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The Hauptvermutung for $C^\infty$ homeomorphisms II.
A proof valid for open 4-manifolds


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THE HAUPTVERMUTUNG FOR $C^\infty$ HOMEOMORPHISMS II
A PROOF VALID FOR OPEN 4-MANIFOLDS

M. G. Scharlemann and L. C. Siebenmann

Introduction

It has often been observed that every twisted sphere $M^m = B^m_+ \cup_f B^m_-$ of Milnor is $C^\infty$ homeomorphic to the standard sphere $S^m$, although in general it is not diffeomorphic to $S^m$. Recall that a twisted sphere is put together from copies of the standard hemispheres $B^m_{\pm}$ of $S^m$ by reidentifying boundaries $\partial B^m_\pm$ to $\partial B^m_\mp$ under a diffeomorphism $f$. One obtains a homeomorphism $h: M^m \to S^m$ by setting $h|_{B^m_-} = \text{identity}$ and $h|_{B^m_+} = \{\text{cone on } f: S^{m-1} \to S^{m-1}\}$, the latter regarded as a self-homeomorphism of $B^m_+ = \text{cone}(S^{m-1})$. This is $C^\infty$ and non-singular, except at the origin in $B^m_+ (= \text{cone vertex})$. Composing $h$ with a suitable $C^\infty$ homomorphism $h$ whose derivatives vanish at the origin of $B^m_+$ yields a $C^\infty$ homeomorphism $h: M^m \to S^m$ (Appendix A).

Since the twisted spheres represent the classical obstructions to smoothing a PL homeomorphism to a diffeomorphism, it is not surprising to find (§4 of preprint) that if $M$ is any PL manifold and $\sigma, \sigma'$ are two compatible smoothness structures on it, then one can obtain a $C^\infty$ smooth homeomorphism $h: M_\sigma \to M_{\sigma'}$. It would be reasonable to guess that the same is true for arbitrary smoothings $\sigma, \sigma'$ of $M$. However, we prove the following.

HAUPTVERMUTUNG FOR $C^\infty$ HOMEOMORPHISMS: Let $f: M' \to M$ be a $C^\infty$ homeomorphism of connected metrizable smooth manifolds without boundary. If $M$ and $M'$ are of dimension 4 suppose they are non-compact. Let $M$ and $M'$ be given Whitehead compatible PL structures $[M_{\mu_2}]$.

1 Supported in part by NSF Grant GP-34006.

2 We there used classical smoothing theory and a TOP/$C^\infty$ handle lemma for index $\geq 6$. Surely a more direct proof exists?!?

3 A PL manifold structure $\Sigma$ on $M$ is $(C^\infty)$ Whitehead compatible with the smooth $(C^\infty)$ structure of $M$ if for some PL triangulation of $M_{\Sigma}$ as a simplicial complex, the inclusion of each closed simplex is smooth and nonsingular as a map to $M$. 

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Then there exists a topological isotopy of $f$ to a PL homeomorphism.

Our purpose here is to give a handle by handle proof of this result which uses no obstruction theory and which does succeed with the specified four-manifolds.

Note that the singularities of the differential $Df$ may form a nasty closed set in $M'$ of dimension as high as $m - 1$. The one pleasant property which for us distinguishes $f$ from a mere homeomorphism is the fact that the critical values are meager by the Sard-Brown Theorem [11], both for $f$ and for the composition of $f$ with any smooth map $M \to X$. In fact our result follows with astonishing ease from this fact.

In dimension $\leq 6$ the PL homeomorphism asserted by the $C^\infty$ Hauptvermutung is equivalent to diffeomorphism since there is no obstruction to smoothing a PL homeomorphism [12] [8].

Ordinary homeomorphism in dimension $\leq 6$ does not imply diffeomorphism. Thus the following example may clarify the meaning of our theorem. The second author shows in [17, § 2] how to construct a homeomorphism

$$h: T(\beta) \to T^6$$

of a smooth 1 manifold $T(\beta)$ that is known not to be diffeomorphic to $T^6$. By construction $h$ is a diffeomorphism 2 over the complement of a standard subtorus $T^3 \subseteq T^6$, and also over $T^3$ itself. The $C^\infty$ Hauptvermutung shows that there is no way of making $h$ smooth – say by squeezing towards the singularity set $T^3$ as one does for twisted spheres. The homeomorphisms that disprove the Hauptvermutung are thus measurably more complex than those known previously.

