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On integers generated by a finite number of fixed primes


<http://www.numdam.org/item?id=CM_1974__29_3_273_0>
Let \( p_1, \cdots, p_r \) be different primes, \( r \geq 2 \). Denote the multiplicative semigroup generated by them by \( N \). We arrange the elements of \( N \) in increasing order, \( 1 = n_1 < n_2 < n_3 < \cdots \). It was noted by Pólya [3] that \( \lim_{i \to \infty} n_{i+1}/n_i = 1 \). Later better estimates were obtained for the quotient \( n_{i+1}/n_i \). See [1], [5], [6]. In this paper we investigate the set of quotients \( n_{i+1}/n_i \) (\( i = 1, 2, 3, \cdots \)). Theorem 1 contains a complete characterization of this set in case \( r = 2 \). The situation for \( r > 2 \) is much more complicated.

As a first step we made the following conjecture.

Let \( t \) be fixed, \( 1 \leq t \leq r - 1 \). Then there exist infinitely many pairs \( n_i, n_{i+1} \) such that one of the numbers \( n_i, n_{i+1} \) is composed of \( p_1, \cdots, p_t \) and the other is composed of \( p_{t+1}, \cdots, p_r \).

We prove this conjecture for \( t = 1 \) in Theorem 2 and for \( t = 2 \) in Theorem 3. The case \( t > 2 \) is still open. Since \( t = 1 \) and \( t = 2 \) are equivalent to \( t = r - 1 \) and \( t = r - 2 \) respectively, the conjecture is true for \( r \leq 5 \).

Let \( p_1, \cdots, p_r \) be different primes, \( r \geq 2 \). By the sequence composed of \( p_1, \cdots, p_r \) we mean the monotonically increasing sequence \( N = \{n_i\}_{i=1}^\infty \) of all numbers which are of the form \( p_1^{k_1} \cdots p_r^{k_r} \), where \( k_1, \cdots, k_r \) are non-negative integers. We observe that

\[
(1) \quad d|n_i \quad \text{and} \quad d|n_{i+1} \Rightarrow \frac{n_i}{d} \quad \text{and} \quad \frac{n_{i+1}}{d} \quad \text{are consecutive elements of} \quad N.
\]

Indeed, \( n_i/d < n_j < n_{i+1}/d \) would imply \( n_i < dn_j < n_{i+1} \), which is impossible. We denote the G.C.D. of two integers \( a \) and \( b \) by \( (a, b) \).

We shall use the following lemmas...
LEMMA 1: Let $p_1, \cdots, p_r$ be fixed primes, $r \geq 2$. Let $n_1, n_2, \cdots$ be the sequence composed of these primes. Then there exist positive constants $C_1, C_2$ and $N$ such that

\begin{equation}
\frac{n_i}{(\log n_i)^{C_1}} < n_{i+1} - n_i < \frac{n_i}{(\log n_i)^{C_2}} \quad \text{for } n_i \geq N.
\end{equation}

PROOF. The first inequality is a corollary of [5, Theorem 1]. The second can be found in [6].

LEMMA 2: Let $n_1, n_2, \cdots$ be the sequence composed of the primes $p_1, \cdots, p_r$ with $r \geq 2$. Then

\[ \lim_{i \to \infty} \frac{(n_i, n_{i+1})}{n_i} = 0. \]

PROOF: Let $d_i = (n_i, n_{i+1})$. If $n_j = n_i / d_i$, then, by (1), $n_{j+1} = n_{i+1} / d_i$. Hence, by (2),

\begin{equation}
\frac{1}{(\log n_j)^{C_1}} < \frac{n_{j+1}}{n_j} - 1 = \frac{n_{i+1}}{n_i} - 1 < \frac{1}{(\log n_i)^{C_2}}.
\end{equation}

It follows that

\[ (\log n_i)^{C_2} < \left( \log \frac{n_i}{d_i} \right)^{C_1}. \]

Since the left hand term tends to $\infty$ if $i \to \infty$, we also have $n_i / d_i \to \infty$ if $i \to \infty$.

