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R. TIJDEMAN

H. G. MEIJER

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## ON INTEGERS GENERATED BY A FINITE NUMBER OF FIXED PRIMES

R. Tijdeman and H. G. Meijer

### 1

Let  $p_1, \dots, p_r$  be different primes,  $r \geq 2$ . Denote the multiplicative semigroup generated by them by  $N$ . We arrange the elements of  $N$  in increasing order,  $1 = n_1 < n_2 < n_3 < \dots$ . It was noted by Pólya [3] that  $\lim_{i \rightarrow \infty} n_{i+1}/n_i = 1$ . Later better estimates were obtained for the quotient  $n_{i+1}/n_i$ . See [1], [5], [6]. In this paper we investigate the set of quotients  $n_{i+1}/n_i$  ( $i = 1, 2, 3, \dots$ ). Theorem 1 contains a complete characterization of this set in case  $r = 2$ . The situation for  $r > 2$  is much more complicated. As a first step we made the following conjecture.

Let  $t$  be fixed,  $1 \leq t \leq r-1$ . Then there exist infinitely many pairs  $n_i, n_{i+1}$  such that one of the numbers  $n_i, n_{i+1}$  is composed of  $p_1, \dots, p_t$  and the other is composed of  $p_{t+1}, \dots, p_r$ .

We prove this conjecture for  $t = 1$  in Theorem 2 and for  $t = 2$  in Theorem 3. The case  $t > 2$  is still open. Since  $t = 1$  and  $t = 2$  are equivalent to  $t = r-1$  and  $t = r-2$  respectively, the conjecture is true for  $r \leq 5$ .

### 2

Let  $p_1, \dots, p_r$  be different primes,  $r \geq 2$ . By the *sequence composed of*  $p_1, \dots, p_r$  we mean the monotonically increasing sequence  $N = \{n_i\}_{i=1}^{\infty}$  of all numbers which are of the form  $p_1^{k_1} \dots p_r^{k_r}$ , where  $k_1, \dots, k_r$  are non-negative integers. We observe that

$$(1) \quad d|n_i \quad \text{and} \quad d|n_{i+1} \Rightarrow \frac{n_i}{d} \quad \text{and} \quad \frac{n_{i+1}}{d} \quad \text{are consecutive elements of } N.$$

Indeed,  $n_i/d < n_j < n_{i+1}/d$  would imply  $n_i < dn_j < n_{i+1}$ , which is impossible. We denote the G.C.D. of two integers  $a$  and  $b$  by  $(a, b)$ .

We shall use the following lemmas

LEMMA 1: Let  $p_1, \dots, p_r$  be fixed primes,  $r \geq 2$ . Let  $n_1, n_2, \dots$  be the sequence composed of these primes. Then there exist positive constants  $C_1, C_2$  and  $N$  such that

$$(2) \quad \frac{n_i}{(\log n_i)^{C_1}} < n_{i+1} - n_i < \frac{n_i}{(\log n_i)^{C_2}} \quad \text{for } n_i \geq N.$$

PROOF. The first inequality is a corollary of [5, Theorem 1]. The second can be found in [6].

LEMMA 2: Let  $n_1, n_2, \dots$  be the sequence composed of the primes  $p_1, \dots, p_r$  with  $r \geq 2$ . Then

$$\lim_{i \rightarrow \infty} \frac{(n_i, n_{i+1})}{n_i} = 0.$$

PROOF: Let  $d_i = (n_i, n_{i+1})$ . If  $n_j = n_i/d_i$ , then, by (1),  $n_{j+1} = n_{i+1}/d_i$ . Hence, by (2),

$$\frac{1}{(\log n_j)^{C_1}} < \frac{n_{j+1}}{n_j} - 1 = \frac{n_{i+1}}{n_i} - 1 < \frac{1}{(\log n_i)^{C_2}}.$$

It follows that

$$(\log n_i)^{C_2} < \left( \log \frac{n_i}{d_i} \right)^{C_1}.$$

Since the left hand term tends to  $\infty$  if  $i \rightarrow \infty$ , we also have  $n_i/d_i \rightarrow \infty$  if  $i \rightarrow \infty$ .