The $C^\infty$ Hauptvermutung lends credence to the following seemingly difficult conjecture due to Kirby and Scharlemann [5]. Consider the least pseudo-group MCCG$_n$ of homeomorphisms on $R^n$ which contains all $C^\infty$ homeomorphisms of open subsets of $R^n$.

**Conjecture:** The isomorphism classification of MCCG$_n$ manifolds coincides naturally with the isomorphism classification of PL $n$-manifolds without boundary.

It can be shown that every PL homeomorphism of open subsets of $R^n$ is in MCCG$_n$, see [5], [ ] 3. Thus MCCG can be regarded as an enlargement of PL to contain DIFF, an enlargement which might eventually be useful in dynamics, group action theory, smoothing theory, etc. – especially

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1 In [17, § 2] one can replace PL everywhere by DIFF with no essential change in proofs.
2 In [17, § 2] one should choose the DIFF pseudo-isotopy $H: (I; 0, 1) \times B^2 \times T^n$ to be used to build $h$ constant near 0 and 1 so as to prevent unwanted kinks in $h$.
3 Mistrust this assertion, as no proof has been written down. (Oct. 1974).
cially at points where mere homeomorphism seems too coarse a notion.

The organization of this article is as follows:
Section 1. A $C^\infty$/DIFF handle lemma for index $\leq 3$ in any dimension;
Section 2. A weak $C^\infty$/DIFF handle lemma for index 4 in dimension 4;
Section 3. Proof of an elaborated $C^\infty$ Hauptvermutung;
Appendix A. $C^\infty$-smoothing an isolated singularity;
Appendix B. Potential counterexamples in dimension 4.

1. A $C^\infty$/DIFF handle lemma for index $\leq 3$

The proof of the $C^\infty$ Hauptvermutung will be based on two handle-
smoothing lemmas 1.1 and 2.1 below.

DATA: Let $B^k$ be the unit ball in $R^k$ and let $f: M \to B^k \times R^n$ be a $C^\infty$
homeomorphism which is nonsingular near the boundary.

DEFINITION: A $C^\infty$ isotopy $f_t$, $0 \leq t \leq 1$, of $f$ will be called allowable if
it fixes all points outside some compactum in $(\text{int } B^k) \times R^n$ — i.e. it has
compact support in $(\text{int } B^k) \times R^n$.

1.1. $C^\infty$/DIFF HANDLE LEMMA (index $\leq 3$): For $f: M \to B^k \times R^n$ as
above and $k = 0, 1, 2, 3$, there is an allowable isotopy of $f$ to a $C^\infty$ homeo-


morphism $f_1$ which is non-singular near $f^{-1}(B^k \times 0)$.

Recall that for index $k = 3$, the $C^0$ version of this lemma is false, a key
failure of the $C^0$ Hauptvermutung [6].

PROOF of 1.1: Our first step is to allowably isotop $f$ so that $0 \in R^n$ is a
regular value of the projection $p_2 f: M \to R^n$. Choose a regular value $y_0$
in $R^n$ with $|y_0| < \frac{1}{2}$. Let $\psi_t$, $0 \leq t \leq 1$, be a diffeotopy (non-singular $C^\infty$
isotopy) of $\text{id}|R^n$ with support in $B^n$ carrying $y_0$ to 0. Let $\gamma: B^k \to [0, 1]$ be
a $C^\infty$ map such that $\gamma = 0$ near $\partial B^k$ and $f$ is nonsingular over $\{\gamma^{-1}[0, 1]\}$
$x B^n$. Now

$$\Psi_t: B^k \times R^n \to B^k \times R^n$$

defined by $\Psi_t(x, y) = (x, \psi_{t(y)}(y))$ for $0 \leq t \leq 1$ gives an allowable isotopy
$f_t = \Psi_t f$ as desired. See figure 1a, which illustrates this manoeuvre for
$k = n = 1$.

