

# COMPOSITIO MATHEMATICA

ENRICO ARBARELLO

## **Weierstrass points and moduli of curves**

*Compositio Mathematica*, tome 29, n° 3 (1974), p. 325-342

[http://www.numdam.org/item?id=CM\\_1974\\_\\_29\\_3\\_325\\_0](http://www.numdam.org/item?id=CM_1974__29_3_325_0)

© Foundation Compositio Mathematica, 1974, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## WEIERSTRASS POINTS AND MODULI OF CURVES

Enrico Arbarello\*

### Introduction

This paper is devoted to the study of a particular class of subvarieties of the moduli space of stable curves of genus  $g$ . These varieties are the loci of moduli of special curves distinguished by the presence of a Weierstrass point of a particular type. These subloci of  $\bar{M}_g$  (cf. sec. 3) can be viewed as a natural generalization of the hyperelliptic locus. They are defined in the following way. For each  $n$ ,  $2 \leq n \leq g$ , we let  $\bar{W}_{n,g} =$  (closure in  $\bar{M}_g$  of the sublocus of moduli of curves  $C$  possessing a Weierstrass point  $x$  such that  $\dim H^0(\mathcal{O}_C(nx)) \geq 2$ ). These subvarieties of  $\bar{M}_g$  (more precisely their preimages in the Teichmüller space) were studied by Rauch [9] and Farkas [4]. Their main result is the following. (cf. sec. 3 for the definition of Weierstrass sequence).

**THEOREM (1) (Rauch):** *The totality of closed Riemann (Torelli, Teichmüller) surfaces of genus  $g$  possessing Weierstrass points whose Weierstrass sequences begin with a fixed  $n \leq g$  form a complex analytic (possibly disconnected) submanifold of dimension  $2g + n - 3$  of the modulus (Torelli, Teichmüller) space in the (general) case when  $n + 1$  is a gap. When  $n + 1$  is not a gap, they form a complex analytic subvariety of dimension  $2g + n - 4$  of a complex analytic submanifold of dimension  $2g + n - 3$ .*

Rauch observes that ‘the distinction of cases in Theorem 1 represents the discovery of a new sort of “singularity” in the set of Riemann surfaces’ and he adds, ‘It must be mentioned that it is not altogether clear that the singularity in the exceptional case of Theorem 1 is genuine and not the product of the method of proof’.

We shall analyze the above subloci from a point of view which differs substantially from the one of Rauch and Farkas. Our approach has been inspired by Fulton’s construction of the Hurwitz space  $H^{n,w}$ , whose points parametrize the set of  $n$ -sheeted coverings of  $\mathbb{P}^1$  with  $w$  simple ramification points. We will in fact introduce a Weierstrass-Hurwitz

\* Partially supported by NSF Grant 36269

space  $WH^{n,w}$  whose points parametrize the set of *simple Weierstrass coverings of type  $(n, w)$* , (cf. Definition 2.1). The analytic manifold  $WH^{n,w}$  defines then an analytic subvariety of  $\bar{M}_g$  whose closure we denote by  $\bar{W}_{n,g}$ . In Theorem 3.11 we prove that  $\bar{W}_{n,g}$  is irreducible and that  $\dim W_{n,g} = 2g + n - 3$ . In this theorem the irreducibility statement plays a fundamental role in the proof of the dimensionality statement. It is from this point of view that it seems more convenient to work with the moduli space  $\bar{M}_g$  rather than the Teichmüller space  $T^g$  (for instance it is not at all clear that the preimages of the  $\bar{W}_{n,g}$ 's in  $T^g$  are connected). We also notice that the dimensionality statement on  $\bar{W}_{n,g}$  implicitly proves that, indeed, the first case in Rauch's theorem is the "generic" one.

In Theorem 3.18 we then prove that  $\bar{W}_{n,g}$  is the  $n^{\text{th}}$  level of a filtration

$$\bar{W}_{2,g} \subset \bar{W}_{3,g} \subset \cdots \subset \bar{W}_{g-1,g} \subset \bar{W}_{g,g} = \bar{M}_g.$$

The proof of this fact is based on a degeneration argument which shows that any simple Weierstrass covering of type  $(n-1, w-1)$  can be thought of as the 'limit' of simple Weierstrass coverings of type  $(n, w)$ .

A second use of such a degeneration argument comes in the proof of Theorem 3.27. There we prove that given any algebraic curve  $S$  in  $\bar{W}_{n,g}$  then either some points of  $S$  correspond to a curve of genus less than  $g$  or  $S$  has non-trivial intersection with  $\bar{W}_{n-1,g}$ . The proof of this fact is based on the following observation. Given any non trivial one-parameter, algebraic, family of  $n$ -sheeted Weierstrass coverings, then there are points in the parameter space for which the corresponding  $n$ -sheeted covering is degenerate, in the sense that the point of total ramification and another ramification point 'come together'. One then proves that, when this phenomenon occurs, then either the corresponding curve acquires a node or the Weierstrass point increases its weight. When  $n = g-1$ , Theorem 3.27 asserts that the divisor  $D = \bar{W}_{g-1,g} \cup (\text{boundary of } \bar{M}_g)$  is pseudo-ample in  $\bar{M}_g$ . The pseudo-amplicity of  $D$  acquires a particular significance in light of the fact that the boundary of  $\bar{M}_g$  is not pseudo-ample in  $\bar{M}_g$ , nor is, in general,  $\bar{W}_{g-1,g}$  as we shall prove in Theorem 3.28.

I wish to express my gratitude to Lipman Bers and Herbert Clemens for the many suggestions and the stimulating discussions on the topic of this paper. I finally wish to thank Frans Oort for very helpful conversations and correspondence.

## 1. The Hurwitz Space

Let  $X$  be a compact Riemann surface. Let  $f : X \rightarrow \mathbb{P}^1$  be an analytic map of  $X$  onto the Riemann sphere. Given any point  $x \in X$  there exists

a neighborhood  $U$  of  $x$  on which  $f$  is conformally equivalent to a map  $z \mapsto z^{e(x)}$  of the unit disc  $\{z \in \mathbb{C} : |z| < 1\}$  onto itself. The integer  $e(x)$  is called the *ramification index of  $f$  at  $x$* . The set  $R = \{x \in X : e(x) \geq 2\}$  is called the *ramification locus of  $f$* . The set  $f(R) \subset \mathbb{P}^1$  is called the *branch locus of  $f$*  and it is denoted by  $\Delta(f)$ . The branch locus is the support of the divisor

$$(1.1) \quad \delta(f) = \sum_{y \in \mathbb{P}^1} \left( \sum_{x \in f^{-1}(y)} (e(x) - 1) \right) y$$

which is called the *discriminant of  $f$* . If  $\sum_{x \in f^{-1}(y)} e(x) = n$ ,  $y \in \mathbb{P}^1$ , we say that  $f$  is an  *$n$ -sheeted branched covering of  $\mathbb{P}^1$* . A point  $x \in X$  is a point of *simple ramification* for  $f$  if  $e(x) = 2$  and if  $x$  is the only point of  $f^{-1}f(x)$  at which  $f$  ramifies. A point  $x \in X$  is a point of *total ramification* for the  $n$ -sheeted branched covering  $f$  if  $e(x) = n$ . If  $f : X \rightarrow \mathbb{P}^1$  is an  $n$ -sheeted branched covering and if  $g = \text{genus}(X)$ , then the well known *Hurwitz formula* gives

$$(1.2) \quad 2n + 2g - 2 = \text{degree}(\delta(f)).$$

Two  $n$ -sheeted branched coverings  $f : X \rightarrow \mathbb{P}^1$  and  $g : X' \rightarrow \mathbb{P}^1$  are said to be *equivalent* if there exists a biholomorphic map  $\varphi : X \rightarrow X'$  such that  $g \circ \varphi = f$ . The equivalence class containing  $f$  will be denoted by  $(f)$ .