### 3

We need several elementary results from the theory of continued fractions. Let  $\xi > 0$  be an irrational number with simple continued fraction  $[a_0, a_1, a_2, \dots]$ . The  $n$ -th convergent  $[a_0, \dots, a_n]$  to  $\xi$  is denoted by  $A_n/B_n$ . It is well known that the denominators  $B_n$  form a monotonically increasing sequence of integers for  $n \geq 1$ , that the sequence  $A_0/B_0, A_2/B_2, A_4/B_4, \dots$  is monotonically increasing to  $\xi$  and  $A_1/B_1, A_3/B_3, A_5/B_5, \dots$  is monotonically decreasing to  $\xi$ . The convergents  $A_n/B_n$  are the best approximations to  $\xi$  in the sense of Lemma 3(a). For our convenience we give a slightly different form of this assertion in Lemma 3(b).

LEMMA 3: (a) *The convergents to  $\xi$  are just the fractions  $A/B$  having the property that every fraction  $r/s$  with  $0 < |r - s\xi| < |A - B\xi|$  satisfies  $s > B$ .*

(b) *If  $A_n/B_n$  is a convergent to  $\xi$ , then every fraction  $r/s$  with  $0 < |r - s\xi| < |A_n - B_n\xi|$  satisfies  $s \geq B_{n+1}$ .*

PROOF: See [2, Satz 2.18, 2.17].

Apart from the convergents to  $\xi$  we shall consider a larger set of fractions. We recall

$$(3) \quad \begin{aligned} A_{n+1} &= a_{n+1}A_n + A_{n-1}, \\ B_{n+1} &= a_{n+1}B_n + B_{n-1}, \end{aligned} \quad \text{for } n \geq 0.$$

We call a fraction

$$\frac{A}{B} = \frac{jA_n + A_{n-1}}{jB_n + B_{n-1}} \quad \text{with } j \in \{1, 2, \dots, a_{n+1}\}$$

a *one-sided convergent* to  $\xi$  (Naherung). We call it a *left convergent* if  $A/B < \xi$  and a *right convergent* if  $A/B > \xi$ . We can arrange the one-sided convergents to  $\xi$  with increasing denominators. Part of this sequence reads as follows

$$\frac{A_n}{B_n}, \frac{A_n + A_{n-1}}{B_n + B_{n-1}}, \dots, \frac{a_{n+1}A_n + A_{n-1}}{a_{n+1}B_n + B_{n-1}} = \frac{A_{n+1}}{B_{n+1}}, \frac{A_{n+1} + A_n}{B_{n+1} + B_n}.$$

It follows immediately from the construction that

$$(jA_n + A_{n-1})/(jB_n + B_{n-1}) \quad (j = 1, \dots, a_{n+1})$$

are on the same side of  $\xi$ , but  $A_n/B_n$  and  $(A_{n+1} + A_n)/(B_{n+1} + B_n)$  are on the opposite side of  $\xi$ .

In [2, Satz 2.21, 2.22] a complete characterization of the one-sided convergents is given. The second theorem states the following.

LEMMA 4: *If a fraction  $A/B$  with positive denominator has the property that every fraction between  $\xi$  and  $A/B$  has a denominator greater than  $B$ , then  $A/B$  is a one-sided convergent to  $\xi$ .*

We shall use Lemma 4 to derive a slightly different characterization which is more analogous to Lemma 3(a) and more appropriate for our purposes.

LEMMA 5:

(a) *The left convergents to  $\xi$  are just the fractions  $A/B$  having the property that every fraction  $r/s$  with  $A - B\xi < r - s\xi < 0$  satisfies  $s > B$ .*

(b) *The right convergents to  $\xi$  are just the fractions  $A/B$  having the property that every fraction  $r/s$  with  $0 < r - s\xi < A - B\xi$  satisfies  $s > B$ .*

PROOF: Since the proofs of both parts are almost identical we only prove the second assertion.

Let  $A/B$  have the property that every fraction  $r/s$  with

$$0 < r - s\xi < A - B\xi$$

satisfies  $s > B$ . Then every fraction  $r/s$  with  $\xi < r/s < A/B$  satisfies  $s > B$ . Indeed, if  $r/s$  were a fraction with  $s \leq B$  and  $\xi < r/s < A/B$  then it would follow that

$$0 < r - s\xi = s \left( \frac{r}{s} - \xi \right) \leq B \left( \frac{A}{B} - \xi \right) = A - B\xi,$$

which is a contradiction. It follows from Lemma 4 that  $A/B$  is a right convergent.