We need several elementary results from the theory of continued fractions. Let $\xi > 0$ be an irrational number with simple continued fraction $[a_0, a_1, a_2, \cdots]$. The $n$-th convergent $[a_0, \cdots, a_n]$ to $\xi$ is denoted by $A_n/B_n$. It is well known that the denominators $B_n$ form a monotonically increasing sequence of integers for $n \geq 1$, that the sequence $A_0/B_0$, $A_2/B_2$, $A_4/B_4$, $\cdots$ is monotonically increasing to $\xi$ and $A_1/B_1$, $A_3/B_3$, $A_5/B_5$, $\cdots$ is monotonically decreasing to $\xi$. The convergents $A_n/B_n$ are the best approximations to $\xi$ in the sense of Lemma 3(a). For our convenience we give a slightly different form of this assertion in Lemma 3(b).
Lemma 3: (a) The convergents to \( \zeta \) are just the fractions \( A/B \) having the property that every fraction \( r/s \) with \( 0 < |r - s\zeta| < |A - B\zeta| \) satisfies \( s > B \).

(b) If \( A_n/B_n \) is a convergent to \( \zeta \), then every fraction \( r/s \) with \( 0 < |r - s\zeta| < |A_n - B_n\zeta| \) satisfies \( s \geq B_{n+1} \).

Proof: See [2, Satz 2.18, 2.17].

Apart from the convergents to \( \zeta \) we shall consider a larger set of fractions. We recall

\[
A_{n+1} = a_{n+1} A_n + A_{n-1}, \quad B_{n+1} = a_{n+1} B_n + B_{n-1},
\]

for \( n \geq 0 \).

We call a fraction

\[
\frac{A}{B} = \frac{jA_n + A_{n-1}}{jB_n + B_{n-1}} \quad \text{with} \quad j \in \{1, 2, \ldots, a_{n+1}\}
\]

a one-sided convergent to \( \zeta \) (Näherung). We call it a left convergent if \( A/B < \zeta \) and a right convergent if \( A/B > \zeta \). We can arrange the one-sided convergents to \( \zeta \) with increasing denominators. Part of this sequence reads as follows

\[
\frac{A_n}{B_n}, \frac{A_n + A_{n-1}}{B_n + B_{n-1}}, \ldots, \frac{a_{n+1} A_n + A_{n-1}}{a_{n+1} B_n + B_{n-1}} = \frac{A_{n+1}}{B_{n+1}}, \frac{A_{n+1} + A_n}{B_{n+1} + B_n}.
\]

It follows immediately from the construction that

\[
(jA_n + A_{n-1})/(jB_n + B_{n-1}) \quad (j = 1, \ldots, a_{n+1})
\]

are on the same side of \( \zeta \), but \( A_n/B_n \) and \((A_{n+1} + A_n)/(B_{n+1} + B_n)\) are on the opposite side of \( \zeta \).

In [2, Satz 2.21, 2.22] a complete characterization of the one-sided convergents is given. The second theorem states the following.

Lemma 4: If a fraction \( A/B \) with positive denominator has the property that every fraction between \( \zeta \) and \( A/B \) has a denominator greater than \( B \), then \( A/B \) is a one-sided convergent to \( \zeta \).

We shall use Lemma 4 to derive a slightly different characterization which is more analogous to Lemma 3(a) and more appropriate for our purposes.
LEMMA 5:
(a) The left convergents to $\zeta$ are just the fractions $A/B$ having the property that every fraction $r/s$ with $A-B\zeta < r-s\zeta < 0$ satisfies $s > B$.
(b) The right convergents to $\zeta$ are just the fractions $A/B$ having the property that every fraction $r/s$ with $0 < r-s\zeta < A-B\zeta$ satisfies $s > B$.

PROOF: Since the proofs of both parts are almost identical we only prove the second assertion.

Let $A/B$ have the property that every fraction $r/s$ with

$$0 < r-s\zeta < A-B\zeta$$

satisfies $s > B$. Then every fraction $r/s$ with $\zeta < r/s < A/B$ satisfies $s > B$. Indeed, if $r/s$ were a fraction with $s \leq B$ and $\zeta < r/s < A/B$ then it would follow that

$$0 < r-s\zeta = s\left(\frac{r}{s} - \zeta\right) \leq B\left(\frac{A}{B} - \zeta\right) = A-B\zeta,$$

which is a contradiction. It follows from Lemma 4 that $A/B$ is a right convergent.

