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THE POINT-OUTERTHICKNESS OF COMPLETE n-PARTITE GRAPHS

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A graph G is said to have property F_n, $n \geqq 1$, if G has no subgraphs homeomorphic from the complete graph K_{n+1} or the complete bipartite graph $K([(n+2)/2], \{(n+2)/2\})$. For a real number x, $[x]$ denotes the greatest integer not exceeding x, and $\{x\}$ is the least integer not less than x. For $n = 1, 2, 3, 4$ graphs with property F_n correspond respectively with totally disconnected, acyclic, outerplanar, and planar graphs. In [3] Chartrand, Geller, and Hedetniemi defined *the point-partition number* $f_n(G)$, $n \geqq 1$, of a graph G as the minimum number of pairwise disjoint subsets into which the point set of G can be partitioned such that each set induces a graph with property F_n. Such a partition is called an F_n partition. The parameter f_1 is the famous chromatic number, and f_2 is the more recently introduced point-arboricity. (See, for example, [4], [5], or [8].) In this paper we consider f_3, the point-outerthickness.

By replacing the word 'point' in the definition of $f_n(G)$, $n \geqq 2$, with 'line' we obtain the line-partition number $f'_n(G)$. Nash-Williams [9] developed an exact formula for $f'_2(G)$, the arboricity of G. The parameter $f'_4(G)$ is called the thickness of G. The precise value of $f'_4(K_p)$ is known for all p (See [7] and [6]). Beineke, Harary, and Moon [2] and Beineke [1] have determined $f'_4(K(m, n))$ for most, but not all, values of m and n.

Before beginning our investigation of $f_3(G)$, which henceforth is denoted simply $f(G)$ we need some additional definitions and notation. The cardinality of set S is denoted by $|S|$. Let $V_1, V_2, \cdots, V_n$ be finite, nonvoid, mutually disjoint sets with $|V_i| = p_i$, $1 \leqq i \leqq n$, and $p_1 \leqq p_2 \leqq \cdots \leqq p_n$, the *complete n-partite* graph $G = K(p_1, p_2, \cdots, p_n)$ has point set $\bigcup_1^n V_i$ and two points of G are adjacent if and only if they are in different V_i. The V_i are called *partite sets* of G. The complete bipartite graph $K(1, n)$ is called a *star*. Now, in four theorems we develop an exact formula for the point-outerthickness of any complete n-partite graph and also give the desired decomposition. Chartrand, Kronk, and Wall, [4], developed the analogous formula for point-arboricity.

We begin with a number of observations.

REMARK 1: For every positive integer p, $f(K_p) = \{p/3\}$.

REMARK 2: A complete n-partite graph G, $n \geqq 2$, is outerplanar if and only if G is isomorphic to one of the following: $K(1, 1, 2)$, $K(2, 2)$, $K(1, 1, 1)$, or $K(1, m)$ where m is any positive integer.

REMARK 3: Let S be a set of at least five points of a complete n-partite graph G. If the graph induced by S is outerplanar, then it either has no lines or is a star, and S has all but possibly one point from a single partite set.

Throughout the remainder of the paper we use the following notation:

$G = K(p_1, p_2, \cdots, p_n)$

$p_0 = 0$

a = least positive integer such that $\sum_{i=1}^{a} p_i \geqq n-a$.

$r = \max \{i: p_i \leqq 2\}$

$k = \max \{i: p_i \leqq 1\}$

$s = \{(\sum_{1}^{r} p_i + 3(n-r))/4\}$ if $(k+r-n) \leqq (2/3)(2r-n)$ and $p_{a+1} \leqq 2$.

$s = \{(2n-r)/3\}$ if $(k+r-n) > (2/3)(2r-n)$ and $p_{a+1} \leqq 2$.

THEOREM 1: *If* $p_{a+1} \geqq 3$, *then* $f(G) = n - \max \{b: \sum_{1}^{b} p_i \leqq n-b\}$.

PROOF: We consider two cases and in each case show that the desired result is an upper bound for the point-outerthickness of G. Then, combining the two cases, we verify that there is no smaller outerplanar partition of $V(G)$.

Case (i) *Suppose* $\sum_{1}^{a} p_i = n-a$. We can partition $V(G)$ into $n-a$ sets $S_1, S_2, \cdots, S_{n-a}$, where $S_j = V_{n+1-j} \cup \{v_j\}$, $1 \leqq j \leqq n-a$, and each v_j is an element of $\bigcup_{1}^{a} V_i$. Since each S_j induces a star we have that $f(G) \leqq n-a = n - \max \{b: \sum_{1}^{b} p_i \leqq n-b\}$.