Let  $(\mathbb{P}^1)^{(w)}$  denote the  $w$ -fold symmetric product of  $\mathbb{P}^1$ . Let  $\Delta \subset (\mathbb{P}^1)^{(w)}$  be the *discriminant locus*, i.e., the subset of  $(\mathbb{P}^1)^{(w)}$  formed by those  $w$ -tuples which contain fewer than  $w$  distinct points. Since  $(\mathbb{P}^1)^{(w)}$  may be identified with the complex projective  $w$ -space  $\mathbb{P}^w$ , instead of  $(\mathbb{P}^1)^{(w)} - \Delta$  we shall write  $\mathbb{P}^w - \Delta$ .

For each  $A \in \mathbb{P}^w - \Delta$ , let  $H(n, A)$  denote the set of equivalence classes of  $n$ -sheeted branched coverings of  $\mathbb{P}^1$  whose branch locus is equal to  $A$ . Let  $H(n, w)$  denote the set of equivalence classes of  $n$ -sheeted branched coverings of  $\mathbb{P}^1$  with  $w$  branch points, and let

$$(1.3) \quad \Lambda : H(n, w) \rightarrow \mathbb{P}^w - \Delta$$

be the function defined by  $\Lambda((f)) = \Delta(f)$ . We then have  $\Lambda^{-1}(A) = H(n, A)$ . It is an easy matter to introduce a topology on  $H(n, w)$  in such a way that  $\Lambda$  becomes a *topological covering*. For a proof of this fact we refer the reader to ([5], page 545). Via the covering  $\Lambda$  the topological space  $H(n, w)$  can be equipped with the analytic structure induced from that of  $\mathbb{P}^w - \Delta$ .

**DEFINITION (1.4):** *The complex manifold  $H(n, w)$  is called the Hurwitz space of type  $(n, w)$ .*

We shall also use the following notation. Let  $A$  be a finite subset of  $\mathbb{P}^1$ , let  $y \in \mathbb{P}^1 - A$ . Let  $\mathfrak{S}_n$  be the symmetric group on  $n$  letters acting on the set  $\{1, \dots, n\}$ . Let  $\overline{\text{Hom}}(\pi_1(\mathbb{P}^1 - A, y), \mathfrak{S}_n)$  denote the set of equivalence classes of homomorphisms from  $\pi_1(\mathbb{P}^1 - A, y)$  to  $\mathfrak{S}_n$ , where two homomorphisms are equivalent if they differ by an inner automorphism. For each  $f : X \rightarrow \mathbb{P}^1$ ,  $(f) \in H(n, A)$  we define

$$(1.5) \quad \Phi((f)) \in \overline{\text{Hom}}(\pi_1(\mathbb{P}^1 - A, y), \mathfrak{S}_n)$$

in the following manner. Let  $\alpha : \{1, \dots, n\} \rightarrow f^{-1}(y)$  be a numbering of  $f^{-1}(y)$ . Let  $\gamma \subset \mathbb{P}^1 - A$  be a loop with initial point at  $y$ . Let  $\tilde{\gamma}_k$  be the lifting of  $\gamma$  to  $X - f^{-1}(A)$  with initial point at  $\alpha(k) \in f^{-1}(y)$ . Then one can easily see (cf. [5], (1.2)) that setting  $\Phi((f))(\gamma)(k) = \alpha^{-1}(\text{end point of } \tilde{\gamma}_k)$ , for each  $k \in \{1, \dots, n\}$ , gives a well-defined element of

$$\overline{\text{Hom}}(\pi_1(\mathbb{P}^1 - A, y), \mathfrak{S}_n).$$

We finish this section by fixing, once and for all, a standard way to choose a system of generators for  $\pi_1(\mathbb{P}^1 - A, y)$ , (cf. [5]). Let us fix an orientation on the complex manifold  $\mathbb{P}^1$ .

**DEFINITION (1.6):** *Let  $A$  be a finite subset of  $\mathbb{P}^1$ . Let  $y \in \mathbb{P}^1 - A$ . Let  $L$  be a simple, closed, oriented arc containing the points of  $A$ . Let  $G$  (resp.  $G'$ ) be the region of  $\mathbb{P}^1$  on the right (resp. left) side of  $L$ . Let  $L$  be oriented in such a way that  $y \in G$ . Let  $(a_1, \dots, a_w) = A$  be an ordering of the points of  $A$  such that  $L$  passes successively through  $a_1, \dots, a_w$  and returns to  $a_1$ . Choose non intersecting simple arcs  $\ell_i$ , in  $G$ , from  $y$  to  $a_i$ . Let  $G_i$  be the simply connected region of  $G$  enclosed by the lines  $\ell_i, \ell_{i+1}$  and the arc of  $L$  from  $a_i$  to  $a_{i+1}$ . Let  $\sigma_i$  be a loop which begins at  $y$ , travels along  $\ell_i$  to a point near  $a_i$ , makes a small clockwise loop around  $a_i$  and returns to  $y$  along  $\ell_i$ . The loops  $\sigma_1, \dots, \sigma_w$  generate  $\pi_1(\mathbb{P}^1 - A, y)$  and satisfy  $\prod_i \sigma_i = \text{id}$ . Such a system of generators is called a standard system of generators for  $\pi_1(\mathbb{P}^1 - A, y)$ .*

## 2. The Weierstrass-Hurwitz space

We shall now restrict our attention to a particular component of  $H(n, w)$ .

We recall that a point  $x$  on a Riemann surface  $X$  is called a Weierstrass point if  $\dim H^0(\mathcal{O}_X(n_x)) \geq 2$  for some positive integer  $n$  such that  $n \leq \text{genus}(X)$ . The following terminology is therefore suggested.

DEFINITION (2.1): An  $n$ -sheeted covering  $f : X \rightarrow \mathbb{P}^1$  with  $w$  branch points is called a simple Weierstrass covering of type  $(n, w)$  if there exists a point  $x \in X$  such that  $e(x) = n$  and such if every ramification point  $x' \in X - \{x\}$  is simple.

We observe that if  $f : X \rightarrow \mathbb{P}^1$  is a simple Weierstrass covering of type  $(n, w)$  then the Hurwitz formula (1.2) determines the genus of  $X$ , namely

$$(2.2) \quad \text{genus}(X) = \left(\frac{1}{2}\right)(w - n).$$

We denote by  $WH^{n,w}$  the set of equivalence classes of simple Weierstrass coverings of type  $(n, w)$ . It is easy to see that  $WH^{n,w}$  is both open and closed in  $H(n, w)$  and it is therefore a union of connected components of  $H(n, w)$ . It follows that the covering (1.3) restricts to a covering

$$(2.3) \quad A : WH^{n,w} \rightarrow \mathbb{P}^w - \Delta.$$

DEFINITION (2.4): The complex manifold  $WH^{n,w}$  is called the Weierstrass Hurwitz space of type  $(n, w)$ .

By using an argument which is essentially due to Lüroth and Clebsch one can prove

THEOREM (2.5):  $WH^{n,w}$  is connected.

For a proof of this fact we refer the reader to ([1], (2.7); [5], (1.5)). We just observe that given a point  $A \in \mathbb{P}^w - \Delta$  and considering the covering (2.3) one proves that  $WH^{n,w}$  is connected by showing that the action of  $\pi_1(\mathbb{P}^w - \Delta, A)$  on  $A^{-1}(A)$  is transitive. The proof of this fact turns out to be of a combinatorial nature. It consists, in fact, in showing that, given  $(f) \in A^{-1}(A)$ , there is a way of assigning to each ramification point  $x$ , of  $f$ , a permutation among the sheets ‘coming together’ at  $x$ . The result can be stated in the following manner.

