) also involves the convergence of $\sum A_{nk}$ for each n.
To aid the determination of $(w_p(X), c(Y))$ we first prove

LEMMA 1: *Let $0 < p < 1$, and suppose $(B_k) \in B(X, Y)$. Then $\sum B_k x_k$ converges, whenever $\bar{x} \in w_p(X)$ if and only if*

$$M = \sup \sum_{r=0}^{\infty} 2^{r/p} \max_r \|B_k^* f\| < \infty, \tag{5}$$

where the supremum is taken over $f \in S^*$ and $\max_r$ *is over* $2^r \leqq k < 2^{r+1}$.

PROOF: First suppose that (5) holds. Write T_m for the mth tail of (B_k). Then for $s \geqq 0$ and $m \geqq 2^s$ we have

$$\|T_m\| \leqq \|T_{2^s}\| = \sup \|\sum_{k=2^s}^{i} B_k x_k\| \tag{6}$$

$$\leqq \sup \sum_{k=2^s}^{i} |(f, B_k x_k)|, \qquad \text{for some } f \in S^*,$$

$$\leqq \sup \sum_{k=2^s}^{i} \|B_k^* f\| \, \|x_k\|$$

$$\leqq \sum_{r=s}^{\infty} \sum_{r} \|B_k^* f\|$$

$$\leqq M \cdot 2^{s/q}, \qquad \text{where } q = p/(p-1) < 0.$$

In (6) above $\|T_m\|$ denotes the group norm of the sequence T_m and the supremum is taken over all $i \geqq 2^s$ and all $\|x_k\| \leqq 1$. It now follows that (5) implies

$$\|T_m\| \to 0 \qquad (m \to \infty). \tag{7}$$

From (7) we see immediately that $\sum B_k$ converges.

Now suppose that $\sum_r \|x_k - l\|^p = o(2^r)$ as $r \to \infty$. Since $\sum B_k l$ converges we have to show that $\sum B_k(x_k - l)$ converges. Take m and $s \geqq 0$ and let $n \geqq 2^m$. Then for some $f \in S^*$ we have

$$\left\| \sum_{k=n}^{n+s} B_k(x_k - l) \right\| \leqq \sum_{k=n}^{n+s} \|B_k^* f\| \, \|x_k - l\|$$

$$\leqq \sum_{r=m}^{\infty} \max_{r} \|B_k^* f\| \; \sum_{r} \|x_k - l\|$$

$$\leqq \sum_{r=m}^{\infty} 2^{r/p} \max_{r} \|B_k^* f\| \cdot \left(2^{-r} \sum_{r} \|x_k - l\|^p\right)^{1/p}.$$

By (5) it is now clear that $\sum B_k(x_k - l)$ converges. This proves the sufficiency.

Conversely let $\sum B_k x_k$ converge for all $\bar{x} \in w_p(X)$. Take $f \in Y^*$. Then $\sum (f, B_k x_k)$ converges for all $\bar{x} \in w_p(X)$. Now by definition of $\|B_k^* f\|$ we may choose z_k in the closed unit sphere in X such that $\|B_k^* f\| \leqq 2|(f, B_k z_k)|$. Let us take any complex sequence (a_k) such that $a_k \to o(w_p)$. Then $(a_k z_k) \in w_p(X)$; and so $\sum a_k (f, B_k z_k)$ converges whenever $a_k \to o(w_p)$. It follows from [3] that

$$\sum_{r=0}^{\infty} 2^{r/p} \max_{r} |(f, B_k z_k)| < \infty$$

and so

$$\sum_{r=0}^{\infty} 2^{r/p} \max_{r} \|B_k{}^* f\| < \infty \tag{8}$$

for each $f \in Y^*$. Now let $q_n(f)$ be the nth partial sum of the series in (8). Then each q_n is a continuous seminorm on the Banach space Y^*, whence by a version of the Banach-Steinhaus theorem [4, corollary to Theorem

11, p, 114] the series in (8) is also a continuous seminorm on Y^*, so that (5) holds.

4.

We now present the main result.

THEOREM 2: *Let* $0 < p < 1$. *Then* $A \in (w_p(X), c(Y))$ *if and only if*:

(9) *There exists* $\lim_n A_{nk} = A_k$ *(each* k*)*,

$$M_1 = \sup \sum_{r=0}^{\infty} 2^{r/p} \max_r \|A_{nk}^* f\| < \infty, \tag{10}$$

$$M_2 = \sup \sum_{r=0}^{\infty} 2^{r/p} \max_r \|(A_{nk}^* - A_k^*) f\| < \infty, \tag{11}$$

where in (10) *and* (11) *the supremum is taken over all* $n \geqq 1$ *and all* $f \in S^*$.