Let $A/B$ be any right convergent. By definition $A/B$ can be written in the form

$$\frac{A}{B} = \frac{jA_n + A_{n-1}}{jB_n + B_{n-1}}, \quad j \in \{1, 2, \ldots, a_{n+1}\},$$

where $A_{n-1}/B_{n-1}$ and $A_n/B_n$ are convergents to $\zeta$ with

$$\frac{A_n}{B_n} < \zeta < \frac{A_{n-1}}{B_{n-1}}.$$

Define $A^*/B^*$ by

$$A^*-B^*\zeta = \min_{r-s\zeta > 0, s \leq B} (r-s\zeta)$$

Since $\zeta$ is irrational, $A^*$ and $B^*$ are uniquely determined. It is obvious that there does not exist a fraction $r/s$ with $s \leq B^*$ and $0 < r-s\zeta < A^*-B^*\zeta$. Hence, by the first part of the proof, $A^*/B^*$ is a right convergent. It follows from (6) and (5) that $0 < A^*-B^*\zeta \leq A_{n-1} - B_{n-1} \zeta$. On applying Lemma
3(b) we obtain $B^* \geq B_{n-1}$. Since $A^*/B^*$ is a right convergent to $\zeta$ and $B^* \leq B$, we obtain

$$\frac{A^*}{B^*} = \frac{IA_n + A_{n-1}}{iB_n + B_{n-1}}, \quad \text{where } i \in \{0, 1, \ldots, j\}. \quad (7)$$

We have, by (7), (5) and (4),

$$A^* - B^* \zeta = i(A_n - B_n \zeta) + (A_{n-1} - B_{n-1} \zeta)$$

$$\geq j(A_n - B_n \zeta) + (A_{n-1} - B_{n-1} \zeta) = A - B \zeta,$$

while equality holds if and only if $i = j$. By (6), $A^* - B^* \zeta \leq A - B \zeta$. Hence, $i = j$ and $A^*/B^* = A/B$. In view of (6) this completes the proof of Lemma 5(b).

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Let $\alpha$ and $\beta$ be real numbers with $\alpha > \beta > 1$. By the sequence composed of $\alpha$ and $\beta$ we mean the monotonically increasing sequence $N = \{n_i\}_{i=1}^\infty$ of all numbers of the form $\alpha^k \beta^l$, where $k$ and $l$ are non-negative integers. The following theorem gives a complete characterization of the set of quotients $\{n_{i+1}/n_i\}_{i=1}^\infty$.

**THEOREM 1**: Let $\alpha$ and $\beta$ be real numbers with $\alpha > \beta > 1$, and such that $\zeta = \log \beta / \log \alpha$ is irrational. Let $n_1, n_2, \cdots$ be the sequence composed of $\alpha$ and $\beta$. If $S = \{n_{i+1}/n_i|i = 1, 2, \cdots\}$, then $S$ is the set of all products $\alpha^k \beta^l$ and $\alpha^k \beta^{-l}$ which are greater than 1 and such that $k/l$ is a one-sided convergent to $\zeta$.

**REMARK**: In view of Theorem 1 one can define a natural generalization of the continued fractions as follows. Let $\alpha_1, \cdots, \alpha_m$ be real numbers all greater than 1. Let $n_1, n_2, \cdots$ be the sequence composed of $\alpha_1, \cdots, \alpha_m$. Put $S = \{n_{i+1}/n_i|i = 1, 2, \cdots\}$. We would be very interested in a characterization of $S$ like Theorem 1 does in case $m = 2$.

**PROOF**: Let $k/l$ be a one-sided convergent to $\zeta$. We shall prove that $\alpha^k$ and $\beta^l$ are consecutive elements of $N$. This implies that $k/l$ belongs to $S$.

Assume $k/l$ is a left convergent to $\zeta$. Then $\alpha^k < \beta^l$. Suppose there exists an element $\alpha^r \beta^s$ such that $\alpha^k < \alpha^r \beta^s < \beta^l$. Hence, $l > s \geq 0$. We have

$$k < r + s \zeta < l \zeta,$$
or, equivalently,
\[ k - l \xi < r - (l - s) \xi < 0. \]

This is a contradiction with Lemma 5(a).