LEMMA (2.6): Let  $f : X \rightarrow \mathbb{P}^1$  be a simple Weierstrass covering of type  $(n, w)$ . Then it is possible to find a standard system of generators  $\sigma_1, \dots, \sigma_w$  for  $\pi_1(\mathbb{P}^1 - A(f), y)$ , (see Def. (1.6)), such that

$$(2.7) \quad \Phi((f))(\sigma_1, \dots, \sigma_w) = ((123 \dots n), \underbrace{(12), \dots, (12)}_{2g+1 \text{ times}}, (23), \dots, (n-1 \ n)),$$

(where  $\Phi((f))$  is defined as in (1.5)).

REMARK (2.8): With the notation of the preceding lemma, let  $\gamma_i$  be the only connected component of  $f^{-1}(\sigma_{i+1} \cdot \sigma_{i+2})$ ,  $1 \leq i \leq 2g$ , which is a two-sheeted covering of  $\sigma_{i+1} \cdot \sigma_{i+2}$ . Then the 1-cycles  $\gamma_1, \dots, \gamma_{2g}$  form a basis for  $H_1(X)$ . Also observe that Lemma 2.6 gives the classical result that any Riemann surface can be obtained from a hyperelliptic surface of the same genus by ‘attaching spheres’.

Given the connected manifold  $WH^{n,w}$ , parametrizing simple Weierstrass coverings of type  $(n, w)$ , it is easy to construct, analytically, when  $n > 2$ , a ‘universal family’ of Riemann surfaces parametrized by  $WH^{n,w}$  (cf. [1], (3.10)). By this we mean that there exists a pair  $(Y, h)$  of an analytic manifold  $Y$  and an analytic map

$$(2.9) \quad h : Y \rightarrow \mathbb{P}^1 \times WH^{n,w}$$

having the following properties

- (i) for each  $s \in WH^{n,w}$  the fiber  $Y_s = h^{-1}(\mathbb{P}^1 \times \{s\})$  is a Riemann surface of genus  $g = (\frac{1}{2})(w-n)$ .
- (ii) for each  $s \in WH^{n,w}$ ,  $h_s = h|_{Y_s}$  is an  $n$ -sheeted covering such that  $(h_s)_* = s$ .

The universal family (2.9) is obtained by first constructing families  $f_\alpha : Y_\alpha \rightarrow \mathbb{P}^1 \times V_\alpha$  for an open cover  $\{V_\alpha\}_{\alpha \in I}$  of  $WH^{n,w}$  (see [1], (3.1)) and then by patching together these local data.

We finish this section by observing that the Weierstrass-Hurwitz space  $WH^{n,w}$  satisfies a ‘universal property’ in the following sense. If  $V$  and  $Y$  are connected analytic manifolds and  $h : Y \rightarrow \mathbb{P}^1 \times V$  an analytic map such that, for each  $s \in V$ ,  $h_s : Y_s \rightarrow \mathbb{P}^1 \times \{s\}$  is a simple Weierstrass covering of type  $(n, w)$ , then there exists an analytic map  $\varphi : V \rightarrow WH^{n,w}$  such that  $\varphi(s) = (h_s)$ , for each  $s \in V$ . For a proof of this fact we refer the reader to ([5], (1.7)) where the word ‘simple covering’ should be replaced by ‘simple Weierstrass covering’. In exactly the same way one can prove that  $WH^{n,w}$  is an algebraic variety ([5], (6.4), (7.3)).

### 3. A filtration of the moduli space

We denote by  $M_g$  the moduli space of smooth, irreducible algebraic curves of genus  $g$  defined over the complex numbers. If  $C$  is a smooth algebraic curve of genus  $g$  we let  $m(C) \in M_g$  denote the point in  $M_g$  corresponding to  $C$ . Also we let  $\bar{M}_g$  denote the Mumford-Deligne [3] compactification of  $M_g$ , i.e., the moduli space of stable curves of genus  $g$  (see Def. 3.13 below).

For an irreducible algebraic curve  $C$  and a positive divisor  $D$ , of

degree  $n$ , on  $C$ , we shall denote by  $|D|$  the *complete linear series* determined by  $D$ , and by  $\ell(D)$  the dimension of  $H^0(\mathcal{O}_C(D))$ , so that  $\dim |D| = \ell(D) - 1$ . Also we shall occasionally denote by  $g'_n(D)$  (or simply  $g'_n$ ) a linear series of dimension  $r$  contained in  $|D|$ .

Consider now, for  $n > 2$ , the Weierstrass-Hurwitz space  $WH^{n,w}$  together with the universal family

$$(3.1) \quad h : Y \rightarrow \mathbb{P}^1 \times WH^{n,w}$$

described in section 2. Let  $g = (\frac{1}{2})(w - n)$ . Then the *universal property* of  $M_g$ , (cf. [8] page 99), gives a morphism

$$(3.2) \quad m : WH^{n,w} \rightarrow M_g$$

such that, for each  $s \in WH^{n,w}$ ,  $m(s) = m(Y_s)$ .

For  $n > 2$  we set  $W_{n,g} = m(WH^{n,w})$  and we let  $W_{2,g}$  = (*hyperelliptic locus*)  $\subset M_g$ . Also we let  $\bar{W}_{n,g}$  denote the *closure* of  $W_{n,g}$  in  $\bar{M}_g$ .

DEFINITION (3.3): *The subvariety  $\bar{W}_{n,g}$  of  $M_g$  is called the Weierstrass space of type  $(n, g)$ .*

We recall that given a point  $x$  on a curve of genus  $g$ , a positive integer  $n$ ,  $n \leq 2g - 1$ , is said to be a *gap* for  $x$  if  $\ell(nx) = \ell((n - 1)x)$  and that the sequence  $n_1, n_2, \dots$  of *non-gaps* is called *Weierstrass sequence* of  $x$ . As we mentioned in the Introduction, the spaces  $\bar{W}_{n,g}$  (more precisely their preimages in the Teichmüller space) were analyzed by Rauch and Farkas. Following their language we can say that

$$(3.4) \quad \bar{W}_{n,g} \supset \{ \text{space of moduli of curves of genus } g \text{ having} \\ \text{a } W. \text{ pt. whose first non-gap is } n \}$$

The relation (3.4) is intuitively clear since, again intuitively, any  $n$ -sheeted Weierstrass covering of genus  $g$ ,  $f_0 : Y_0 \rightarrow \mathbb{P}^1$  (not necessarily simple), is the ‘*limit*’ of *simple* Weierstrass coverings. To be more precise one should show that there exists a smooth surface  $Y$  and an analytic map  $h : Y \rightarrow \mathbb{P}^1 \times V$ ,  $V = \{t \in \mathbb{C} : |t| < 1\}$ , such that for all  $t \in V - \{0\}$ ,  $(h_t) \in WH^{n,w}$  and  $h_0 = f_0$ . In the proof of Theorem 3.18 we will prove the existence of such families in a very similar setting. That proof, with only minor changes, can be used to prove (3.4).

We shall use the following result due to Rauch [9]. Let  $T^g$  be the *Teichmüller space* of genus  $g$ .

THEOREM (3.5) (Rauch): *The sublocus of  $T^g$  consisting of Riemann*

surfaces having a Weierstrass point  $x$  for which  $v$  and  $v + 1$  are respectively the first and second non-gap, is a subvariety of  $T^g$  of dimension  $2g + v - 4$ .