Let  $A/B$  be any right convergent. By definition  $A/B$  can be written in the form

$$(4) \quad \frac{A}{B} = \frac{jA_n + A_{n-1}}{jB_n + B_{n-1}}, \quad j \in \{1, 2, \dots, a_{n+1}\},$$

where  $A_{n-1}/B_{n-1}$  and  $A_n/B_n$  are convergents to  $\xi$  with

$$(5) \quad \frac{A_n}{B_n} < \xi < \frac{A_{n-1}}{B_{n-1}}.$$

Define  $A^*/B^*$  by

$$(6) \quad A^* - B^*\xi = \min_{\substack{r - s\xi > 0 \\ s \leq B}} (r - s\xi)$$

Since  $\xi$  is irrational,  $A^*$  and  $B^*$  are uniquely determined. It is obvious that there does not exist a fraction  $r/s$  with  $s \leq B^*$  and  $0 < r - s\xi < A^* - B^*\xi$ . Hence, by the first part of the proof,  $A^*/B^*$  is a right convergent. It follows from (6) and (5) that  $0 < A^* - B^*\xi \leq A_{n-1} - B_{n-1}\xi$ . On applying Lemma

3(b) we obtain  $B^* \geq B_{n-1}$ . Since  $A^*/B^*$  is a right convergent to  $\xi$  and  $B^* \leq B$ , we obtain

$$(7) \quad \frac{A^*}{B^*} = \frac{iA_n + A_{n-1}}{iB_n + B_{n-1}}, \quad \text{where } i \in \{0, 1, \dots, j\}.$$

We have, by (7), (5) and (4),

$$\begin{aligned} A^* - B^*\xi &= i(A_n - B_n\xi) + (A_{n-1} - B_{n-1}\xi) \\ &\geq j(A_n - B_n\xi) + (A_{n-1} - B_{n-1}\xi) = A - B\xi, \end{aligned}$$

while equality holds if and only if  $i = j$ . By (6),  $A^* - B^*\xi \leq A - B\xi$ . Hence,  $i = j$  and  $A^*/B^* = A/B$ . In view of (6) this completes the proof of Lemma 5(b).

#### 4

Let  $\alpha$  and  $\beta$  be real numbers with  $\alpha > \beta > 1$ . By the *sequence composed of  $\alpha$  and  $\beta$*  we mean the monotonically increasing sequence  $N = \{n_i\}_{i=1}^\infty$  of all numbers of the form  $\alpha^k\beta^l$ , where  $k$  and  $l$  are non-negative integers. The following theorem gives a complete characterization of the set of quotients  $\{n_{i+1}/n_i\}_{i=1}^\infty$ .

**THEOREM 1:** *Let  $\alpha$  and  $\beta$  be real numbers with  $\alpha > \beta > 1$ , and such that  $\xi = \log \beta / \log \alpha$  is irrational. Let  $n_1, n_2, \dots$  be the sequence composed of  $\alpha$  and  $\beta$ . If  $S = \{n_{i+1}/n_i | i = 1, 2, \dots\}$ , then  $S$  is the set of all products  $\alpha^{-k}\beta^l$  and  $\alpha^k\beta^{-l}$  which are greater than 1 and such that  $k/l$  is a one-sided convergent to  $\xi$ .*

**REMARK:** In view of Theorem 1 one can define a natural generalization of the continued fractions as follows. Let  $\alpha_1, \dots, \alpha_m$  be real numbers all greater than 1. Let  $n_1, n_2, \dots$  be the sequence composed of  $\alpha_1, \dots, \alpha_m$ . Put  $S = \{n_{i+1}/n_i | i = 1, 2, \dots\}$ . We would be very interested in a characterization of  $S$  like Theorem 1 does in case  $m = 2$ .

**PROOF:** Let  $k/l$  be a one-sided convergent to  $\xi$ . We shall prove that  $\alpha^k$  and  $\beta^l$  are consecutive elements of  $N$ . This implies that  $k/l$  belongs to  $S$ .

Assume  $k/l$  is a left convergent to  $\xi$ . Then  $\alpha^k < \beta^l$ . Suppose there exists an element  $\alpha^r\beta^s$  such that  $\alpha^k < \alpha^r\beta^s < \beta^l$ . Hence,  $l > s \geq 0$ . We have

$$k < r + s\xi < l\xi,$$

or, equivalently,

$$k - l\xi < r - (l - s)\xi < 0.$$

This is a contradiction with Lemma 5(a).

If  $k/l$  is a right convergent to  $\xi$ , then  $\beta^l < \alpha^k$  and a similar argument gives that  $\beta^l$  and  $\alpha^k$  are consecutive elements of  $N$ .