Revert to $f$ as notation for $f_1 = \Psi_1 f$.

As a second step we will allowably isotop $f$ by a squeeze so that the
structure imposed by $f$ on $B^k \times R^n$ is a product along $R^n$ near $B^k \times 0$.
Choose a small closed $\epsilon$-ball $B_{\epsilon}$ about 0 in $R^n$ such that $p_2 f$ is nonsingular
over $B_{\epsilon}$, hence a (trivial) smooth bundle projection over $B_{\epsilon}$. Choose a
trivialization $\phi$ of this bundle in a commutative diagram
With the help of a collar of $\partial N$ we can arrange that on a neighborhood of $\partial N \times B_t$, $\Phi$ coincides with $f^{-1}$. See figure 1b, which illustrates the behavior of $f\Phi(x \times B_t)$ for 5 values of $x$ in $N$.

Figure 1a.

Let $\lambda: [0, \infty) \to [0, \infty)$ be a smooth map such that $\lambda([0, \varepsilon/2]) = 0$ while $\lambda: (\varepsilon/2, \infty) \to (0, \infty)$ is a diffeomorphism equal to the identity on $[\varepsilon, \infty)$; then define a $C^\infty$ homotopy $\lambda_t: R^n \to R^n$, $0 \leq t \leq 1$, by

$$\lambda_t(y) = (1-t)y + t \frac{\lambda(|y|)}{|y|} y$$

where $\lambda(|y|)/|y|$ is understood to be zero for $y = 0$. Define an allowable isotopy (see figure 1c)

$$f_t: M \to B^k \times R^n$$

to be fixed outside $f^{-1}(B^k \times B_t)$ and to send $\varphi(x, y) = \varphi(N \times B_t) = f^{-1}(B^k \times B_t)$ to $(p_1 f(x, \lambda_t(y)), y) \in B^k \times R^n$. It is not difficult to see that this completes the second step. Again revert to $f$ as notation for $f_1$.

The handle lemma is now clearly reduced to the handle problem posed by $f^{-1}(B^k \times 0) \to B^k \times 0$. Thus it remains only to prove
1.2. Lemma: If \( f : M \rightarrow B^k \), \( k = 0, 1, 2, 3 \), is a \( C^\infty \) homeomorphism which is nonsingular near \( \partial M \), then \( f \) is \( C^\infty \) isotopic rel \( \partial M \) to a diffeomorphism.

Proof of Lemma 1.2: By relative uniqueness of smooth structures in dimension \( \leq 3 \), there is a diffeomorphism \( \alpha : B^k \rightarrow M \) which is inverse to \( f \) near the boundary. Then \( f' = f\alpha : B^k \rightarrow B^k \) extends by the identity map to a \( C^\infty \)-homeomorphism \( S^k \rightarrow S^k \) where we identify \( B^k \) to \( B^k \) in \( S^k \). This map in turn extends to a \( C^\infty \)-homeomorphism \( B^{k+1} \rightarrow B^{k+1} \) by the smoothing lemma of Appendix A.

We now have a \( C^\infty \)-homeomorphism \( B^{k+1} \rightarrow B^{k+1} \) which is the identity near \( B^k \subset \partial B^{k+1} \) and \( f\alpha \) on \( B^k_+ \subset \partial B^{k+1} \). Let \( \theta : B^{k+1} \rightarrow B^k \times I \) be a homeomorphism which sends \( B^k_+ \) onto \( B^k \times \{0\} \) and is a diffeomorphism except where corners are added in \( B^k \). Then \( \theta F\theta^{-1} : B^k \times I \rightarrow B^k \times I \) is the identity near \( B^k \times \{0\} \cup \partial B^k \times I \) and hence a \( C^\infty \)-homeomorphism everywhere. Now \( \theta F\theta^{-1}(\alpha^{-1} \times \text{id}_I) \) is the required \( C^\infty \)-isotopy from \( f \) to a diffeomorphism.