Consider now the universal family (3.1). For each  $s \in WH^{n,w}$  we denote by  $y(s) \in Y_s$  the point of total ramification of  $h_s$ . Also consider the finite topological covering (2.3). Let  $(X_0, X_1)$  be projective coordinates in  $\mathbb{P}^1$  and denote by  $\infty \in \mathbb{P}^1$  the point of coordinates  $(0, 1)$ . Let

$$\mathbb{P}^w_\infty - \Delta = \{(a_1, \dots, a_w) \in \mathbb{P}^w - \Delta : a_i = \infty \text{ for some } i\}.$$

We let

$$(3.6) \quad WH^{n,w}_\infty = \{s \in \Lambda^{-1}(\mathbb{P}^w_\infty - \Delta) : h_s(y(s)) = (\infty, s)\}.$$

One can easily see that  $WH^{n,w}_\infty$  is both open and closed in  $\Lambda^{-1}(\mathbb{P}^w_\infty - \Delta)$  and it is therefore a  $(w - 1)$ -dimensional (possibly disconnected) submanifold of  $WH^{n,w}$ . Moreover  $\Lambda$  restricts to a finite topological covering  $\Lambda : WH^{n,w}_\infty \rightarrow \mathbb{P}^w_\infty - \Delta$ . We shall denote by

$$(3.7) \quad h : Y \rightarrow \mathbb{P}^1 \times WH^{n,w}_\infty$$

the restriction to  $WH^{n,w}_\infty$  of the universal family (3.1), and by

$$(3.8) \quad m : WH^{n,w}_\infty \rightarrow M_g$$

the induced map.

LEMMA 3.9:  $m(WH^{n,w}_\infty) = m(WH^{n,w}) = W_{n,g}$ .

PROOF: Let  $p \in m(WH^{n,w})$ , so that  $p = m(s) = m(Y_s)$  for some  $s \in WH^{n,w}$ . Let  $\chi \in PGL(1)$  be such that  $\chi(h_s(y(s))) = \infty$ . Then  $s' = (\chi \cdot h_s) \in WH^{n,w}_\infty$  and  $m(s) = m(s')$ .

LEMMA (3.10): Let  $g \geq n > 2$ . Let  $p \in W_{n,g} = m(WH^{n,w}_\infty)$ . Let  $C$  be a smooth curve of genus  $g$  such that  $p = m(C)$ . Let  $x_1, \dots, x_N$  be the Weierstrass points of  $C$  which satisfy the following conditions

- (i)  $\ell(nx_i) \geq 2, i = 1, \dots, N$
  - (ii) there exists  $f_i \in H^0(\mathcal{O}_C(nx_i))$  such that  $(f_i) \in WH^{n,w}_\infty, i = 1, \dots, N$ .
- Let  $\ell = \max \ell(nx_i)$ . Then  $\dim m^{-1}(p) = \ell$ .

PROOF: Let us fix our attention on one of the vector spaces  $H^0(\mathcal{O}_C(nx_i))$ , and let, for simplicity,  $x_i = x$  and  $\ell(nx) = r$ . Let  $g_1, \dots, g_r$  be a basis for

$H^0(\mathcal{O}_C(nx))$  and let  $V_x = \{(\lambda_1, \dots, \lambda_r) \in \mathbb{C}^r : \sum \lambda_j g_j \in WH_\infty^{n,w}\}$ . By assumption  $V_x \neq \emptyset$  and it is easy to see that  $V_x$  is a Zariski open in  $\mathbb{C}^r$ . Also we define a family of simple Weierstrass coverings on  $C$ ,  $F : C \times V_x \rightarrow \mathbb{P}^1 \times V_x$ , by letting, for each  $\lambda = (\lambda_1, \dots, \lambda_r) \in V_x$ ,  $F_\lambda = \sum \lambda_j g_j : C \times \{\lambda\} \rightarrow \mathbb{P}^1 \times \{\lambda\}$ . Let  $V = m^{-1}(p) = m^{-1}(m(C)) \subset WH_\infty^{n,w}$ . Then by the universal property of  $WH^{n,w}$  we get an analytic map  $\varphi : V_x \rightarrow V$  defined by  $\varphi(\lambda) = (F_\lambda)$ , for each  $\lambda \in V_x$ . Since the automorphism group of  $C$  is finite it follows that the map  $\varphi$  is finite to one, so that  $\dim \varphi(V_x) = r$ . By repeating this argument for all the points  $x_i, i = 1, \dots, N$ , we conclude, with an obvious notation, that  $V \supset \cup_i \varphi_i(V_{x_i})$  and that  $\dim \varphi_i(V_{x_i}) = \ell(nx_i)$ . It now suffices to prove that  $V = \cup_i \varphi_i(V_{x_i})$ . Let  $s \in V$ . Consider  $h_s : Y_s \rightarrow \mathbb{P}^1 \times (s)$ , (3.7). Since  $s \in V$  there exists an isomorphism  $\psi_s : Y_s \rightarrow C$  so that  $h_s \cdot \psi_s^{-1} \in H^0(\mathcal{O}_C(nx_i))$  for some  $i$ . But  $(h_s \cdot \psi_s^{-1}) = (h_s) = s$  proving that  $s \in \varphi_i(V_{x_i})$  for some  $i$ .

**THEOREM (3.11):**  $\bar{W}_{n,g}$  is an irreducible subvariety of  $\bar{M}_g$  of dimension  $2g + n - 3$ .

**PROOF:** Consider the morphism  $m : WH^{n,w} \rightarrow M_g$ . Let  $\overline{WH}^{n,w}$  be a non-singular compactification of  $WH^{n,w}$ . It follows then from a theorem of Borel (cf. [2]: Theorem A and 3.10) that  $m$  extends to a morphism  $m : \overline{WH}^{n,w} \rightarrow \bar{M}_g$ , so that  $\bar{W}_{n,g} = m(\overline{WH}^{n,w})$ . The irreducibility of  $\bar{W}_{n,g}$  now follows from Theorem 2.5.

As is well known  $\dim \bar{W}_{2,g} = 2g - 1$ . We shall prove the theorem by induction. Suppose the theorem true for  $v \leq n - 1$ . Consider the map (3.8) and let  $p$  be a generic point in  $\bar{W}_{n,g}$ . We have

$$\dim \bar{W}_{n,g} = \dim WH_\infty^{n,w} - \dim m^{-1}(p).$$

Since  $\dim WH_\infty^{n,w} = w - 1$ , (2.2) gives  $\dim \bar{W}_{n,g} = 2g + n - 1 - \ell$ , where  $\ell = \dim m^{-1}(p)$ . Let  $C$  be a smooth curve of genus  $g$  such that  $m(C) = p$ . It follows from Lemma 3.10 that there exists a Weierstrass point  $x \in C$  with  $\ell(nx) = \ell$ . In view of the same lemma it now suffices to prove that  $\ell = 2$ . Suppose, on the contrary, that  $\ell > 2$ . Let  $v$  be the smallest positive integer (first non gap) such that  $\ell(vx) = 2$ . We then have  $v \leq n - (\ell - 2)$ . Since  $v$  is the first non gap for  $x \in C$  and since  $m(C)$  is a generic point in  $\bar{W}_{n,g}$  it follows from (3.4) that  $\bar{W}_{n,g} \subset \bar{W}_{v,g}$ . The induction hypothesis gives  $\dim \bar{W}_{v,g} = 2g + v - 3$ , and therefore

$$2g + n - 1 - \ell = \dim \bar{W}_{n,g} \leq 2g + v - 3.$$

This inequality together with the preceding one gives

$$(3.12) \quad 2g + n - 1 - \ell = 2g + v - 3.$$

But  $\bar{W}_{n,g}$  and  $\bar{W}_{v,g}$  are both *irreducible* so that  $\bar{W}_{n,g} = \bar{W}_{v,g}$ . Also (3.12) gives  $v = n - (\ell - 2)$  which in turn implies  $\ell((v + 1)x) = 3$ . Therefore  $v$  and  $v + 1$  are both non gaps for  $x$ . The conclusion is that there exists a  $(2g + v - 3)$ -dimensional subvariety of  $M_g$  consisting of birational classes of curves having a Weierstrass point for which  $v$  and  $v + 1$  are, respectively, the first and the second non gap. But this contradicts Theorem 3.5.