In order to prove that every element of  $S$  is of the required form, put  $n_i = \alpha^{r_i} \beta^{s_i}$ ,  $n_{i+1} = \alpha^{r_{i+1}} \beta^{s_{i+1}}$ . Since  $\alpha > \beta$ , we have

$$\alpha^{r_{i+1}} \beta^{s_i} > \alpha^{r_i} \beta^{s_{i+1}} \geq n_{i+1},$$

and, hence, either  $r_{i+1} \leq r_i$  or  $s_{i+1} < s_i$ . Since both cases are treated in similar ways, we only deal with the first. Assume  $r_{i+1} \leq r_i$ . Then  $s_{i+1} > s_i$ . Put  $k = r_i - r_{i+1}$ ,  $l = s_{i+1} - s_i$ . We have  $\alpha^{-k} \beta^l = n_{i+1}/n_i > 1$ . We shall prove that  $k/l$  is a left convergent to  $\xi$ . We have  $k/l < \log \beta / \log \alpha = \xi$ . Suppose there exists a fraction  $r/s$  with  $s \leq l$  and

$$k - l\xi < r - s\xi < 0.$$

Then

$$(8) \quad \alpha^{r-k+r_i} \beta^{l-s+s_i} = n_i e^{(r-k) \log \alpha + (l-s) \log \beta} > n_i.$$

Since  $r - k + r_i = r + r_{i+1} > 0$  and  $l - s + s_i \geq s_i > 0$ , we obtain

$$(9) \quad \alpha^{r-k+r_i} \beta^{l-s+s_i} \in N.$$

On the other hand,

$$(10) \quad \alpha^{r-k+r_i} \beta^{l-s+s_i} = n_{i+1} e^{r \log \alpha - s \log \beta} < n_{i+1}.$$

The contradiction (8), (9), (10), proves by Lemma 5(a) that  $k/l$  is a left convergent to  $\xi$ . (In case  $s_{i+1} < s_i$  the fraction  $k/l$  turns out to be a right convergent to  $\xi$ ).

## 5

It would be very valuable to have a characterization like Theorem 1 for sequences composed of  $r$  multiplicatively independent positive

numbers,  $r > 2$ . This would solve the conjecture in the introduction immediately. We now prove case  $t = 1$  of this conjecture.

**THEOREM 2:** *Let  $n_1, n_2, \dots$  be the sequence composed of the primes  $p_1, \dots, p_r$  ( $r \geq 2$ ). Let  $p$  be one of these primes. Then there exists an infinite number of pairs  $n_i, n_{i+1}$  such that  $n_i$  is a pure power of  $p$  and  $n_{i+1}$  is not divisible by  $p$ .*

**PROOF:** Without loss of generality we may assume  $p = p_1$ . Let  $k$  be a positive integer and  $n_{j_k} = p^k$ . Let  $n_{j_{k+1}} = p_1^{l_1} \cdots p_r^{l_r}$  and  $n_{i_k} = p_1^{k-1}$ . It follows from (1) that  $n_{i_{k+1}} = p_2^{l_2} \cdots p_r^{l_r}$ . Since, by Lemma 2,

$$n_{i_k} = \frac{n_{j_k}}{(n_{j_k}, n_{j_{k+1}})} \rightarrow \infty \quad \text{for } k \rightarrow \infty,$$

we obtain infinitely many different pairs  $n_{i_k}, n_{i_{k+1}}$  with the required property.

**REMARK:** In the same way one can prove the existence of infinitely many pairs  $n_i, n_{i+1}$  such that  $n_{i+1}$  is a pure power of  $p$  and  $n_i$  is not divisible by  $p$ .

## 6

Finally we prove case  $t = 2$  of our conjecture.

**THEOREM 3:** *Let  $p_1, \dots, p_r$  be  $r > 2$  different primes. Let  $M = \{m_1, m_2, \dots\}$  be the sequence composed of these primes. Let  $p$  and  $q$  be two primes from  $p_1, \dots, p_r$ . Then there exist infinitely many pairs  $m_i, m_{i+1}$  such that one of the numbers  $m_i, m_{i+1}$  is composed of  $p$  and  $q$  and the other is neither divisible by  $p$  nor by  $q$ .*

The proof is based on two lemmas.