This completes the proof of Lemma 1.2 and with it the proof of the \( C^\infty /\text{DIFF} \) handle lemma for index \( \leq 3 \).

Assertion: In the above proofs the use of relative uniqueness theorems for smooth structures in dimension \( \leq 3 \) can be replaced by the smooth Alexander-Schoenflies theorems in dimension \( \leq 3 \) (the latter are easily proved, c.f. Cerf [1, Appendix]).

Proof of Assertion: First note that these Schoenflies theorems suffice to prove Lemma 1.2 in case \( M \) is known to embed smoothly and nonsingularly in \( R^k \).

Next suppose the assertion established for index \( < k \). (It is trivial for index 0.) Then deal with index \( k \) by establishing Lemma 1.2 for index \( k \) using the smooth Schoenflies theorem in dimension \( k \), as follows. Smoothly triangulate \( B^k \) so finely that

\[ (*) \quad \text{For each } k\text{-simplex } \sigma \text{ of } B^k, f^{-1}(\sigma) \text{ lies in a co-ordinate chart of } M. \]

The index \( < k \) case suffices to get a \( C^\infty \) isotopy of \( f \) rel \( \partial M \) to an \( f_1 \) that is nonsingular over the \((k-1)\)-skeleton and still satisfies \((*)\). Then the smooth Schoenflies theorem suffices, by our first remark, to establish Lemma 1.2 for index \( k \).

2. A weak \( C^\infty /\text{DIFF} \) handle lemma for index 4

The \( C^\infty /\text{DIFF} \) handle problem for index 4 and dimension 4 admits a

\[ ^1 \text{ i.e. isotopic fixing a neighborhood of } \partial M. \]
weak solution based on the weak Schoenflies theorem for dimension 4 (given by Rourke and Sanderson [14, 3.38])

**Theorem:** Let $S \subset \mathbb{R}^4 - 0$ be a smoothly embedded 3-sphere, and let $T$ be the closure of the bounded component of $\mathbb{R}^4 - S$. Then $T - 0$ is diffeomorphic to $B^4 - 0$.

**Definition:** We call a homotopy $h_t$, $0 \leq t \leq 1$, *almost compact* if, for each $\tau < 1$, the homotopy $h_t$, $0 \leq t \leq \tau$, has compact support.

2.1. **Proposition:** Suppose $M^4$ is a smooth submanifold of $\mathbb{R}^4$, and $f : M \to B^4$ is a $C^\infty$ homeomorphism which is a diffeomorphism over a neighborhood of the boundary $\partial B^4$. Then there is an isotopy rel boundary $f_t : M \to B^4$, $0 \leq t \leq 1$, such that:

(i) $f_0 = f$ and $f_1$ is a diffeomorphism over $B^k - \{p\}$ for some point $p \in \text{int} B^4$.

(ii) $f_t$ restricts to a $C^\infty$ almost compact isotopy $M - f^{-1}\{p\} \to B^k - \{p\}$.

(iii) $f_t$ is fixed over some smooth path from $p$ to $\partial B^4$.

**Proof of 2.1:** Without loss of generality we may assume there is a radius of $B^4$ over which $f$ is nonsingular. In this case we will make $p = \{0\} \in B^4$ and cause the path mentioned in (iii) to be this radius. By the weak Schoenflies theorem, we can find a homeomorphism $\alpha : B^4 \to M$ such that $\alpha \circ f : B^4 \to B^4$ restricts to a diffeomorphism $(B^4 - 0) \to (B^4 - 0)$ and is the identity near $\partial B^4$. We can alter $\alpha$ rel boundary by a diffeotopy of $(B^4 - 0) \to M^4 - f^{-1}\{0\}$, so that $\alpha$ is also the identity on the chosen radius. This requires just a proper version, applied to $\alpha|(\text{open radius})$, of Whitney’s (ambient) isotopy theorem cf. [2].