Case (ii) *Assume* $\sum_{1}^{a} p_i > n-a$. Since $\sum_{1}^{a-1} p_i < n-a+1$, the number of elements in $\bigcup_{1}^{a-1} V_i$ is less than the number of sets in the collection $\{V_a, V_{a+1}, \cdots, V_n\}$. We form $r = \sum_{1}^{a-1} p_i$ mutually disjoint subsets $S_1, S_2, \cdots, S_r$ of $V(G)$, with $S_j = V_{n+1-j} \cup \{v_j\}$, $1 \leqq j \leqq r$, and where each v_j is an element of $\bigcup_{1}^{a-1} V_k$. Next, form mutually disjoint point sets $S_{r+1}, \cdots, S_{n-a}$ where, for $k = r+1, \cdots, n-a$, $S_k = V_{n+1-k} \cup \{v_k\}$ and the v_k are distinct elements of V_a. Since $\sum_{1}^{a} p_i > n-a$, we have some points of V_a which are not in any $S_j, j = 1, \cdots, n-a$. Call this set of points S_{n-a+1}. The sets $S_1, \cdots, S_{n-a}$ each induce a star and the set S_{n-a+1} induces a totally disconnected graph. It follows that $f(G) \leqq n-a+1 = n - \max \{b: \sum_{1}^{b} p_i \leqq n-b\}$.

In each of the aforementioned cases denote the upper bound by z and suppose $f(G) = t < z$. Then $V(G)$ has an outerplanar partition $T_1, T_2, \cdots, T_t$ where $|T_i| \geqq |T_{i+1}|$. Let h be the largest integer such that $|T_h| > |S_h|$.

Then

$$|\bigcup_1^h T_i|-h > |\bigcup_1^h S_i|-h.$$

From the formulation of the various S_i it follows that the cardinality of S_h is at least four. For $i < h$, $|T_i| \geqq |T_h| > |S_h| \geqq 4$. Remark 3 implies that each T_i, $i \leqq h$, has all but at most one point from a single partite set. If such a point exists for a given T_i, denote it by w_i. Then, for $i \leqq h$, define $T_i' = T_i - \{w_i\}$ for all i for which w_i exists and $T_i' = T_i$, otherwise. This implies that the set $\bigcup_1^h T_i'$ has all of its points in h or fewer partite sets. However,

$$|\bigcup_{n-h+1}^{n} V_i| = |\bigcup_1^h S_i|-h.$$

Thus the union of any h partite sets has at most $|\bigcup_1^h S_i|-h$ points, but

$$|\bigcup_1^h S_i|-h < |\bigcup_1^h T_i|-h \leqq |\bigcup_1^h T_i'|$$

implies that $\bigcup_1^h T_i'$ cannot have all of its points in h or fewer partite sets. We have a contradiction and $f(G) = z$ in both cases.

THEOREM 2: *If $p_{a+1} \leqq 2$, then $V(G)$ can be partitioned into outerplanar sets $S_1, S_2, \cdots, S_s$, where $|S_i| \geqq |S_{i+1}|$.*

PROOF: We exhibit an outerplanar partition of $V(G)$ into the desired number of subsets. The inequality $r > a$ implies that $\sum_1^a p_i \geqq n-a > n-r$. Thus there are more elements in the set $\bigcup_1^a V_i$ than sets in the collection $\{V_{r+1}, V_{r+2}, \cdots, V_n\}$. We form $n-r$ mutually disjoint sets $S_1, S_2, \cdots, S_{n-r}$ where $S_j = V_{n+1-j} \cup \{v_j\}$, $1 \leqq j \leqq n-r$ and $v_j \in \bigcup_1^a V_i$. Moreover, the points v_j are always selected successively from the set V_i with i minimum such that V_i has points remaining.

Each of the S_i induces a star with at least four points, and there are $\sum_1^r p_i-(n-r) > 0$ points of G not in any S_i. Each of these points is contained in a partite set of G which consists of at most two elements.

Case (i) *Suppose* $k+r-n \leqq (2/3)(2r-n)$. If $k-(n-r)$ is positive, we have $k+r-n$ unused one-point partite sets of G. In defining the S_i we used points from at most $2(n-r)$ partite sets of G. Thus, there are at least $n-2(n-r) = 2r-n$ partite sets of G which are disjoint from each S_i, $i = 1, \cdots, n-r$. Since $k+r-n \leqq (2/3)(2r-n)$, we form mutually disjoint sets $S_{n-r+1}, \cdots, S_q$, each consisting of two one-point partite sets and one two-point partite set until we have at most one unused singleton partite set. All remaining partite sets have precisely two points. If $k+r-n$ is not positive, then there are only two-point partite sets of G remaining and perhaps one more point which is an element of a two-point partite set. Thus, in either case, we have two-point partite sets remaining, and possibly one extra point. With the remaining points, we may form mutually disjoint sets which consist of the unit of two of the remaining two-point partite sets until there are at most three points remaining. These points form an outerplanar set. Thus, we have partitioned $V(G)$ into

$$\{(\sum_{1}^{r} p_i+3(n-r))/4\} = s$$

outerplanar sets, each of which, with at most one exception, has at least four points.