**LEMMA 6:** *Let  $r > 2$ . Let  $M = \{m_1, m_2, \dots\}$  be the sequence composed of the different primes  $p_1, \dots, p_r$  and  $N = \{n_1, n_2, \dots\}$  the sequence composed of  $p_1$  and  $p_2$ . Suppose there exists an  $i_0$  such that for every  $i \geq i_0$*

$$m_i \in N \Rightarrow (m_{i-1}, p_1 p_2) > 1 \quad \text{and} \quad (m_{i+1}, p_1 p_2) > 1.$$

*Then there exists an  $i_1$  such that for every  $i \geq i_1$*

- (a) if  $m_i \in N$  and  $m_i^2 \leq m_{i-1} m_{i+1}$ , then  $m_{i-1} \in N$ ,  
 (b) if  $m_i \in N$  and  $m_i^2 \geq m_{i-1} m_{i+1}$ , then  $m_{i+1} \in N$ .

PROOF: We know from Lemma 2 that

$$\frac{m_{i-1}}{(m_{i-1}, m_i)} \rightarrow \infty \quad \text{as } i \rightarrow \infty.$$

We choose  $i_1$  such that

$$\frac{m_{i-1}}{(m_{i-1}, m_i)} > m_{i_0} \quad \text{for } i \geq i_1.$$

In the sequel we only consider  $i$  with  $i \geq i_1$ .

Assume  $m_i \in N$ . Let  $m_i = p_1^a p_2^b$ . Put  $m_{i-1} = p_1^{k_1} \cdots p_r^{k_r}$  and  $m_{i+1} = p_1^{l_1} \cdots p_r^{l_r}$ . Then

$$m_{i-1} < \frac{m_{i-1} m_{i+1}}{m_i} < m_{i+1}.$$

Hence, we have either

$$(11) \quad m_{i-1} m_{i+1} / m_i = m_i$$

or

$$(12) \quad m_{i-1} m_{i+1} / m_i \notin M.$$

We note  $m_{i-1} m_{i+1} / m_i = p_1^{k_1+l_1-a} p_2^{k_2+l_2-b} p_3^{k_3+l_3} \cdots p_r^{k_r+l_r}$ . If (11) holds, then  $k_3+l_3 = \cdots = k_r+l_r = 0$ , and, hence,  $k_3 = \cdots = k_r = 0$  and  $l_3 = \cdots = l_r = 0$ . In this case both  $m_{i-1} \in N$  and  $m_{i+1} \in N$ . If (12) holds, then

$$(13) \quad k_1+l_1-a < 0 \quad \text{or} \quad k_2+l_2-b < 0.$$

Suppose  $k_1 \leq a$  and  $k_2 \leq b$ . By (1),  $p_1^{a-k_1} p_2^{b-k_2}$  is preceded in  $M$  by  $p_3^{k_3} \cdots p_r^{k_r}$ . Since

$$p_1^{a-k_1} p_2^{b-k_2} = \frac{m_i}{(m_{i-1}, m_i)} > m_{i_0},$$

this is a contradiction with the condition of the lemma. Hence,  $k_1 > a$  or

$k_2 > b$ . Similarly,  $l_1 > a$  or  $l_2 > b$ . Without loss of generality we may assume  $k_2 > b$ . Then, by (13),  $k_1 < a$  and  $l_1 < a$ . Thus  $l_2 > b$ . So we obtain

$$(14) \quad k_1 < a, \quad l_1 < a, \quad k_2 > b, \quad l_2 > b.$$

We define a sequence of positive integers  $\{a_j\}_{j=0}^\infty$  by

$$m_{a_j} = p_1^a p_2^j \quad \text{for } j = 0, 1, 2, \dots$$

We have, by (1) and (14),

$$m_{a_{j-1}} = p_1^{k_1} p_2^{k_2-b+j} p_3^{k_3} \dots p_r^{k_r} \quad \text{and} \quad m_{a_{j+1}} = p_1^{l_1} p_2^{l_2-b+j} p_3^{l_3} \dots p_r^{l_r},$$

for  $j = 0, 1, \dots, b$ . Consider the pairs of quotients

$$(15) \quad \left( \frac{m_{a_{j-1}}}{m_{a_j}}, \frac{m_{a_{j+1}}}{m_{a_j}} \right) \quad \text{for } j = 0, 1, 2, \dots$$