Identifying $B^4 - \{0\}$ naturally to $\partial B^4 \times \mathbb{R}^+ = \partial B^4 \times [0, \infty)$ we are only required to find, for a certain $\{q\} \in \partial B^4$, an almost compact $C^\infty$-isotopy $f_t$, $0 \leq t \leq 1$, fixing $\{q\} \times \mathbb{R}^+$ and a neighborhood of $\partial B^4 \times \{0\}$, from $f' = f \circ \alpha : \partial B^4 \times \mathbb{R}^+ \to \partial B^4 \times \mathbb{R}^+$ to a diffeomorphism. Once this is accomplished the required isotopy $f_t$ of $f$ will be $f_t(f^{-1}(0)) = 0$ and $f_t(x) = f_t' \circ \alpha^{-1}(x)$ for $x \in M - f^{-1}(0)$.

Let $\mu_t : [0, \infty) \to [0, \infty)$ be an almost compact smooth (into) isotopy from the identity to a diffeomorphism $\mu_t : [0, \infty) \to [0, \varepsilon)$. (Only $\mu_1$ is not onto.) Let $\varepsilon > 0$ be so small that $f'$ is a diffeomorphism on $S^3 \times [0, \varepsilon)$. Define $f_t' : \partial B^4 \times [0, \infty) \to \partial B^4 \times [0, \infty)$ to be

\[
(\text{id} | \partial B^4) \times \mu_t \circ f' \circ (\text{id} | \partial B^4) \times \mu_t^{-1}.
\]

1 It is a down to earth version of Mazur’s proof of the topological Schoenflies theorem [9].
It clearly has the right properties and completes the proof of Proposition 2.1.

3. Proof of an elaborated $C^\infty$ Hauptvermutung

3.1. **THEOREM**: ($C^\infty$ Hauptvermutung). Consider a $C^\infty$ homeomorphism $f : M' \to M$ of smooth $m$-dimensional manifolds equipped with Whitehead triangulations. Suppose $f$ is also a PL equivalence over a neighborhood of some closed subset $C$ of $M$.

In case $\dim M = 4$ or $\dim \partial M = 4$ we make some provisos. If $\dim M = 4$ we suppose that each component of the complement of $C$ in $M$ has noncompact closure in $M$. In case $\dim \partial M = 4$ we suppose that each component of $\partial M - C$ has noncompact closure in $\partial M$.

(I) Then, for $m \leq 4$, there exists a $C^\infty$ isotopy rel $C$ from $f$ to a diffeomorphism.

(II) For $m = 5$ or $6$, there exists a topological isotopy rel $C$ from $f$ to a diffeomorphism.

(III) For all $m$, there exists a topological isotopy rel $C$ from $f$ to a PL homeomorphism.

The salient advance beyond [15] is clearly the case of open 4-manifolds in (III). Note that (II) is implied by (III) and classical smoothing theory (but we naturally get to (II) first).

**REMARK 1**: If $f$ is a $C^\infty$ homeomorphism which is a PL equivalence near $C$, then $f$ will be non-singular near $C$. Indeed $f$ PL implies that for each (closed) principal simplex $\sigma$ of a suitable subdivision of $M'$, $f$ maps $\sigma$ linearly into a principal simplex of $M$, hence $C^\infty$ non-singularly with rank $m$ into $M$ as a $C^\infty$ manifold. Thus, in the above theorem, $f$ is actually nonsingular near $C$.

**REMARK 2**: The provisos concerning dimension 4 can be eliminated if and only if the smooth 4-dimensional Schoenflies conjecture is true. (See Appendix B and Lemma 1.2.)

**REMARK 3**: It is easy to believe that in (II) the isotopy can be $C^\infty$.

**REMARK 4**: The isotopies produced by 3.1 can be made as small as we please for the strong (majorant) topology – except possibly where dimension 4 manifolds or boundaries intervene. This is accomplished merely by using sufficiently fine Whitehead $C^1$ triangulations in the proofs to follow.
3.2. Proof of 3.1 Part I: Manifolds of dimension \( \leq 4 \).

This is by far the most delicate part.

Exploit smooth collars of \( \partial M' \) and \( \partial M \) corresponding under \( f \) near \( C \) to \( C^\infty \) isotope \( f \rel C \) by a classical squeezing argument (cf. proof of 1.1) so that \( f \) becomes a product near the boundary along the collaring interval factor. This property is to be preserved carefully through all changes of \( f \).