If \( k/l \) is a right convergent to \( \xi \), then \( \beta^l < \alpha^k \) and a similar argument gives that \( \beta^l \) and \( \alpha^k \) are consecutive elements of \( N \).

In order to prove that every element of \( S \) is of the required form, put
\[ n_i = \alpha^r \beta^{s_i}, \quad n_{i+1} = \alpha^{r_i+1} \beta^{s_{i+1}}. \]
Since \( \alpha > \beta \), we have
\[ \alpha^{r_i+1} \beta^{s_i} > \alpha^r \beta^{s_{i+1}} \geq n_{i+1}, \]
and, hence, either \( r_{i+1} \leq r_i \) or \( s_{i+1} < s_i \). Since both cases are treated in similar ways, we only deal with the first. Assume \( r_{i+1} \leq r_i \). Then \( s_{i+1} > s_i \).

Put \( k = r_i - r_{i+1}, \ l = s_{i+1} - s_i \). We have \( \alpha^{-k} \beta^l = n_{i+1}/n_i > 1 \). We shall prove that \( k/l \) is a left convergent to \( \xi \). We have \( k/l < \log \beta/\log \alpha = \xi \).

Suppose there exists a fraction \( r/s \) with \( s \leq l \) and
\[ k - l \xi < r - s \xi < 0. \]

Then
\[ \alpha^{r - k + r_i} \beta^{l - s + s_i} = n_i e^{(r - k) \log \alpha + (l - s) \log \beta} > n_i. \]

Since \( r - k + r_i = r + r_{i+1} > 0 \) and \( l - s + s_i \geq s_i > 0 \), we obtain
\[ \alpha^{r - k + r_i} \beta^{l - s + s_i} \in N. \]

On the other hand,
\[ \alpha^{r - k + r_i} \beta^{l - s + s_i} = n_{i+1} e^{r \log \alpha - s \log \beta} < n_{i+1}. \]

The contradiction (8), (9), (10), proves by Lemma 5(a) that \( k/l \) is a left convergent to \( \xi \). (In case \( s_{i+1} < s_i \) the fraction \( k/l \) turns out to be a right convergent to \( \xi \).

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It would be very valuable to have a characterization like Theorem 1 for sequences composed of \( r \) multiplicatively independent positive
numbers, $r > 2$. This would solve the conjecture in the introduction immediately. We now prove case $t = 1$ of this conjecture.

**Theorem 2:** Let $n_1, n_2, \cdots$ be the sequence composed of the primes $p_1, \cdots, p_r$ $(r \geq 2)$. Let $p$ be one of these primes. Then there exists an infinite number of pairs $n_i, n_{i+1}$ such that $n_i$ is a pure power of $p$ and $n_{i+1}$ is not divisible by $p$.

**Proof:** Without loss of generality we may assume $p = p_1$. Let $k$ be a positive integer and $n_{jk} = p^k$. Let $n_{jk+1} = p_1^{l_1} \cdots p_r^{l_r}$ and $n_{ik} = p_1^{k-1}$. It follows from (1) that $n_{ik+1} = p_2^{l_2} \cdots p_r^{l_r}$. Since, by Lemma 2,

$$n_{ik} = \frac{n_{jk}}{(n_{jk}, n_{jk+1})} \rightarrow \infty \quad \text{for } k \rightarrow \infty,$$

we obtain infinitely many different pairs $n_{ik}, n_{ik+1}$ with the required property.

**Remark:** In the same way one can prove the existence of infinitely many pairs $n_i, n_{i+1}$ such that $n_{i+1}$ is a pure power of $p$ and $n_i$ is not divisible by $p$.

Finally we prove case $t = 2$ of our conjecture.

**Theorem 3:** Let $p_1, \cdots, p_r$ be $r > 2$ different primes. Let $M = \{m_1, m_2, \cdots\}$ be the sequence composed of these primes. Let $p$ and $q$ be two primes from $p_1, \cdots, p_r$. Then there exist infinitely many pairs $m_i, m_{i+1}$ such that one of the numbers $m_i, m_{i+1}$ is composed of $p$ and $q$ and the other is neither divisible by $p$ nor by $q$.

The proof is based on two lemmas.