We know

$$\frac{m_{a_{j-1}}}{m_{a_j}} = p_1^{k_1-a} p_2^{k_2-b} p_3^{k_3} \dots p_r^{k_r} \quad \text{and} \quad \frac{m_{a_{j+1}}}{m_{a_j}} = p_1^{l_1-a} p_2^{l_2-b} p_3^{l_3} \dots p_r^{l_r}$$

for  $j = 0, \dots, b$ . Let  $J_0$  be the smallest value of  $j$  for which one of the quotients in (15) assumes another value. This  $J_0$  exists, since, by Lemma 1,  $m_{i+1}/m_i$  tends to 1 as  $i \rightarrow \infty$ . We assume that the first quotient changes firstly. Thus

$$(16) \quad 1 > \frac{m_{a_{J-1}}}{m_{a_J}} > \frac{m_{a_{J-1}-1}}{m_{a_{J-1}}} = \dots = \frac{m_{a_0-1}}{m_{a_0}} = \frac{m_{i-1}}{m_i}$$

and

$$(17) \quad \frac{m_{a_{J-1}+1}}{m_{a_{J-1}}} = \dots = \frac{m_{a_0+1}}{m_{a_0}} = \frac{m_{i+1}}{m_i}.$$

Put  $m_{a_{J-1}} = p_1^{\kappa_1} \dots p_r^{\kappa_r}$  and  $m_{a_{J+1}} = p_1^{\lambda_1} \dots p_r^{\lambda_r}$ . The following argument shows  $\kappa_2 = 0$ . If  $\kappa_2 > 0$ , then, by (1),  $m_{a_{J-1}}/p_2$  is the precursor of  $m_{a_J}/p_2 = m_{a_{J-1}}$ , and, hence,  $m_{a_{J-1}}/m_{a_J} = m_{a_{J-1}-1}/m_{a_{J-1}}$  in contradiction with (16). Since we know from the argument preceding formula (14) that both  $\kappa_1 \leq a$  and  $\kappa_2 \leq J$  is impossible, we have

$$(18) \quad \kappa_1 > a.$$

Consider

$$m = \frac{m_{a_J-1+1} m_{a_J-1}}{m_{a_J}}.$$

We have, by (17)

$$\begin{aligned} m &= \frac{m_{a_J-1+1}}{m_{a_J-1}} \cdot m_{a_J-1} \cdot \frac{m_{a_J-1}}{m_{a_J}} = \frac{m_{i+1}}{m_i} \cdot \frac{m_{a_J-1}}{m_{a_J}} \cdot m_{a_J-1} \\ &= p_1^{l_1+\kappa_1-a} p_2^{l_2+\kappa_2-b-1} p_3^{l_3+\kappa_3} \dots p_r^{l_r+\kappa_r}. \end{aligned}$$

From (18) and (14) we see that  $m \in M$ . Moreover,

$$m_{a_J-1+1} > m > \frac{m_{a_J-1+1} m_{a_J-1-1}}{m_{a_J-1}} > m_{a_J-1-1}.$$

Hence,

$$(19) \quad m = m_{a_J-1}.$$

This implies  $l_3 + \kappa_3 = \dots = l_r + \kappa_r = 0$ . Thus  $l_3 = \dots = l_r = 0$  and  $m_{i+1} \in N$ . Furthermore, in view of (17), (19), the definition of  $m$  and (16),

$$\begin{aligned} \frac{m_{i+1}}{m_i} &= \frac{m_{a_J-1+1}}{m_{a_J-1}} = \frac{m_{a_J-1+1}}{m} \\ &= \frac{m_{a_J}}{m_{a_J-1}} < \frac{m_{a_J-1}}{m_{a_J-1-1}} = \frac{m_i}{m_{i-1}}. \end{aligned}$$

Similarly, the assumption that the second quotient in (15) changes firstly leads to  $m_{i-1} \in N$  and  $m_{i-1}/m_i > m_i/m_{i+1}$ . This completes the proof of the lemma.

LEMMA 7: Let  $r > 2$ . Let  $M = \{m_1, m_2, \dots\}$  be the sequence composed of the primes  $p_1, \dots, p_r$ . Let  $p$  and  $q$  be two arbitrary primes from  $p_1, \dots, p_r$  with  $p > q$  and let  $N = \{n_1, n_2, \dots\}$  be the sequence composed of  $p$  and  $q$ . Suppose there exists an  $i_0$  such that for every  $i \geq i_0$

$$m_i \in N \Rightarrow (m_{i-1}, pq) > 1 \quad \text{and} \quad (m_{i+1}, pq) > 1.$$

Then there exists a monotonically increasing, unbounded sequence  $T_1, T_2, T_3, \dots$  such that no interval  $[T_h, qT_h]$  contains an element of  $M \setminus N$ .