Before proving our next result we recall the following, (cf. [3] page 76).

DEFINITION (3.13): *Let  $V$  be a variety. Let  $g \geq 2$ . A stable curve of genus  $g$  over  $V$  is a proper flat morphism  $\pi : Y \rightarrow V$  such that the fibers  $Y_s$  of  $\pi$ ,  $s \in V$ , are reduced connected curves such that*

- (i)  $Y_s$  has only ordinary double points.
- (ii) no non-singular rational component of  $Y_s$  meets other components at less than 3 points.
- (iii)  $\dim H^1(\mathcal{O}_{Y_s}) = g$ .

We will use the following result (cf. [7] Proposition 1).

THEOREM 3.14: *Let  $S_1 \subset \bar{M}_g$  be an irreducible curve. Then there exists a non-singular curve  $S$ , a finite morphism  $p : S \rightarrow S_1$ , and a stable curve of genus  $g$  over  $S$ ,  $\pi : Y \rightarrow S$ , such that*

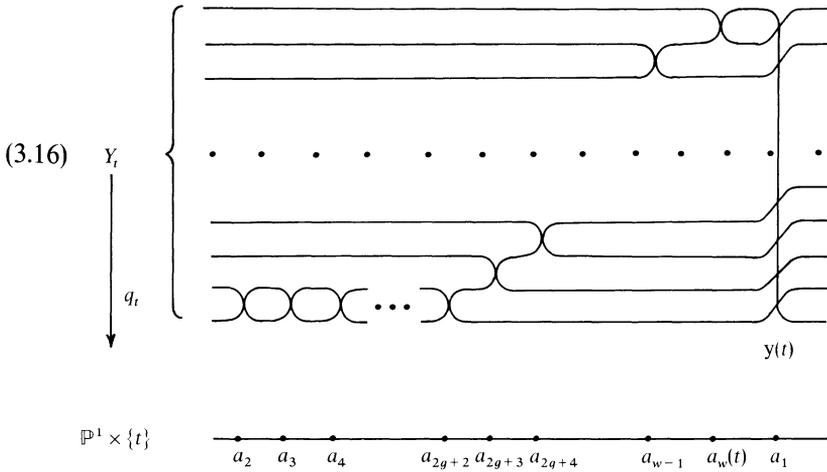
- (i)  $Y$  is a non-singular surface.
- (ii) for all  $s \in S$ ,  $p(s) = m(\pi^{-1}(s))$ .

We shall now prove

$$(3.15) \quad (\text{Hyperelliptic locus}) = \bar{W}_{2,g} \subset \bar{W}_{3,g} \subset \dots \subset \bar{W}_{g-1,g} \subset \bar{W}_{g,g} = \bar{M}_g.$$

We obtain in this way a *filtration* of  $\bar{M}_g$  by means of *irreducible* subvarieties such that  $\dim \bar{W}_{n,g} = \dim \bar{W}_{n-1,g} + 1$ .

In order to prove (3.15) we will show that any simple Weierstrass covering of type  $(n - 1, w - 1)$  can be thought of as a ‘*limit*’ of simple Weierstrass coverings of type  $(n, w)$ . We will first construct a *stable curve*  $\pi : Y \rightarrow V$ ,  $V = \{t \in \mathbb{C} : |t| < 1\}$  and then a family of *simple Weierstrass coverings*  $q : Y - \pi^{-1}(0) \rightarrow \mathbb{P}^1 \times V - \{0\}$  which can be represented, by the following picture (cf. (2.7)).



Here  $q_t : Y_t \rightarrow \mathbb{P}^1 \times \{t\}$  is a simple Weierstrass covering of type  $(n, w)$  varying analytically with  $t$  and such that  $\Lambda(q_t) = (a_1, \dots, a_{w-1}, a_w(t))$ . As  $t$  tends to 0,  $a_w(t)$  tends to  $a_1$ , and, as this picture suggests, the curve  $Y_t$  tends to a (non stable) curve  $\tilde{Y}_0 = \mathbb{P}^1 \cup Y'_0$ , while the covering  $q_t$  tends to a map  $q_0 : \tilde{Y}_0 \rightarrow \mathbb{P}^1$  such that  $q_0 : Y'_0 \rightarrow \mathbb{P}^1$  is a simple Weierstrass covering of type  $(n-1, w-1)$ . This phenomenon should be interpreted in the following way. If we denote by  $y(t) \in Y_t, t \in V$ , the varying Weierstrass point, then, when  $t = 0$ , the linear series  $|ny(t)|$  acquires a fixed point at  $y(0)$  and the curve  $\tilde{Y}_0$  is obtained by blowing up  $Y_0$  at  $y(0)$ .

**REMARK (3.17):** Let  $S$  be a reduced irreducible curve. Let  $\pi : Y \rightarrow S$  be a stable curve of genus  $g$  over  $S$ . Let  $\Gamma$  be a reduced curve contained in  $Y$  and such that the restriction of  $\pi$  to  $\Gamma$  is an  $N$ -sheeted branched covering. It is then easy to see that there exists a finite branched covering

$$\varphi : \tilde{S} \rightarrow S$$

and a stable curve

$$\tilde{\pi} : \tilde{Y} \rightarrow \tilde{S}$$

such that

- (i) the pull back  $\tilde{\Gamma}$  of  $\Gamma$  to  $\tilde{Y}$  consists of  $N$  distinct components  $\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_N$ , such that  $\tilde{\pi} : \tilde{\Gamma}_i \rightarrow \tilde{S}$  is an isomorphism.
- (ii) for each  $\tilde{s} \in \tilde{S}$  the curves  $\tilde{\pi}^{-1}(\tilde{s})$  and  $\pi^{-1}(\varphi(\tilde{s}))$  are isomorphic.

The stable curve  $\tilde{\pi} : \tilde{Y} \rightarrow \tilde{S}$ , is said to be obtained by *unwinding*  $Y$  along  $\Gamma$ .

THEOREM (3.18):  $\bar{W}_{n-1,g} \subset \bar{W}_{n,g}$ .

PROOF: Let  $p$  be a *generic* point in  $\bar{W}_{n-1,g}$ . The Theorem will be proved by constructing a *stable curve*

$$(3.19) \quad \pi : Y \rightarrow S$$

over a non singular curve  $S$  and a neighborhood  $V$  of a point  $s_0 \in S$  such that, for all  $s \in V - \{s_0\}$ ,  $m(\pi^{-1}(s)) \in W_{n,g}$  and  $m(\pi^{-1}(s_0)) = p$ . Let

$$(3.20) \quad f : X \rightarrow \mathbb{P}^1$$

be such that  $(f) \in WH_\infty^{n-1, w-1}$  and  $m((f)) = p$  (cf. (3.8)). Let

$$A(f) = A = \{a_1, \dots, a_{w-1}\} \in \mathbb{P}_\infty^{w-1} - A,$$

with  $a_1 = \infty$ . Let  $y \in \mathbb{P}^1 - A$  and  $\sigma_1, \dots, \sigma_{w-1}$  a *standard system of generators* for  $\pi_1(\mathbb{P}^1 - A, y)$ , (cf. (1.6)), such that