**Lemma 6:** Let $r > 2$. Let $M = \{m_1, m_2, \cdots\}$ be the sequence composed of the different primes $p_1, \cdots, p_r$ and $N = \{n_1, n_2, \cdots\}$ the sequence composed of $p_1$ and $p_2$. Suppose there exists an $i_0$ such that for every $i \geq i_0$

$$m_i \in N \implies (m_{i-1}, p_1 p_2) > 1 \quad \text{and} \quad (m_{i+1}, p_1 p_2) > 1.$$

Then there exists an $i_1$ such that for every $i \geq i_1$
(a) if \( m_i \in \mathbb{N} \) and \( m_i^2 \leq m_{i-1}m_{i+1} \), then \( m_{i-1} \in \mathbb{N} \),
(b) if \( m_i \in \mathbb{N} \) and \( m_i^2 \geq m_{i-1}m_{i+1} \), then \( m_{i+1} \in \mathbb{N} \).

**Proof:** We know from Lemma 2 that

\[
\frac{m_{i-1}}{(m_{i-1}, m_i)} \to \infty \quad \text{as} \quad i \to \infty.
\]

We choose \( i_1 \) such that

\[
\frac{m_{i-1}}{(m_{i-1}, m_i)} > m_{i_0} \quad \text{for} \quad i \geq i_1.
\]

In the sequel we only consider \( i \) with \( i \geq i_1 \).

Assume \( m_i \in \mathbb{N} \). Let \( m_i = p_1^{a_1}p_2^{b_1} \). Put \( m_{i-1} = p_1^{k_1} \cdots p_r^{k_r} \) and \( m_{i+1} = p_1^{l_1} \cdots p_r^{l_r} \). Then

\[
m_{i-1} < m_{i-1}m_{i+1}/m_i < m_{i+1}.
\]

Hence, we have either

(11) \[
m_{i-1}m_{i+1}/m_i = m_i
\]

or

(12) \[
m_{i-1}m_{i+1}/m_i \notin M.
\]

We note \( m_{i-1}m_{i+1}/m_i = p_1^{k_1+l_1-a}p_2^{k_2+l_2-b}p_3^{k_3+l_3} \cdots p_r^{k_r+l_r} \). If (11) holds, then \( k_3 + l_3 = \cdots = k_r + l_r = 0 \), and, hence, \( k_3 = \cdots = k_r = 0 \) and \( l_3 = \cdots = l_r = 0 \). In this case both \( m_{i-1} \in \mathbb{N} \) and \( m_{i+1} \in \mathbb{N} \). If (12) holds, then

(13) \[
k_1 + l_1 - a < 0 \quad \text{or} \quad k_2 + l_2 - b < 0.
\]

Suppose \( k_1 \leq a \) and \( k_2 \leq b \). By (1), \( p_1^{a-k_1}p_2^{b-k_2} \) is preceeded in \( M \) by \( p_3^{k_3} \cdots p_r^{k_r} \). Since

\[
p_1^{a-k_1}p_2^{b-k_2} = \frac{m_i}{(m_{i-1}, m_i)} > m_{i_0},
\]

this is a contradiction with the condition of the lemma. Hence, \( k_1 > a \) or
Similarly, \( k_2 > b \). Without loss of generality we may assume \( k_2 > b \). Then, by (13), \( k_1 < a \) and \( l_1 < a \). Thus \( l_2 > b \). So we obtain

\[
(14) \quad k_1 < a, \quad l_1 < a, \quad k_2 > b, \quad l_2 > b.
\]

We define a sequence of positive integers \( \{a_j\}_{j=0}^{\infty} \) by

\[
m_{a_j} = p_1^a p_2^b \quad \text{for } j = 0, 1, 2, \cdots.
\]

We have, by (1) and (14),

\[
m_{a_j-1} = p_1^{k_1} p_2^{k_2-b+j} p_3^{k_3} \cdots p_r^{k_r} \quad \text{and} \quad m_{a_j+1} = p_1^{l_1} p_2^{l_2-b+j} p_3^{l_3} \cdots p_r^{l_r},
\]

for \( j = 0, 1, \cdots, b \). Consider the pairs of quotients

\[
(15) \quad \left( \frac{m_{a_j-1}}{m_a}, \frac{m_{a_j+1}}{m_{a_j}} \right) \quad \text{for } j = 0, 1, 2, \cdots.
\]