Select a smooth Whitehead triangulation of \( M \) so fine that \( f \) is non-singular over a subcomplex containing \( C \), and the preimage of each 4-simplex lies in a co-ordinate chart. With no loss of generality we suppose now that \( C \) is a subcomplex.

Apply the \( C^\infty/\text{DIFF} \) handle lemma 1.1, around the smooth open \( k \)-simplices \( \tilde{\sigma} \cong R^k \) of \( M \) in order of increasing dimension for \( k = 0, 1, 2, 3 \), to make \( f \) nonsingular over a neighborhood of the 3-skeleton of \( M \). When \( \tilde{\sigma} \) lies in \( \partial M \) the handle lemma gives a \( C^\infty \) isotopy of \( f: \partial M' \to \partial M \) which we must damp out along the collaring interval factor to get a \( C^\infty \) isotopy of \( f \). The proof is now complete for \( m \leq 3 \).

Suppose now that \( m = 4 \). It is easy to choose the handles so near to the open simplices that for each 4-simplex \( \sigma \), the preimage of \( \sigma \) remains in its co-ordinate chart throughout the isotopy constructed thus far.

Using the index 4 weak \( C^\infty/\text{DIFF} \) handle lemma 2.1, we could give an isotopy of \( f \) over smooth 4-handles in the open 4-simplices to obtain a homeomorphism which is a diffeomorphism on the complement of center points of these 4-handles. There is a well-known trick that then provides a diffeomorphism homotopic to \( f \) when \( M \) is open. But, to ensure the \( C^\infty \) isotopy asserted by 3.1 we must now take some care and execute the isotopy and the trick simultaneously.

After making \( f \) nonsingular over a neighborhood of the 3-skeleton, we have a \( C^\infty \) homeomorphism \( f: M' \to M \) which is nonsingular except well within the interior of the preimage of a smooth 4-handle \( B_i \) inside each 4-simplex \( \tilde{\sigma}_i \). We extend the smooth arcs given by the weak \( C^\infty/\text{DIFF} \) handle lemma 2.1 obtaining, for each 4-handle \( B_i \), a point \( p_i \) in \( \text{int} B_i \) and a smooth arc \( \alpha_i \) from \( p_i \) to \( \infty \) in the complement of \( C \). Here we use the curious proviso that these components are unbounded in \( M \). We can arrange that \( \alpha_i \cap \partial M = \phi \), that \( \alpha_i \cap \alpha_j = \phi = \alpha_i \cap B_j \) for \( i \neq j \) and that the union of the \( \alpha_i \) is a properly embedded smooth submanifold of \( M \).

The weak index 4 handle lemma provides an isotopy \( f_i : M' \to M \) such that

(a) \( f_0 = f \) and \( f_1 \) is a diffeomorphism over \( M - \bigcup_i \{p_i\} \)
(b) \( f_i(M' - \bigcup_i f^{-1}(\{p_i\})) \) is an almost compact \( C^\infty \) isotopy in \( M - \bigcup_i \{p_i\} \).
(c) \( f_i \) is constant over each smooth arc \( \alpha_i \).


Extend the smooth arcs $\alpha_i$ and $f_i^{-1}\alpha_i = f^{-1}\alpha_i$ slightly to smooth arcs $\beta_i: R_+ \to M$ and $\beta_i': R_+ \to M'$ parametrized so that $\beta_i(1) = p_i$.

Choose disjoint closed tubular neighborhoods $\bar{\beta}_i: R_+ \times B^3 \to M$ and $\bar{\beta}_i': R_+ \times B^3 \to M'$ of $\beta_i$ and $\beta_i'$ such that their sum over $i$ is a properly embedded submanifold of $M$ and $M'$ respectively.

Define an isotopy $g_i: M \to M$, $0 \leq t \leq 1$, by

(i) $g_i(x) = x$ if $t = 0$ or $x$ is outside the normal tubes $\text{Im}(\bar{\beta}_i)$.