$\Phi((f))(\sigma_1, \dots, \sigma_{w-1}) = ((123 \dots n-1), (12), \dots, (12), (23), \dots, (n-2 \ n-1))$ , (cf. (2.6), (2.7)). Let  $D \subset \mathbb{P}^1 - \{a_2, \dots, a_{w-1}\}$  be a small disc centered at  $a_1$ . Let  $z_1 \in D - \{a_1\}$ , and  $\sigma_w \subset \mathbb{P}^1$  a loop such that  $\sigma_1, \dots, \sigma_{w-1}, \sigma_w$  is a *standard system of generators* for  $\pi_1(\mathbb{P}^1 - A \cup \{z_1\}, y)$ . Let

$$(3.21) \quad f_1 : X_1 \rightarrow \mathbb{P}^1$$

be such that  $(f_1) \in WH_\infty^{n,w}$ ,  $A(f_1) = A \cup \{z_1\}$  and such that  $\Phi((f_1))(\sigma_1, \dots, \sigma_w)$  satisfies (2.7). Let  $R \subset \mathbb{P}_\infty^w - A$  be the curve

$$R = \{(a_1, \dots, a_{w-1}, z) \in \mathbb{P}_\infty^w - A : z \in \mathbb{P}^1 - A\},$$

and let  $\tilde{R} \subset WH_\infty^{n,w}$  be the connected component of  $A^{-1}(R)$  containing  $(f_1)$ . Let  $S_1 \subset \bar{W}_{n,g}$  be the closure of  $m(\tilde{R}) \subset W_{n,g}$ . It easily follows from the definition of  $S_1$  and from Lemma 3.10 that  $\dim S_1 = 1$  and that  $S_1 \not\subset \bar{W}_{n-1,g}$ . Let now  $p : S \rightarrow S_1$  be the *finite* morphism and

$$(3.22) \quad \pi : Y \rightarrow S$$

the *stable curve* of genus  $g$  given by Theorem 3.14. Consider the Zariski open  $S' \subset S$  defined by  $S' = p^{-1}(m(\tilde{R}) - m(\tilde{R})) \cap \bar{W}_{n-1,g}$ . We claim that

after unwinding  $Y$ , if necessary, one can assume that there exist *distinct* sections

$$(3.23) \quad y_1 = r_1, r_2, \dots, r_w : S \rightarrow Y$$

such that for each  $s \in S'$

- (i)  $\ell(ny_1(s)) = \hat{2}$
- (ii) there exists  $z \in \mathbb{P}^1 - A$  and (a unique)  $f_s \in H^0(\mathcal{O}_{Y_s}(ny_1(s)))$  with  $A(f_s) = (a_1, \dots, a_w, z)$
- (iii)  $r_1(s), \dots, r_w(s)$  are the ramification points of the  $g_n^1(ny_1(s))$ .

It follows in fact from the definition of  $S'$  that the set

$$\Gamma = (\text{closure in } Y \text{ of } \{y \in Y : y \text{ satisfies (i) and (ii)}\})$$

defines a curve in  $Y$ . By unwinding  $Y$  along  $\Gamma$  (cf. Remark 3.17) we may assume that there exists a section  $y_1 : S \rightarrow Y$  satisfying (i) and (ii). It is then easy to check that, by further unwinding  $Y$ , one may assume the existence of sections  $r_i$ ,  $1 \leq i \leq w$ , satisfying (i), (ii), and (iii).

We then define an analytic map

$$(3.24) \quad q : \pi^{-1}(S') \rightarrow \mathbb{P}^1 \times S'$$

by setting  $q_s = f_s$  for each  $s \in S'$ . From the universal property of  $WH^{n,w}$  we get a surjective analytic map  $\varphi : S' \rightarrow \tilde{R}$  such that, for each  $s \in S'$ ,  $\varphi(s) = (q_s)$ . It follows from our construction that there is an open set  $V \subset S$ , homeomorphic to the unit disc, and a point  $s_0 \in V$ , such that

- (a)  $V - \{s_0\} \subset S'$ , i.e.,  $m(\pi^{-1}(s)) \in W_{n,q}$  for each  $s \in V - \{s_0\}$ .
- (b)  $(f_1) \in \varphi(V - \{s_0\})$ , (cf. (3.21)).
- (c)  $A \cdot \varphi(V - \{s_0\}) = \{(a_1, \dots, a_{w-1}, z) : z \in D\}$ .

We shall prove that  $m(Y_{s_0}) = p$ . We first show that  $Y_{s_0}$  is *non singular*. It follows in fact from (a), (b), (c) and Lemma 2.6 that, for each  $s \in V - \{s_0\}$ , there exists a *standard system of generators*  $\sigma_1(s), \dots, \sigma_w(s)$  for  $\pi_1(\mathbb{P}^1 - A(q_s), y)$  such that  $\sigma_i(s) = \sigma_i$  for  $2 \leq i \leq w-1$ , and such that

$$\begin{aligned} \Phi((q_s)(\sigma_1(s), \sigma_2, \dots, \sigma_{2g+2}, \sigma_{2g+3}, \dots, \sigma_{w-1}, \sigma_w(s))) \\ = ((123 \cdots n), (12), \dots, (12), (23), \dots, (n-2 \ n-1), (n-1, n)). \end{aligned}$$

Let now  $\gamma_i(s)$ ,  $1 \leq i \leq 2g$  be the only connected component of  $q_s^{-1}(\sigma_{i+1} \cdot \sigma_{i+2})$  which is a two-sheeted covering of  $\sigma_{i+1} \cdot \sigma_{i+2}$ . The 1-cycles  $\gamma_1(s), \dots, \gamma_{2g}(s)$  form a basis of  $H_1(Y_s)$  which varies continuously

with  $s$  (cf. Remark 2.8). Therefore for  $s \in V - \{s_0\}$  the *monodromy map*  $M : H_1(Y_s) \rightarrow H_1(Y_s)$ , (cf. [10], VI.6), is the identity map. This fact together with the stability of  $Y_{s_0}$  implies that  $Y_{s_0}$  is a non singular curve of genus  $g$ .

We must now show that  $m(Y_{s_0}) = p$ . Consider the map (3.24) and let  $pr : \mathbb{P}^1 \times S' \rightarrow \mathbb{P}^1$  be the projection. Let  $C_t = \{\text{closure in } Y \text{ of } (pr \cdot q)^{-1}(t)\}$ , for each  $t \in \mathbb{P}^1$ . Since for  $s \in S'$  the algebraic system  $\{C_t\}_{t \in \mathbb{P}^1}$  cuts out on  $Y_s$  the linear series  $g_n^1(ny_1(s))$ , it follows ([10], V.1) that the algebraic system  $\{C_t\}_{t \in \mathbb{P}^1}$  is *linear*. Therefore  $\{C_t\}_{t \in \mathbb{P}^1}$  cuts out on  $Y_{s_0}$  a linear series of *degree*  $n$  which we denote by  $g_n^1$ . We claim that the  $g_n^1$  has *one fixed point*. Suppose not, then one can extend  $q : \pi^{-1}(V - \{s_0\}) \rightarrow \mathbb{P}^1 \times (V - \{s_0\})$  to an analytic map  $q : \pi^{-1}(V) \rightarrow \mathbb{P}^1 \times V$ . Since for each  $s \in V - \{s_0\}$ ,  $q_s(r_i(s)) = a_i \neq a_j$ , for  $1 \leq i < j \leq w-1$ , (cf. (iii)), it follows that  $r_i(s_0) \neq r_j(s_0)$ ,  $1 \leq i < j \leq w-1$ . From this and from the fact that  $r_1(s_0) = r_w(s_0)$  it is easy to show that

$$\delta(q_{s_0}) = na_1 + \sum_{i=1}^{w-1} a_i.$$

So that *degree*  $(\delta(q_{s_0})) = n + w - 1$  contradicting the Hurwitz formula (1.2).