PROOF: Consider the necessity. That (9) is necessary is trivial. Let us show that (11) is necessary – the necessity of (10) may be shown by similar reasoning. Define $T_n(x) = \sum A_{nk} x_k$, the series converging for each n and all $\bar{x} \in w_p(X)$. By the Banach-Steinhaus theorem each T_n is a continuous linear operator on $w_p(X)$ into Y, whence for each triple (m, n, h) of positive integers the set

$$E(m, n, h) = \{\bar{x} \in w_p(X): \|(T_n - T_m)(x)\| \leqq h\}$$

is closed. Hence $E(h)$, the intersection over all (m, n), is closed, and $w_p(X)$ is the union of the $E(h)$. Since $w_p(X)$ is of the second category a standard type of argument yields the existence of an absolute constant H such that $\sup_{m,n} \|(T_n - T_m)(x)\| \leqq H\|\bar{x}\|^{1/p}$ for all $\bar{x} \in w_p(X)$.

Now let s be a positive integer, θ the zero of X and consider only those sequences $\bar{x}$ such that $x_k = \theta$ for $k \geqq 2^{s+1}$. Also, write $B_{nk} = A_{nk} - A_k$. Then, using (9), but letting $m \to \infty$, we obtain

$$\Big\| \sum_{k=1}^{2^{s+1}-1} B_{nk} x_k \Big\| \leqq H\|\bar{x}\|^{1/p},$$

so that, for every $f \in Y^*$,

$$\Big| \sum_{k=1}^{2^{s+1}-1} (f, B_{nk} x_k) \Big| \leqq \|f\| \cdot H \cdot \|\bar{x}\|^{1/p}. \tag{12}$$

Next we determine $\|z_{nk}\| \leqq 1$ such that $\|B_{nk}^* f\| \leqq 2|(f, B_{nk} z_{nk})|$. Then by suitable choice of a complex sequence (a_k), with w_p norm equal 1 (see [4, p. 173]) we have $(a_k z_{nk}) \in w_p(X)$, so from (12) we get

$$\sum_{r=0}^{s} 2^{r/p} \max_r |(f, B_{nk} z_{nk})| \leqq \|f\| \cdot H,$$

whence

$$\sup_n \sum_{r=0}^{\infty} 2^{r/p} \max_r \|B_{nk}^* f\| \leqq 2\|f\| \cdot H$$

for every $f \in Y^*$, which implies (11).

Conversely, let (9), (10) and (11) hold. Let R_{nm} be given by (1) and write R_m for the mth tail of the sequence (A_k). By the argument used to prove that (5) implied (7), we see that (11) implies

(13) $$\sup_n \|R_{nm} - R_m\| \to 0 \qquad (m \to \infty).$$

Also, (10) implies

(14) $$\|R_{nm}\| \to 0 \qquad (\mathrm{m} \to \infty, \text{ each } n).$$

Now let $x_k \to l(w_p(X))$. Then by (10) and the argument of the sufficiency part of Lemma 1,

(15) $$\|\sum_{k=m}^{m+s} A_{nk}(x_k - l)\| \leqq \varepsilon M_1,$$

for each $\varepsilon > 0$, for all (n, s) and all sufficiently large m. Letting $n \to \infty$ in (15) we get

(16) $$\|\sum_{k=m}^{m+s} A_k(x_k - l)\| \leqq \varepsilon M_1.$$

Hence by (16),

$$\|\sum_{k=m}^{m+s} A_k x_k\| \leqq \varepsilon M_1 + \|\sum_{k=m}^{m+s} (A_k - A_{1k})l\| + \|\sum_{k=m}^{m+s} A_{1k} l\|$$
$$\leqq \varepsilon M_1 + \|R_{1m} - R_m\| \cdot \|l\| + \|R_{1m}\| \cdot \|l\|,$$

so by (13) and (14) we see that $\sum A_k x_k$ converges for each $\bar{x} \in w_p(X)$.

Is is now a simple matter to show that (9), (13) and (14) are sufficient for A to be in $(l_\infty(X), c(Y))$, and that for such A,

(17) $$\lim_n \sum \mathrm{A}_{nk} \mathrm{x}_k = \sum \mathrm{A}_k \mathrm{x}_k$$

for each $\bar{x} \in l_\infty(X)$. We remark that (9), (13) and (14) are also necessary for

A to be in $(l_\infty(X), c(Y))$, but the proof is not completely trivial, and we do not require the necessity for our present purpose.

Finally, let us write

$$\sum A_{nk}x_k = \sum A_k x_k + \sum (A_{nk}-A_k)(x_k - l) + \sum (A_{nk}-A_k)l.$$

Then (9) and (13) imply $\sum (A_{nk}-A_k)(x_k-l) \to \theta$ $(n \to \infty)$, and (17) implies $\sum (A_{nk}-A_k)l \to \theta$ $(n \to \infty)$. Hence

$$\lim_n \sum A_{nk}x_k = \sum A_k x_k \tag{18}$$

for every $\bar{x} \in w_p(X)$. This proves the theorem.

Next we give the generalization of Kuttner's theorem.