We know

\[
\frac{m_{a_j-1}}{m_{a_j}} = p_1^{k_1} p_2^{k_2-b} p_3^{k_3} \cdots p_r^{k_r} \quad \text{and} \quad \frac{m_{a_j+1}}{m_{a_j}} = p_1^{l_1} p_2^{l_2-b} p_3^{l_3} \cdots p_r^{l_r}
\]

for \( j = 0, \cdots, b \). Let \( J_0 \) be the smallest value of \( j \) for which one of the quotients in (15) assumes another value. This \( J_0 \) exists, since, by Lemma 1, \( m_{i+1}/m_i \) tends to 1 as \( i \to \infty \). We assume that the first quotient changes firstly. Thus

\[
(16) \quad 1 > \frac{m_{a_j-1}}{m_{a_j}} > \frac{m_{a_j-1}}{m_{a_j-1}} = \cdots = \frac{m_{a_0-1}}{m_{a_0}} = \frac{m_{i-1}}{m_i}
\]

and

\[
(17) \quad \frac{m_{a_j-1}+1}{m_{a_j-1}} = \cdots = \frac{m_{a_0+1}}{m_{a_0}} = \frac{m_{i+1}}{m_i}.
\]

Put \( m_{a_j-1} = p_1^{k_1} \cdots p_r^{k_r} \) and \( m_{a_j+1} = p_1^{l_1} \cdots p_r^{l_r} \). The following argument shows \( \kappa_2 = 0 \). If \( \kappa_2 > 0 \), then, by (1), \( m_{a_j-1}/p_2 \) is the predecessor of \( m_{a_j}/p_2 = m_{a_j-1} \), and, hence, \( m_{a_j-1}/m_{a_j} = m_{a_j-1}/m_{a_j-1} \) in contradiction with (16). Since we know from the argument preceding formula (14) that both \( \kappa_1 \leq a \) and \( \kappa_2 \leq J \) is impossible, we have
Consider

\[ m = \frac{m_{a_j-1} + 1}{m_{a_j}}. \]

We have, by (17)

\[ m = \frac{m_{a_j-1} + 1}{m_{a_j}} \cdot \frac{m_{a_j} - 1}{m_{a_j}} = \frac{m_{i+1}}{m_i} \cdot \frac{m_{a_j-1}}{m_{a_j}} \]

\[ = p_1^{l_1+\kappa_1} p_2^{l_2+\kappa_2} \cdots p_r^{l_r+\kappa_r}. \]

From (18) and (14) we see that \( m \in M \). Moreover,

\[ m_{a_j-1} + 1 > m > \frac{m_{a_j-1} + 1}{m_{a_j}} > m_{a_j-1}. \]

Hence,

\[ (19) \]

\[ m = m_{a_j-1}. \]

This implies \( l_3 + \kappa_3 = \cdots = l_r + \kappa_r = 0 \). Thus \( l_3 = \cdots = l_r = 0 \) and \( m_{i+1} \in N \). Furthermore, in view of (17), (19), the definition of \( m \) and (16),

\[ \frac{m_{i+1}}{m_i} = \frac{m_{a_j-1} + 1}{m_{a_j-1}} = \frac{m_{a_j-1} + 1}{m} \]

\[ = \frac{m_{a_j}}{m_{a_j-1}} < \frac{m_{a_j-1}}{m_{a_j-1} - 1} = \frac{m_i}{m_{i-1}}. \]

Similarly, the assumption that the second quotient in (15) changes firstly leads to \( m_{i-1} \in N \) and \( m_{i-1}/m_i > m_i/m_{i+1} \). This completes the proof of the lemma.

**Lemma 7**: Let \( r > 2 \). Let \( M = \{m_1, m_2, \cdots\} \) be the sequence composed of the primes \( p_1, \cdots, p_r \). Let \( p \) and \( q \) be two arbitrary primes from \( p_1, \cdots, p_r \) with \( p > q \) and let \( N = \{n_1, n_2, \cdots\} \) be the sequence composed of \( p \) and \( q \). Suppose there exists an \( i_0 \) such that for every \( i \geq i_0 \)

\[ m_i \in N \Rightarrow (m_{i-1}, pq) > 1 \quad \text{and} \quad (m_{i+1}, pq) > 1. \]
Then there exists a monotonically increasing, unbounded sequence $T_1, T_2, T_3, \cdots$ such that no interval $[T_H, qT_H]$ contains an element of $M \setminus N$.