Notice that since  $ny_1(s_0) \in g_n^1$ , the point  $y_1(s_0)$  is the only fixed point of the  $g_n^1$ . Also observe that  $q$  can be extended to  $\pi^{-1}(V) - \{y_1(s_0)\}$ . From these two facts it easily follows that  $r_i(s_0) \neq y_1(s_0)$  for  $2 \leq i \leq w-1$ . Therefore the  $g_n^1$  gives rise to a  $v$ -sheeted simple Weierstrass covering  $q_0 : Y_{s_0} \rightarrow \mathbb{P}^1$  such that  $\delta(q_0) = va_1 + \sum_{i=1}^{w-1} a_i$ . Since  $Y_{s_0}$  is a curve of *genus*  $g$  it follows from (1.2) that  $v = n - 1$ . It now follows from our construction that  $Y_{s_0} \cong X$  (cf. 3.20), and the theorem is proved.

REMARK (3.25): By using an argument which is essentially due to Severi, it can be shown that for  $2 \leq v \leq 5$  and  $g \geq 2$ ,  $\bar{W}_{v,g}$  is *unirational* ([1], (4.63)).

We now proceed to analyze *degenerating families of Weierstrass coverings*. Before proving our next result we recall the following.

DEFINITION (3.26): Let  $V$  be an algebraic variety. An effective divisor  $D \subset V$  is said to be *pseudo-ample* in  $V$  if, for every algebraic curve  $S \subset V$ ,  $D \cap S \neq \emptyset$ .

We will denote by  $\partial \bar{M}_g$  the divisor  $(\bar{M}_g - M_g)$  in  $\bar{M}_g$ . Points in  $\partial \bar{M}_g$  correspond, essentially, to curves of genus less than  $g$  with ‘marked points’. It is well known ([6]) that  $\partial \bar{M}_g$  is not pseudo-ample in  $\bar{M}_g$ . We shall prove that  $\bar{W}_{g-1,g} \cup \partial \bar{M}_g$  is pseudo-ample in  $\bar{M}_g$ . More generally

we shall study the problem of degenerating families of curves in each level of the filtration (3.15).

For  $2 \leq n \leq g$  we set  $\partial \bar{W}_{n,g} = \bar{W}_{n,g} \cap \partial \bar{M}_g$ .

**THEOREM (3.27):**  $\bar{W}_{n-1,g} \cup \partial \bar{W}_{n,g}$  is pseudo-ample in  $\bar{W}_{n,g}$ .

**PROOF:** Suppose that the statement of the Theorem is false. Then there exists an irreducible complete curve  $S_1 \subset \bar{W}_{n,g}$  such that  $S_1 \cap (\bar{W}_{n-1,g} \cup \partial \bar{W}_{n,g}) = \emptyset$ . Let  $p : S \rightarrow S_1$  be the finite morphism and  $\pi : Y \rightarrow S$  be the stable curve of genus  $g$  given by Theorem 3.14. It follows from our assumption on  $S_1$  that, for each  $s \in S$ ,  $Y_s$  is a non singular curve of genus  $g$  possessing a Weierstrass point  $y_s$  such that  $\ell(ny_s) \geq 2$ . Since  $m(Y_s) \not\subset \bar{W}_{n-1,g}$  we actually have  $\ell(ny_s) = 2$ . By proceeding as in the proof of Theorem 3.18, we can now assume, after unwinding  $Y$ , if necessary, that there exist  $w$  sections  $y_1 = r_1, \dots, r_w : S \rightarrow Y$  (not necessarily distinct) such that for each  $s \in S$ ,  $y_1(s)$  is a Weierstrass point with  $\ell(ny_1(s)) = 2$  and such that  $r_1(s), \dots, r_w(s)$  are the ramification points of the  $g_n^1(y_1(s))$ . We now define a Zariski open  $S' \subset S$  by setting  $S' = \{s \in S : r_i(s) \neq r_j(s) \text{ for some fixed } i \text{ and } j, i \neq j, i \neq 1, j \neq 1\}$ . We may as well assume that  $i = 2$  and  $j = 3$ . Let  $f_s$  be the unique element of  $H^0(\mathcal{O}_{Y_s}(ny_1(s)))$  such that  $f_s(r_2(s)) = 0, f_s(r_3(s)) = 1, f_s(r_1(s)) = \infty, s \in S'$ . We then define an analytic map  $q : \pi^{-1}(S') \rightarrow \mathbb{P}^1 \times S'$  by setting  $q_s = f_s$  for each  $s \in S'$ .

Let  $pr : \mathbb{P}^1 \times S' \rightarrow \mathbb{P}^1$  be the projection, and let  $C_t = (\text{closure in } Y \text{ of } (pr \cdot q)^{-1}(t))$  for each  $t \in \mathbb{P}^1$ . As in the proof of Theorem 3.18 one can see that the algebraic system  $\{C_t\}_{t \in \mathbb{P}^1}$  is linear. We now show that  $\{C_t\}_{t \in \mathbb{P}^1}$  has no base point. Suppose in fact that  $y_0 \in Y$  is a base point for  $\{C_t\}_{t \in \mathbb{P}^1}$ . Let  $s_0 = \pi(y_0)$  and denote by  $g_n^1$  the linear series of degree  $n$  cut out on  $Y_{s_0}$  by  $\{C_t\}_{t \in \mathbb{P}^1}$ . Then  $y_0$  is a fixed point for the  $g_n^1$ . Since  $ny_1(s_0) \in g_n^1$  we have  $y_0 = y_1(s_0)$ . It follows that  $m(Y_{s_0}) \subset \bar{W}_{v,g}$  for some  $v < n$ . This together with Theorem 3.18 contradicts our assumption on  $S_1$ . Therefore  $\{C_t\}_{t \in \mathbb{P}^1}$  has no base point. We then get an extension of  $q$  to a regular map  $q : Y \rightarrow \mathbb{P}^1 \times S$ , by setting  $q(y) = (t, \pi(y))$ , where  $C_t$  is the unique curve of  $\{C_t\}_{t \in \mathbb{P}^1}$  passing through  $y$ . Consider now the rational functions  $g_i : S \rightarrow \mathbb{P}^1$  defined by  $g_i = pr \cdot q \cdot r_i, 1 \leq i \leq w$ . Since  $\Lambda(q_s) = (g_1(s), \dots, g_w(s))$  and since the map  $m : S \rightarrow M_g$  is non constant, it follows that not all the  $g_i$ 's are constant. Therefore there exist  $s_0 \in S$  such that  $g_i(s_0) = g_1(s_0)$  for some  $i \neq 1$ . This in turn implies that  $r_i(s_0) = y_1(s_0)$ . It is now an easy matter to check that  $y_1(s_0)$  is a base point for  $\{C_t\}_{t \in \mathbb{P}^1}$ . But we just proved that this cannot be the case. This contradiction shows that our assumption on  $S_1$  is absurd and the Theorem is proved.

It is now natural to ask whether the irreducible divisor  $\bar{W}_{n-1,g}$  is

*pseudo-ample* in  $\bar{W}_{n,g}$ . We will see that, in general, this is not the case.