Case (ii) *Suppose* $k+r-n > (2/3)(2r-n)$. In this case, $2r-n$ is nonnegative, and thus $k+r-n$, the number of unused singleton partite sets, is positive. This implies that for $1 \leqq i \leqq n-r$, $S_i = V_i \cup V_{n+1-i}$, and we have precisely $2r-n$ unused partite sets of G. In this case there are more than twice as many unused partite sets with one point as unused partite sets with two points. It follows that we can form disjoint sets $S_{n-r+1}, \cdots, S_{n-k}$ in such a way that each set consists of four points from three of the remaining partite sets. When this is done, there are $3k-r-n$ points remaining in G. These points induce a complete subgraph and have an outerplanar partition into $\{(3k-r-n)/3\}$ sets. Let the sets in this partition be denoted by $S_{n-k+1}, \cdots, S_s$, $s = n-k+\{(3k-r-n)/3\} = \{(2n-r)/3\}$.

THEOREM 3: *Let* $p_{a+1} \leqq 2$ *and suppose that* $V(G)$ *has an outerplanar partition* $T_1, \cdots, T_t$ *where* $|T_i| \geqq |T_{i+1}|$ *and* $t < s$. *Then there exists a largest positive integer* h *such that* $|T_h| > |S_h|$, *and furthermore* $|T_h| = 4$. *Also if* $m = \max\{i: p_i \leqq 3\}$, *then the* T_i *can be reordered if necessary so that* T_h *does not contain* V_i, $m+1 \leqq i \leqq n$.

PROOF: Since all but perhaps one of the S_i has at least three points, it follows that $|T_h| \geqq 4$. In order to verify the first part of the theorem we assume that $|T_h| > 4$ and obtain a contradiction. Since $|T_h| > |S_h|$, we have

$$|\bigcup_{h+1}^{t} T_i| < |\bigcup_{h+1}^{s} S_i|,$$

which implies that

$$|\bigcup_{1}^{h} T_i|-h > |\bigcup_{1}^{h} S_i|-h.$$

For $i \leqq h$, T_i has five or more points and Remark 3 implies that each such T_i has all but possibly one point from a single partite set. Define T_i', $1 \leqq i \leqq h$ as in Theorem 1. Then the set $\bigcup_1^h T_i'$ has all of its points in h or fewer partite sets. We now consider two cases depending upon h.

Case (i) $h \leqq n-r$. From the fact that each S_i, $1 \leqq i \leqq n-r$, consists of V_{n-i+1} together with one other point it follows that

$$|\bigcup_{n-h+1}^{n} V_i| = |\bigcup_{1}^{h} S_i| - h.$$

Hence, the union of any h partite sets has at most $|\bigcup_i^h S_i| - h$ points. However,

$$|\bigcup_{1}^{h} S_i| - h < |\bigcup_{1}^{h} T_i| - h \leqq |\bigcup_{1}^{h} T_i'|.$$

Thus, $|\bigcup_1^h T_i'|$ cannot have all of its points in h or fewer partite sets, a contradiction.

Case (ii) $h > n-r$. The sets $S_1, \cdots, S_{n-r}$ exhaust all partite sets with three or more points. Since h is necessarily less than s, the sets $S_{n-r+1}, \cdots, S_h$ each use partite sets with one or two points. Without loss of generality, we may assume that these are the partite sets $V_{n+1-(n-r+1)}, \cdots, V_{n+1-h}$. This implies that

$$|\bigcup_{n-h+1}^{n} V_i| < |\bigcup_{1}^{h} S_i| - h.$$

The union of any h partite sets has at most $|\bigcup_{n-h+1}^{n} V_i|$ points. However, the fact that

$$|\bigcup_{n-h+1}^{n} V_i| < |\bigcup_{1}^{h} S_i| - h < |\bigcup_{1}^{h} T_i| - h \leqq |\bigcup_{1}^{h} T_i'|$$

is again a contradiction. Thus $|T_h| = 4$.

For the second part of the Theorem we reorder the T_i, $1 \leqq i \leqq t$, so that, if $|T_i| = |T_j|$ and T_i has more points from some partite set than T_j has from any partite set, then $i < j$.