(ii) For $x$ in $\text{Im}(\bar{\beta}_i)$, say $x = \bar{\beta}_i(u, v)$, where $\bar{\mu}_i: R_+ \to R_+$ is an almost compact smooth nonsingular (into) isotopy with $\bar{\mu}_i(R_+) = [0, 1)$, adjusted to be constant near $t = 0$ and $t = 1$. This is an almost compact smooth into isotopy of $\text{id}|M$ with

$$g_i(x) = \bar{\beta}_i(\bar{\mu}_i(1 - |t|)(u), v).$$

where $\bar{\mu}_i: R_+ \to R_+$ is an almost compact smooth nonsingular (into) isotopy with $\bar{\mu}_i(R_+) = [0, 1)$, adjusted to be constant near $t = 0$ and $t = 1$. This is an almost compact smooth into isotopy of $\text{id}|M$ with

$$g_1 M = M - \bigcup_i \alpha_i = f_1(M' - \bigcup_i \alpha'_i).$$

Define $g'_i: M' \to M'$ similarly.

Consider the composed isotopy $f^*_t = g_t^{-1} \circ f_t \circ g_t': M' \to M$, $0 \leq t \leq 1$. Since $f_1 g'_1 M' = g_1 M$ and $f_t$ is a $C^\infty$ isotopy for $t < 1$ while $f_1$ is a diffeomorphism over $g_1 M$, this $f^*_t$ is a $C^\infty$ isotopy. It runs from $f$ to a diffeomorphism and finally establishes Part I.

3.3. Proof of 3.1, Part II: 5- and 6-manifolds

As in the proof of Part I we can find an isotopy of $f$ rel $C$ to make $f$ a diffeomorphism over a neighborhood of the 3-skeleton of $M$.

If $\dim \partial M = 4$, we can even use Part I to make $f$ a diffeomorphism over a neighborhood of $\partial M$.

As in part I, $f$ can be, near the boundary, always a product along the interval factor of collarings of the boundaries.

Applying a TOP/DIFF handle lemma to handles of index 4, 5, and 6 with cores in the open simplices of $M$ of increasing dimension 4, 5, and 6 we can now topologically isotop $f$ rel $C$ and rel the 3-skeleton to a diffeomorphism. More precisely the TOP/DIFF version of the TOP/PL handle straightening theorem of [6] is to be used. No immersion theory is required; the associated torus problem – presented by an exotic structure

$$(B^k \times T^n)_\Sigma, \quad k + n = m, \quad k = 4, 5, 6,$$

standard near the boundary – may be solved by simply connected surgery. To do this, first use the Product Structure Theorem [7, §5] to reduce to the two cases (i) $k = k + n \geq 5$; (ii) $k = 4, n = 1$. Then for $k = k + n \geq 5$ we solve by the smooth Poincaré Theorem [3]. The remaining case $k = 4$,
\( n = 1 \) is reduced by [18, § 5] to a surgery problem rel boundary with target \( B^4 \times [-1, 1] \) – which is just the smooth Poincaré Theorem for dimension 5 [3]. Compare [16] [4].

### 3.4. Proof 3.1, Part III, the \( C^\infty \) Hauptvermutung

Following the proof for part II, we isotop \( f \) rel \( C \) to make \( f \) a diffeomorphism over a neighborhood of \( C \cup M(6) \). As \( f \) is already PL over a neighborhood of \( C \) the (relative) Whitehead triangulation uniqueness theorem [13] provides an isotopy of \( f \) rel \( C \) making \( f \) PL over a neighborhood of \( C \cup M(6) \).

Now we can further isotop \( f \) rel \( C \cup M(6) \) to a PL homeomorphism using the TOP/PL handle straightening lemma of [6] for handle index values \( \geq 6 \). We note no sophisticated techniques are required here; for example, the Product Structure Theorem of [7] (based on handlebody theory) reduces the straightening lemma of [6] for index \( k \geq 6 \) to the PL Poincaré theorem for a disc of dimension \( k \).

### Appendix A. \( C^\infty \)-smoothing an isolated singularity

The proof of the following proposition was given to us by C. T. C. Wall, when we had proved just a special case sufficient for the \( C^\infty \) Hauptvermutung.