PROOF: Let  $[a_0, a_1, a_2, \dots]$  be the continued fraction of  $\xi := \log q / \log p$ . Put  $A_h/B_h = [a_0, \dots, a_h]$  for  $h = 0, 1, 2, \dots$ . It follows from the Gel'fond-Schneider theorem [4, Satz 14], that  $\xi$  is transcendental. Hence, the sequence  $a_0, a_1, a_2, \dots$  is not periodical [2, Satz 3.1]. There therefore exist infinitely many values  $h$  with  $a_h > 1$ .

Let  $H$  be such that  $a_H > 1$ . It is no loss of generality to assume  $A_H/B_H < \xi$ . We consider the subsequence  $N_1$  of  $N$  beginning with

$$T_H = p^{A_H}q^{B_H-1} \quad \text{and ending with} \quad qT_H = p^{A_H}q^{B_H}.$$

(If  $A_H/B_H > \xi$ , we may choose  $T_H = p^{A_H-1}q^{B_H}$  and consider the interval  $[T_H, pT_H]$ .)

Let  $n_i = p^c q^d$  be in  $N_1$ ,  $n_i \neq qT_H$ . Since  $qT_H < q^{B_H+B_H-1} < p^{A_H+A_H-1}$ , we have

$$(20) \quad c < A_H + A_{H-1} \quad \text{and} \quad d < B_H + B_{H-1}.$$

We distinguish two cases.

(i)  $c \geq A_H$ . We assert that  $n_{i+1} = p^{c-A_H}q^{d+B_H}$ . Since  $A_H/B_H < \xi$ , we have

$$n_i < p^{c-A_H}q^{d+B_H} \in N_1.$$

Suppose

$$n_{i+1} = p^s q^t < p^{c-A_H}q^{d+B_H}.$$

This implies

$$A_H - B_H \xi < (c-s) - (t-d)\xi < 0.$$

By Lemma 3(b),  $|t-d| \geq B_{H+1}$ . Hence,  $d \geq B_{H+1}$  or  $t \geq B_{H+1}$  in contradiction with (20).

(ii)  $c < A_H$ . Since  $d \leq B_{H-1} - 1$  implies  $p^c q^d < p^{A_H}q^{B_H-1} = T_H$ , we have  $d \geq B_{H-1}$ . We assert that  $n_{i+1} = p^{c+A_H-1}q^{d-B_H-1}$ . Since  $A_{H-1}/B_{H-1} > \xi$ , we have

$$n_i < p^{c+A_H-1}q^{d-B_H-1} \in N_1.$$

Suppose

$$n_{i+1} = p^s q^t < p^{c+A_{H-1}} q^{d-B_{H-1}}.$$

Then

$$0 < (s-c)-(d-t)\xi < A_{H-1}-B_{H-1}\xi.$$

By Lemma 3(b),  $|d-t| \geq B_H$ . If  $d-t < 0$ , then  $t > d+B_H \geq B_H+B_{H-1}$  and  $p^s q^t > q^{B_H+B_{H-1}} > qT_H$ , which is false. Hence,  $d-t \geq 0$ . This implies  $s-c > 0$  and  $d \geq B_H+t$ . By Lemma 5(b)  $(s-c)/(d-t)$  is a right convergent to  $\xi$ . Since  $A_H/B_H$  is a left convergent to  $\xi$ , we obtain  $d \geq d-t \geq B_H+B_{H-1}$ , which is impossible in view of (20). Summarizing we see that among the quotients  $n_{i+1}/n_i$  for  $n_i \in N_1$  only  $p^{-A_H} q^{B_H}$  and  $p^{A_{H-1}} q^{-B_{H-1}}$  occur. Note

$$(21) \quad p^{A_{H-1}} q^{-B_{H-1}} > p^{-A_H} q^{B_H} > 1.$$

We now assert that

$$(22) \quad n_{i+1}/n_i = p^{A_{H-1}} q^{-B_{H-1}} \Rightarrow \frac{n_i}{n_{i-1}} = \frac{n_{i+1}}{n_i} \quad \text{or} \quad \frac{n_{i+1}}{n_i} = \frac{n_{i+2}}{n_{i+1}}.$$

Since  $n_i = T_H$  implies  $n_{i+1} = p^{-A_H} q^{B_H} n_i$ , we have  $n_i > T_H$ . Hence,  $n_{i-1} \in N_1$ . Suppose

$$\frac{n_i}{n_{i-1}} = \frac{n_{i+2}}{n_{i+1}} = p^{-A_H} q^{B_H}.$$

Then

$$n_{i+2} = p^{-2A_H+A_{H-1}} q^{2B_H-B_{H-1}} n_{i-1}.$$

By  $a_H \geq 2$ , it follows that  $n_{i+2} \geq q^{B_H+B_{H-1}+B_{H-2}}$ . This is a contradiction.