We now suppose there exists V_{i_1}, $m < i_1 \leqq n$, which is contained in T_h and obtain a contradiction. Since $|T_h| = 4$ and $|V_{i_1}| \geqq 4$, we know that $T_h = V_{i_1}$. From our ordering on the partition $T_1, \cdots, T_t$, it follows that the sets $T_1, \cdots, T_h$ have at most $h-1$ points from one-point partite sets of G. The sets $T_{h+1}, \cdots, T_t$ have at most $|\bigcup_{h+1}^{t} T_i|$ points from one-point partite sets of G. The partition $T_1, \cdots, T_t$ uses all one-point partite sets of G, and the number used must be not more than $h-1+|\bigcup_{h+1}^{t} T_i|$. Thus,

$$h-1+|\bigcup_{h+1}^{t} T_i| \geqq k. \tag{1}$$

The set S_h is the union of three one-point partite sets of G, and thus the sets $S_{h+1}, \cdots, S_s$ each consist of only points from one-point partite sets; that is, the sets $S_{h+1}, \cdots, S_s$ contain $|\bigcup_{h+1}^{s} S_i|$ points from one-point partite sets. However, each of the sets $S_1, \cdots, S_h$ contains at least one point from a one-point partite set. Thus, the partition $S_1, \cdots, S_s$ contains at least $h+|\bigcup_{h+1}^{s} S_i|$ points from one-point partite sets. It follows that

$$k \geqq h+|\bigcup_{h+1}^{s} S_i|. \tag{2}$$

The fact that $|\bigcup_{h+1}^{s} S_i| > |\bigcup_{h+1}^{t} T_i|$, together with (1) and (2), yields a contradiction and completes the proof of Theorem 3.

THEOREM 4: *If* $p_{a+1} \leqq 2$, *then* $f(G) = s$.

PROOF: Suppose that $V(G)$ has an outerplanar partition $T_1, T_2, \cdots, T_t$, $t < s$, with $|T_i| \geqq |T_{i+1}|$. Then the set T_h as given in Theorem 3 has cardinality 4. If $(k+r-n) \leqq (2/3)(2r-n)$, then by the construction in Theorem 2, $4 \leqq |S_h| < |T_h| = 4$. Since this is impossible we need only consider $(k+r-n) > (2/3)(2r-n)$.

Among the outerplanar partitions of $V(G)$ into t sets, select one which has a maximum number, say M, of V_i, $m < i \leqq n$, with the property that each is contained in some set of the partition. Call this partition $T_1, \cdots, T_t$, and order the sets as in the second part of Theorem 3. According to Theorem 3, $|T_h| = 4$. Again let $m = \max\{i: p_i \leqq 3\}$ and consider two cases.

Case (i) Each of the sets $V_{m+1}, \cdots, V_n$ is contained in some T_i. *We may* assume, without loss of generality, that $V_i \subset T_{n+1-i}$, for $i = m+1, \cdots, n$. From the facts that, for $1 \leqq i \leqq n-k$, $S_i = V_{n+1-i} \cup W_i$ where W_i consists of one or two points and S_h consists of three points from three different partite sets, we have that

$$h > n-k. \tag{1}$$

The sets $T_{n-m+1}, T_{n-m+2}, \cdots, T_h$ each have at least four points and therefore at least two points from one partite set. However, all partite sets with at least four points are used in sets $T_1, \cdots, T_{n-m}$. Thus, we need $h-(n-m+1)+1$ partite sets with two or three points, and there are only $m-k$ such partite sets. Hence, using inequality (1) we have a contradiction.

Case (ii) At least one of the partite sets with four or more points, say V_{i_0}, has points in two or more of the sets T_i.

If V_{i_0} has at least three points in one T_j, say T_b, we add all other points of V_{i_0} to T_b. We now have an outerplanar partition of $V(G)$ into t sets such that $M+1$ partite sets with at least four points are contained in various T_j. This is a contradiction.

If V_{i_0} has exactly two points in some T_i, say T_b, then V_{i_0} has one or two points in T_c, $c \neq b$. We add the points of $T_c \cap V_{i_0}$ to T_b and add one point of $T_b - V_{i_0}$ (if such a point exists) to T_c. We have an outerplanar partition of $V(G)$ into t sets such that V_{i_0} has three or more points in one set, and M partite sets V_i, $m < i \leqq n$, are each contained in some T_j. According to the previous paragraph, this leads to a contradiction.

We now suppose that V_{i_0} has each point in a different T_j. Then T_h has at most one point of V_{i_0}. Let w_1, w_2, and w_3 be points in $T_h - V_{i_0}$. Add all points of V_{i_0} to T_h. Since V_{i_0} has at least four points, three of these points must be in distinct T_j different from T_h, say T_{i_1}, T_{i_2}, and T_{i_3}. For $k = 1, 2, 3$, insert w_k into T_{i_k}. As before, this yields a new outerplanar partition of $V(G)$ into t sets. By the second part of Theorem 3, T_h did not contain any partite sets with four or more points, and hence this new partition has $M+1$ sets, each of which contains a V_i, $m < i \leqq n$. This is a contradiction and we have shown that $f(G) = s$.

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