THEOREM 3: *Let $0 < p < 1$ and let T be a fixed non-zero element of $B(X, Y)$. Suppose that $A = (A_{nk})$ is as in Theorem 1. Then there is a sequence in $w_p(X)$ which is not summable A.*

PROOF: Suppose, if possible, that $A \in (w_p(X), c(Y))$. Take $x \in X$ such that $T(x) \neq \theta$. Then by (18) we have $\lim_n \sum A_{nk}x$. But (3) implies $\lim_n \sum A_{nk}x = T(x)$ and (2) implies $\sum A_k x = \theta$, whence $T(x) = \theta$, contrary to the choice of x.

5.

We now briefly consider the case $1 \leqq p < \infty$. The norm

$$\|\bar{x}\| = \sup_r \left(\frac{1}{2^r}\sum_r \|x_k\|^p\right)^{1/p}$$

makes $w_p(X)$ into a Banach space.

The analogue of Lemma 1 is:

LEMMA 2: *Let $1 \leqq p < \infty$ and $(B_k) \in B(X, Y)$. Then $\sum B_k x_k$ converges, whenever $\bar{x} \in w_p(X)$ if and only if*

$$\sum B_k \text{ converges,} \tag{19}$$

$$\sup \sum_{r=0}^{\infty} 2^{r/p}\Big(\sum_r \|B_k^* f\|^q\Big)^{1/q} < \infty, \tag{20}$$

where $1/p + 1/q = 1$, the supremum is taken over $f \in S^*$ and $\sum_r$ denotes a sum over $2^r \leqq k < 2^{r+1}$. The case $p = 1$ of (20) *is interpreted as* (5) *with* $p = 1$.

We remark that (19) and (20) are independent. For example, in the space of complex numbers, if $b_k = (-1)^k/k$ then (19) holds but

$$\sum_{r=0}^{\infty} 2^{r/p}\Big(\sum |b_k|^q\Big)^{1/q} = \infty,$$

so that (20) fails. On the other hand, let $X = Y = \{x \in w_p : x_k \to 0(w_p)\}$. Then by [3, p. 290], $f \in X^*$ if and only if $f(x) = \sum a_k x_k$ where a is such that

$$(21) \qquad \|f\| = \sum_{r=0}^{\infty} 2^{r/p} (\sum_r |a_k|^q)^{1/q} < \infty.$$

Let us define $B_k : X \to X$ by $B_k x = (0, 0, \cdots, x_1, 0, 0, \cdots)$ with x_1 in the kth place. Then $(B_k^* f, x) = (f, B_k x) = a_k x_1$, so that $\|B_k^* f\| = |a_k|$. Hence (21) implies

$$\sum_{r=0}^{\infty} 2^{r/p} (\sum \|B_k^* f\|^q)^{1/q} < \infty$$

for each $f \in X^*$. Thus, by the argument immediately following (8) we see that (20) holds. Now take $x = (1, 0, 0, \cdots)$ and write $y(n) = \sum_{k=1}^{n} B_k x$. Then $z \equiv y(2^{r+1}-1) - y(2^r - 1)$ is a sequence such that $z_k = 1$ for $2^r \leqq k < 2^{r+1}$ and $z_k = 0$ otherwise. Hence $\|z\| = 1$, so that $\sum B_k x$ diverges, which means that (19) fails.

Finally, using arguments similar to those in the proof of Theorem 2 we can establish

THEOREM 4: *Let $1 \leqq p < \infty$. Then $A \in (w_p(X), c(Y))$ if and only if: There exists*

$$(22) \qquad \lim_n A_{nk} \text{ (each } k\text{) and } \lim_n \sum A_{nk},$$

$$(23) \qquad \sup \sum_{r=0}^{\infty} 2^{r/p} (\sum_r \|A_{nk}^* f\|^q)^{1/q} < \infty,$$

$$(24) \qquad \sup \sum_{r=0}^{\infty} 2^{r/p} (\sum_r \|(A_{nk}^* - A_k^*)\|^q)^{1/q} < \infty,$$

the suprema being over all $n \geqq 1$ and all $f \in S^*$.

REFERENCES

[1] B. KUTTNER: Note on strong summability. *J. London Math. Soc., 21* (1946) 118–122.

[2] G. G. LORENTZ and M. S. MACPHAIL: Unbounded operators and a theorem of A. Robinson. *Trans. Royal Soc. of Canada, XLVI, Series III* (1952) 33–37.

[3] I. J. MADDOX: On Kuttner's theorem. *J. London Math. So., 43* (1968) 285–290.

[4] I. J. MADDOX: *Elements of functional analysis* (Cambridge University Press, 1970).

[5] A. ROBINSON: On functional transformations and summability. *Proc. London Math. Soc.*, (2), *52* (1950) 132–160.

(Oblatum 12–IX–1973)

Department of Pure Mathematics,
The Queen's University of Belfast