**Proof**: Let $[a_0, a_1, a_2, \cdots]$ be the continued fraction of $\xi := \log q / \log p$. Put $A_h/B_h = [a_0, \cdots, a_h]$ for $h = 0, 1, 2, \cdots$. It follows from the Gel'fond-Schneider theorem [4, Satz 14], that $\xi$ is transcendental. Hence, the sequence $a_0, a_1, a_2, \cdots$ is not periodical [2, Satz 3.1]. There therefore exist infinitely many values $h$ with $a_h > 1$.

Let $H$ be such that $a_H > 1$. It is no loss of generality to assume $A_H/B_H < \xi$. We consider the subsequence $N_1$ of $N$ beginning with

$$T_H = p^{A_H}q^{B_H-1}$$
and ending with $qT_H = p^{A_H}q^{B_H-1}$.

(If $A_H/B_H > \xi$, we may choose $T_H = p^{A_H-1}q^{B_H}$ and consider the interval $[T_H, pT_H]$.)

Let $n_i = p^c d$ be in $N_1$, $n_i \neq qT_H$. Since $qT_H < q^{B_H+1} < p^{A_H+B_H-1}$, we have

$$c < A_H + A_{H-1} \quad \text{and} \quad d < B_H + B_{H-1}.$$ 

We distinguish two cases.

(i) $c \geq A_H$. We assert that $n_{i+1} = p^{c-A_H} d + B_H$. Since $A_H/B_H < \xi$, we have

$$n_i < p^{c-A_H} d + B_H \in N_1.$$

Suppose

$$n_{i+1} = p^c d' < p^{c-A_H} d + B_H.$$ 

This implies

$$A_H - B_H \xi < (c-s)-(t-d)\xi < 0.$$ 

By Lemma 3(b), $|t-d| \geq B_{H-1}$. Hence, $d \geq B_{H+1}$ or $t \geq B_{H+1}$ in contradiction with (20).

(ii) $c < A_H$. Since $d \leq B_{H-1} - 1$ implies $p^c d < p^{A_H}q^{B_H-1} = T_H$, we have $d \geq B_{H-1}$. We assert that $n_{i+1} = p^{c+A_{H-1}} d - B_{H-1}$. Since $A_{H-1}/B_{H-1} > \xi$, we have

$$n_i < p^{c+A_{H-1}} d - B_{H-1} \in N_1.$$
Suppose
\[ n_{i+1} = p^s q^t < p^{s+AH-1} q^{d-BH-1}. \]

Then
\[ 0 < (s-c) - (d-t) \xi < A_{H-1} - B_{H-1} \xi. \]

By Lemma 3(b), \(|d-t| \geq B_H\). If \(d-t < 0\), then \(t > d + B_H \geq B_H + B_{H-1}\) and \(p^s q^t > q^{B_H+B_{H-1}} > qT_H\), which is false. Hence, \(d-t \geq 0\). This implies \(s-c > 0\) and \(d \geq B_H + t\). By Lemma 5(b) \((s-c)/(d-t)\) is a right convergent to \(\xi\). Since \(A_H/B_H\) is a left convergent to \(\xi\), we obtain \(d \geq d-t \geq B_H + B_{H-1}\), which is impossible in view of (20). Summarizing we see that among the quotients \(n_{i+1}/n_i\) for \(n_i \in N_1\) only \(p^{-AH} q^{BH}\) and \(p^{AH-1} q^{-BH-1}\) occur. Note

(21) \[ p^{AH-1} q^{-BH-1} > p^{-AH} q^{BH} > 1. \]

We now assert that

(22) \[ \frac{n_{i+1}}{n_i} = p^{AH-1} q^{-BH-1} \Rightarrow \frac{n_i}{n_{i-1}} = \frac{n_{i+1}}{n_i} \text{ or } \frac{n_{i+1}}{n_i} = \frac{n_{i+2}}{n_{i+1}}. \]

Since \(n_i = T_H\) implies \(n_{i+1} = p^{-AH} q^{BH} n_i\), we have \(n_i > T_H\). Hence, \(n_{i-1} \in N_1\). Suppose

\[ \frac{n_i}{n_{i-1}} = \frac{n_{i+2}}{n_{i+1}} = p^{-AH} q^{BH}. \]

Then

\[ n_{i+2} = p^{-2AH+AH-1} q^{2BH-BH-1} n_{i-1}. \]

By \(a_H \geq 2\), it follows that \(n_{i+2} \geq q^{BH+B_{H-1}+B_{H-2}}\). This is a contradiction.