We now turn our attention to the subsequence  $M_1$  of  $M$  starting with  $T_H$  and ending with  $qT_H$ . Let  $m_i \in N_1$ ,  $m_i \neq qT_H$ . Put  $m_i = n_j$ . Hence,  $n_{j+1} \in N_1$ . Note that  $n_{j-1} \leq m_{i-1} < m_i < m_{i+1} \leq n_{j+1}$ . The condition of Lemma 7 enables us to apply Lemma 6. Hence,  $m_{i-1} = n_{j-1}$  if  $m_i^2 \leq m_{i-1} m_{i+1}$  and  $m_{i+1} = n_{j+1}$  if  $m_i^2 \geq m_{i-1} m_{i+1}$ . It follows that

$$(23) \quad m_{i-1} = n_{j-1} \quad \text{if} \quad n_j^2 \leq n_{j-1} m_{i+1} \leq n_{j-1} n_{j+1}$$

and

$$(24) \quad m_{i+1} = n_{j+1} \quad \text{if} \quad n_j^2 \geq m_{i-1}n_{j+1} \geq n_{j-1}n_{j+1}.$$

We can now prove that all elements  $m_i$  with  $T_H \leq m_i \leq qT_H$  belong to  $N_1$ . Suppose  $T_H = m_I$  and all integers  $m_I, m_{I+1}, \dots, m_i$  belong to  $N_1$ , while  $m_i < qT_H$ . We shall prove that  $m_{i+1} \in N_1$ . Put  $m_i = n_j$ . We distinguish two cases.

- (i)  $n_{j-1}n_{j+1} \leq n_j^2$ . It follows from (24) that  $m_{i+1} = n_{j+1} \in N_1$ .
- (ii)  $n_{j-1}n_{j+1} \geq n_j^2$ . It follows from formula (21) and the lines before that

$$\frac{n_j}{n_{j-1}} = p^{-A_H}q^{B_H}, \quad \frac{n_{j+1}}{n_j} = p^{A_H-1}q^{-B_H-1}.$$

Since  $n_j = T_H$  implies  $n_{j+1} = p^{-A_H}q^{B_H}n_j$ , we have  $n_j > T_H$  and, hence,  $n_{j-1} \in N_1$ . By (22) we have  $n_{j+2}/n_{j+1} = p^{A_H-1}q^{-B_H-1}$ . Let  $n_{j+1} = m_{i^*}$ . Since  $n_{j+2}/n_{j+1} = n_{j+1}/n_j$ , we obtain from (23) that  $n_j = m_{i^*-1}$ . Hence,  $m_{i^*-1} = m_i$  and  $i^* - 1 = i$ . It follows that  $m_{i+1} = m_{i^*} = n_{j+1} \in N$ .

Since we have constructed an infinite number of  $T_H$ 's such that all integers  $m_i \in [T_H, qT_H]$  belong to  $N$ , the lemma has been proved.

We are now going to prove the main result.

**PROOF OF THEOREM 3:** It is no restriction to assume  $p = p_1, q = p_2, p > q$ . Suppose that there are only a finite number of values  $i$  for which the statement of the theorem holds. Then the condition of Lemma 7 is fulfilled for some  $i_0$ . It follows that there exists an unbounded sequence  $T_1, T_2, T_3, \dots$  such that each element  $m_i \in [T_H, qT_H]$  belongs to the sequence  $N$  composed of  $p$  and  $q$ . Let  $N = \{n_1, n_2, n_3, \dots\}$ . We know from Lemma 1 that  $n_{i+1}/n_i \rightarrow 1$  as  $i \rightarrow \infty$ . Consider the sequence  $p_3n_1, p_3n_2, p_3n_3, \dots$ . These elements belong to  $M \setminus N$ . However,  $p_3n_{i+1}/p_3n_i \rightarrow 1$  as  $i \rightarrow \infty$ . This is a contradiction.

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University of Leiden  
Mathematical Institute  
Leiden, Netherlands

University of Technology  
Department of Mathematics  
Delft, Netherlands