The following notation will be used. Let  $C$  (resp.  $C'$ ) be an *irreducible stable curve* of genus  $g$  such that the only singularity of  $C$  (resp.  $C'$ ) is a *double point*  $x \in C$  (resp.  $x' \in C'$ ). We shall denote by  $\tilde{C}$  (resp.  $\tilde{C}'$ ) the *normalization* of  $C$  (resp.  $C'$ ) and by  $x_1, x_2$  (resp.  $x'_1, x'_2$ ) the preimages in  $\tilde{C}$  (resp.  $\tilde{C}'$ ) of the double point  $x \in C$  (resp.  $x' \in C'$ ). We shall denote by  $(C)^{(2)}$  (resp.  $(C')^{(2)}$ ) the *second symmetric product* of  $\tilde{C}$  (resp.  $\tilde{C}'$ ). If  $\varphi : C \rightarrow C'$  is an isomorphism we shall denote by  $\varphi^{(2)} : (C)^{(2)} \rightarrow (C')^{(2)}$  the isomorphism induced by  $\varphi$ . We also recall that the definition of  $\bar{M}_g$ , [3], implies that  $m(C) = m(C') \in \partial \bar{M}_g$  if and only if there exists an isomorphism  $\varphi : \tilde{C} \rightarrow \tilde{C}'$  such that  $\tilde{\varphi}^{(2)}((x_1, x_2)) = (x'_1, x'_2)$ .

**THEOREM (3.28):**  $\bar{W}_{2,3}$  is not pseudo-ample in  $\bar{W}_{3,3} = \bar{M}_3$ .

**PROOF:** Observe, first of all, that for  $g = 3$  we have  $\bar{W}_{2,3} =$  (*hyperelliptic locus*). We shall construct a curve  $S \subset \bar{W}_{3,3}$  such that  $S \cap \bar{W}_{2,3} = \emptyset$ . For this purpose let

$$(3.29) \quad f : Y \rightarrow \mathbb{P}^1$$

be a *pencil of stable curves* of genus 3 such that for all  $t \in \mathbb{P}^1 - \{t_1, \dots, t_N\}$ ,  $f^{-1}(t) = Y_t$  is a non singular plane quartic and such that  $Y_{t_i}, i = 1, \dots, N$  is an irreducible plane quartic with one *simple node* and no other singularities. Such a '*Lefschetz pencil*' certainly exists. In fact it is not difficult to check that, in the projective space  $\mathbb{P}^{14}$ , whose points correspond to plane quartics, the subvariety of points corresponding to plane quartics with a non-ordinary double point is of codimension two. The pencil (3.29) induces a morphism  $m : \mathbb{P}^1 \rightarrow \bar{M}_g$ . Let  $S = m(\mathbb{P}^1)$ . We shall prove that  $S \cap \bar{W}_{2,3} = \emptyset$ . Notice that for  $t \in \mathbb{P}^1 - \{t_1, \dots, t_N\}$  the plane quartic  $Y_t$  is a non singular *canonical curve* of genus 3. Thus  $Y_t$  is *non-hyperelliptic*. Therefore we only have to prove that for  $i = 1, \dots, N$

$$(3.30) \quad m(Y_{t_i}) \notin \partial \bar{W}_{2,3}.$$

Let us fix a  $t_i \in \{t_1, \dots, t_N\}$ . Set  $C = Y_{t_i}$ . Let  $\tilde{C}$  be a normalization of  $C$  and let  $x_1, x_2$  be the preimages in  $\tilde{C}$  of the double point  $x_0 \in C$ . Let  $\iota : C \rightarrow C$  be the *hyperelliptic involution* on  $C$ . Denote by  $\tilde{\iota}$  the lifting of  $\iota$  to  $\tilde{C}$ . The involution  $\iota$ , on  $C$ , is given by:  $\iota(x) =$  (residual intersection between a line, through  $x$  and  $x_0$ , and  $C$ ). It easily follows from this that

$$(3.31) \quad \tilde{\iota}^{(2)}((x_1, x_2)) \neq (x_1, x_2).$$

Suppose now that (3.30) is false, and let us then assume that  $m(Y'_i) = m(C) \in \partial \bar{W}_{2,3}$ . We now construct a stable curve of genus 3

$$(3.32) \quad g : Y' \rightarrow V = \{t \in \mathbb{C} : |t| < 1\}$$

such that for all  $t \in V - \{0\}$ ,  $m(Y'_i) \in \bar{W}_{2,3}$  and  $m(Y'_0) = m(C)$ . Let  $h : C \rightarrow \mathbb{P}^1$  be the two-sheeted covering induced by the involution  $\iota$ . Let  $e_i$ ,  $i = 1, \dots, 6$  be the affine coordinates of the six branch points of  $h$ . Let  $g_1 : Z \rightarrow V$  be the family of hyperelliptic curves defined by letting, for  $t \in V$ ,  $Z_t = g_1^{-1}(t)$  be the plane projective curve of equation

$$X_2^2 = X_1(X_1 - X_0 t) \prod_{i=1}^6 (X_1 - e_i X_0).$$

We then obtain a stable curve (3.32) by blowing up  $Z_t$  at its 'bad' singularity (point of coordinates  $(0, 1, 0)$ ). Let  $\kappa_1 : Z_0 \rightarrow Z_0$  be the hyperelliptic involution. It is then easy to see that if  $x \in Z_0 \subset \mathbb{P}^2$  has coordinates  $(X_0, X_1, X_2)$  then

$$(3.33) \quad \kappa_1(x) \equiv (X_0, X_1, -X_2).$$

Let now  $x_0$  be the double point of  $Y'_0$  and let  $x'_1, x'_2$  be the preimages of  $x_0$  in a normalization  $\tilde{Y}'_0$  of  $Y'$ . Let  $\tilde{\kappa}$  be the hyperelliptic involution on  $\tilde{Y}'_0$ . Now the way in which we constructed  $Y'_0$  starting from  $Z_0$ , the fact that  $Z_0$  has its double point at the point of coordinates  $(1, 0, 0)$  and (3.33) easily imply that

$$\tilde{\kappa}^{(2)}((x'_1, x'_2)) = (x'_1, x'_2).$$

But this together with (3.31) and the fact that  $m(Y'_0) = m(C) \in \partial \bar{M}_3$  gives a contradiction. It follows that  $m(C) \notin \partial \bar{W}_{2,3}$  and the Theorem is proved.

#### BIBLIOGRAPHY

- [1] ARBARELLO, E. *Thesis*, (Chapters II and III), Columbia University (1973).
- [2] BOREL, A. Some metric properties of arithmetic quotients of Symmetric Spaces and an extension theorem. *Jour. of Diff. Geom. Vol 6, No. 4* (1972) 543–560.
- [3] DELIGNE, P. and MUMFORD, D. The irreducibility of the space of curves of given genus. *Publ. I.H.E.S.* 36 (1970) 75–109.
- [4] FARKAS, H. M. Special divisors and analytic subloci of Teichmüller space. *Amer. Jour. Math.* 88 (1966) 881–901.
- [5] FULTON, W. Hurwitz schemes and irreducibility of moduli of algebraic curves. *Annals of Math.* 90 (1969) 542–575.
- [6] KAS, A. On deformation of a certain type of irregular algebraic surface. *Amer. J. Math.* 90 (1968) 789–804.

- [7] MAYER, A. L. Compactification of the variety of moduli of curves. Seminar on degeneration of algebraic varieties (conducted by P. A. Griffiths). *Notes of the Institute for Advanced Study, Princeton* (1969) 6–15.
- [8] MUMFORD, D. *Geometric Invariant Theory*. Berlin–Heidelberg–New York, Springer, 1965.
- [9] RAUCH, H. E. Weierstrass points, branch points, and the moduli of Riemann surfaces. *Communications on Pure and Applied Mathematics*, 12 (1959) 543–560.
- [10] ZARISKI, O. *Algebraic Surfaces* (second supplemented edition). Springer-Verlag, 1971.

(Oblatum 7-VI-1974 & 15-X-1974)

Harvard University,  
Department of Math.  
Cambridge Mass.  
02138 USA