We now turn our attention to the subsequence \(M_1\) of \(M\) starting with \(T_H\) and ending with \(qT_H\). Let \(m_i \in N_1\), \(m_i \neq qT_H\). Put \(m_i = n_j\). Hence, \(n_{j+1} \in N_1\). Note that \(n_{j-1} \leq m_{i-1} < m_i < m_{i+1} \leq n_{j+1}\). The condition of Lemma 7 enables us to apply Lemma 6. Hence, \(m_{i-1} = n_{j-1}\) if \(m_i^2 \leq m_{i-1} m_{i+1}\) and \(m_{i+1} = n_{j+1}\) if \(m_i^2 \geq m_{i-1} m_{i+1}\). It follows that

(23) \[ m_{i-1} = n_{j-1} \text{ if } n_j^2 \leq n_{j-1} m_{i+1} \leq n_{j-1} n_{j+1} \]
and

\[ m_{i+1} = n_{j+1} \quad \text{if} \quad n_j^2 \geq m_i n_{j+1} \geq n_j n_{j+1}. \]  

We can now prove that all elements \( m_i \) with \( T_H \leq m_i \leq q T_H \) belong to \( N_1 \). Suppose \( T_H = m_I \) and all integers \( m_I, m_{I+1}, \ldots, m_i \) belong to \( N_1 \), while \( m_i < q T_H \). We shall prove that \( m_{i+1} \in N_1 \). Put \( m_i = n_j \). We distinguish two cases.

(i) \( n_j - n_{j+1} \leq n_j^2 \). It follows from (24) that \( m_{i+1} = n_{j+1} \in N_1 \).

(ii) \( n_j - n_{j+1} \geq n_j^2 \). It follows from formula (21) and the lines before that

\[ \frac{n_j}{n_{j-1}} = p^{-A_H} q^{B_H}, \quad \frac{n_{j+1}}{n_j} = p^{A_{H-1}} q^{-B_{H-1}}. \]

Since \( n_j = T_H \) implies \( n_{j+1} = p^{-A_H} q^{B_H} n_j \), we have \( n_j > T_H \) and, hence, \( n_{j-1} \in N_1 \). By (22) we have \( n_{j+2}/n_{j+1} = p^{A_{H-1}} q^{-B_{H-1}} \). Let \( n_{j+1} = m_{i*} \).

Since \( n_{j+2}/n_{j+1} = n_{j+1}/n_j \), we obtain from (23) that \( n_i = m_{i*} \). Hence, \( m_{i*} = m_i \) and \( i* - 1 = i \). It follows that \( m_{i+1} = m_{i*} = n_{j+1} \in N \).

Since we have constructed an infinite number of \( T_H \)'s such that all integers \( m_i \in [T_H, q T_H] \) belong to \( N \), the lemma has been proved.

We are now going to prove the main result.

**Proof of Theorem 3:** It is no restriction to assume \( p = p_1, q = p_2, p > q \). Suppose that there are only a finite number of values \( i \) for which the statement of the theorem holds. Then the condition of Lemma 7 is fulfilled for some \( i_0 \). It follows that there exists an unbounded sequence \( T_1, T_2, T_3, \cdots \) such that each element \( m_i \in [T_H, q T_H] \) belongs to the sequence \( N \) composed of \( p \) and \( q \). Let \( N = \{n_1, n_2, n_3, \cdots \} \). We know from Lemma 1 that \( n_{i+1}/n_i \to 1 \) as \( i \to \infty \). Consider the sequence \( p_3 n_1, p_3 n_2, p_3 n_3, \cdots \). These elements belong to \( M \setminus N \). However, \( p_3 n_{i+1}/p_3 n_i \to 1 \) as \( i \to \infty \). This is a contradiction.
REFERENCES


(Oblatum 19-III-1974)