**Proposition A.1:** Let \( f: \mathbb{R}^r \to \mathbb{R}^s \) be a continuous map that is \( C^\infty \) on \( \mathbb{R}^r - 0 \). There exists a \( C^\infty \) homeomorphism \( \mu: [0, \infty) \to [0, \infty) \) (depending on \( f \)) such that the map \( h: \mathbb{R}^r \to \mathbb{R}^s, h(x) = \mu(||x||^2)f(x) \) is a \( C^\infty \) mapping.

**Proof of A.1:** Write

\[
N_{n,r}(f) = \sup \left\{ \left| \frac{\partial^lf}{\partial x^I} \right| : \frac{1}{n+1} \leq ||x||^2 \leq \frac{1}{n-1}, |I| = r \right\}.
\]

Choose a decreasing sequence \( c_n \) with \( c_n N_{n,r}(f) \to 0 \) as \( n \to \infty \) for each \( r \) (easily done by diagonal process). If \( \mu \) is \( C^\infty \)-homeomorphism of \([0, \infty)\), nonsingular on \((0, \infty)\) and flat at 0, with \( \mu^{(s)}(y)/c_n y \to 0 \) as \( y \to 0 \) for all \( s \) (where \( n \) depends on \( y \) by \( 1/(n+1) \leq y \leq 1/(n-1) \) then \( g(x) = \mu(||x||^2) \) is \( C^\infty \), and as

\[
D^J(fg) = \Sigma(D^Jf D^Kg : J + K = I) \quad \text{by Leibnitz' theorem},
\]

\( D^Jf \) is estimated by an \( N_{n,r}(f) \) and \( D^Kg \) by \( c_n^s ||x||^2 \), we have \( D^J(fg) \to 0 \) as \( ||x|| \to 0 \). Thus by induction if we define \( h(x) = f(x)g(x) (x \neq 0) \) \( h(0) = 0 \), \( h \) is flat at 0 as required.
We construct $\mu(y) = \int_0^y \mu'$ defining first $\mu'$ so that, for small $y$,

$$\mu'(y) = \sum_{n=2}^{\infty} 2^{-n} c_{n+1} B((n^2-1)y-n),$$

where $B(x) > 0$ for $|x| < 1$ and $= 0$ otherwise. At most 2 terms in the summation can be nonzero, and since each $B^{\mu}(x)$ is bounded, the desired estimates follow easily.

**Appendix B. Potential Counterexamples in Dimension 4**

It is clear that a positive solution to the smooth (or PL) Schoenflies conjecture in dimension 4 would eliminate the conditions concerning dimension 4 in the $C^\infty$ Hauptvermutung. Conversely we show now that a counterexample to this conjecture would give a counterexample to the $C^\infty$ Hauptvermutung for compact (even closed) 4-manifolds.

**Proposition:** Suppose $S$ is a smoothly embedded 3-sphere in $\mathbb{R}^4 - 0$, and $T$ is the closure in $\mathbb{R}^4$ of the bounded component of $\mathbb{R}^4 - S$. Then there exist $C^\infty$ homeomorphisms $B^4 \to T$ and $T \to B^4$ each with one singular point, at 0.

**Discussion:** The 4-dimensional smooth Schoenflies conjecture asserts that every such $T$ is in fact diffeomorphic to $B^4$, (equivalently PL isomorphic to $B^4$, cf [12]). So it is immediate that a counterexample $T$ to this conjecture would yield a counterexample $T \to B^4$ to the $C^\infty$ Hauptvermutung. By capping off with 4-discs it also yields a counterexample $M = T \cup B^4_+ \times S^4$ for closed 4-manifolds. In each case there is just one singularity.

**Proof of Proposition:** Mazur’s Schoenflies argument (as reworked in [14, 3.38] yields a diffeomorphism $f:(\mathbb{R}^4 - 0) \to (\mathbb{R}^4 - 0)$ with $f(B^4 - 0) = T - 0$. Then application of Lemma A.1 to $f$ and $f^{-1}$ respectively yields the asserted homeomorphisms.

**BIBLIOGRAPHY**


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