

# COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 30, n° 2 (1975), p. 169-180

[http://www.numdam.org/item?id=CM\\_1975\\_\\_30\\_2\\_169\\_0](http://www.numdam.org/item?id=CM_1975__30_2_169_0)

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## NON-ARCHIMEDEAN INVARIANT MEANS

W. H. Schikhof

### Introduction

Let  $K$  be any complete valued field and let  $G$  be a locally compact group. The  $K$ -vector space  $BC(G \rightarrow K)$  consisting of all  $K$ -valued bounded continuous functions on  $G$  is a Banach space under the norm  $f \mapsto \|f\| = \sup \{|f(x)| : x \in G\}$ . A left invariant mean (l.i.m.) is a  $K$ -linear function  $M : BC(G \rightarrow K) \rightarrow K$  satisfying

- (1)  $M(1) = 1$
- (2)  $\|M\| \leq 1$  (i.e.,  $|M(f)| \leq \|f\|$ ) for all  $f \in BC(G \rightarrow K)$ )
- (3)  $M(f_s) = M(f)$  for all  $f \in BC(G \rightarrow K)$  and  $s \in G$ .

(Here the symbol 1 is used for the constant function one, for the unit element of  $K$ , and also for the real number 1;  $f_s$  is defined by  $f_s(x) = f(sx)$  for  $x \in G$ ).  $G$  is called  $K$ -amenable if there exists a l.i.m. on  $BC(G \rightarrow K)$ .

It is well known that  $IR$ -amenability in the above sense is the same as ‘amenability’ as it occurs in the literature: for  $K = IR$  the properties (1), (2), (3) are equivalent to (1), (3), and positivity of  $M$ . (For general  $K$  we cannot use a positivity condition in the definition of a l.i.m., since an ordering is not always available in  $K$ ). It is also easy to see that  $IR$ -amenability is equivalent to  $\mathbb{C}$ -amenability. So in order to get something new we must have that  $K$  is not isomorphic to either  $IR$  or  $\mathbb{C}$ , which implies that the valuation on  $K$  is non-archimedean (i.e.,  $|x+y| \leq \max(|x|, |y|)$  for all  $x, y \in K$ ). (See [2], 1.2). It turns out that the only interesting groups to consider are 0-dimensional.

As a first example, let  $G = \mathbb{Z}$  (with discrete topology). The function  $f : \mathbb{Z} \rightarrow K$  defined by  $f(n) = n$  is bounded(!), hence in  $BC(\mathbb{Z} \rightarrow K)$ . If  $M$  were a l.i.m. on  $BC(\mathbb{Z} \rightarrow K)$  then  $1 = M(1) = M(f_1) - M(f) = 0$ . So  $\mathbb{Z}$  is not  $K$ -amenable. Another typical non-archimedean feature is presented by the case  $G = C_p$  (group of  $p$  elements) and  $K = \mathbb{Q}_p$ . If  $f$  is the characteristic function of an element of  $C_p$ , and  $M$  is a l.i.m. on  $BC(C_p \rightarrow \mathbb{Q}_p)$  then  $M(f) = 1/p$ , and  $|M(f)| = |1/p| > 1$ , which contradicts

(2). The reason why it goes wrong is different for both cases:  $\mathbb{Z}$  is not 'torsional' and  $C_p$  is not ' $p$ -free' (3.2 and 1.3).

It is a rather surprising fact that one can find necessary and sufficient conditions (formulated in terms of properties of  $G$  and its topology) for  $K$ -amenability. (Theorems 2.1 and 3.6).

In 'classical' analysis one often uses the fact that an  $f \in BC(G \rightarrow \mathbb{R})$  has precompact image rather than its boundedness. This leads to another non-archimedean candidate for function space namely  $PC(G \rightarrow K)$ , the space of all  $f \in BC(G \rightarrow K)$  such that  $f(G)$  is precompact.  $G$  is called weakly  $K$ -amenable if there is a l.i.m. on  $PC(G \rightarrow K)$ . Corollary 5.3 gives necessary and sufficient conditions for weak  $K$ -amenability in case the characteristic of the residue class field of  $K$  is non-zero.

In [5] A. C. M. van Rooij studies  $K$ -amenability for discrete abelian semigroups. Further, he proves (Theorem 7.1) that there exists a l.i.m. on  $UC(G \rightarrow K)$  iff  $G$  is  $p$ -free. (Here  $G$  is an abelian zerodimensional torsional (see [5], 7) group, not necessarily locally compact;  $K$  is spherically complete;  $UC(G \rightarrow K)$  is the space of the bounded uniformly continuous functions:  $G \rightarrow K$ ). The intersection of the theory of [5] and the results of this paper ( $G$  abelian, locally compact, torsional) is rather trivial.

*Note:* for detailed information on facts of non-archimedean analysis needed here (for instance Ingleton's theorem: the non-archimedean form of the Hahn-Banach theorem) we refer to [2] and [4]. We use the symbols  $\mathbb{Q}_p$ ,  $\mathbb{F}_p$ ,  $\mathbb{Q}$ . They stand for the field of the  $p$ -adic numbers, the field with  $p$  elements, and the field of the rationals, respectively.

### 1. Non-archimedean amenability

For a topological group  $G$  and a non-archimedean complete valued field  $K$  (trivial valuation included) we define  $BC(G \rightarrow K)$  to be the  $K$ -vector space of all bounded continuous functions  $f : G \rightarrow K$ , normed via  $f \mapsto \|f\| = \sup \{|f(x)| : x \in G\}$ . For  $f \in BC(G \rightarrow K)$  and  $s \in G$  we put  $f_s(x) = f(sx)$ . Then  $f_s \in BC(G \rightarrow K)$ . The ( $K$ -valued) characteristic function of a clopen (= closed and open) subset  $U$  of  $G$  is in  $BC(G \rightarrow K)$  and we denote it by  $\xi_U$ . Many times we write 1 instead of  $\xi_G$ . (The symbol 1 will also be used for the unit element of  $K$  and for the unit element of  $\mathbb{R}$ ). The characteristic of a field  $L$  is denoted by  $\chi(L)$ .

1.1 DEFINITION: A left invariant mean (l.i.m.) on  $BC(G \rightarrow K)$  is a  $K$ -linear function  $M : BC(G \rightarrow K) \rightarrow K$  satisfying

- (1)  $M(1) = 1$
- (2)  $|M(f)| \leq \|f\|$  for all  $f \in BC(G \rightarrow K)$

(3)  $M(f_s) = M(f)$  for all  $f \in BC(G \rightarrow K)$  and  $s \in G$ .

$G$  is called  $K$ -amenable if there exists a l.i.m. on  $BC(G \rightarrow K)$ .

We shall be concerned only with locally compact groups  $G$ . Since  $K$  is totally disconnected there is a natural isomorphism

$$BC(G \rightarrow K) \rightarrow BC(G/C \rightarrow K),$$

where  $C$  is the connected component of the group identity.  $G/C$  is a totally disconnected locally compact group, hence 0-dimensional ([1], 3.5): when studying amenability of locally compact groups we may restrict ourselves to locally compact 0-dimensional groups  $G$ . Note that such groups have small open subgroups ([1], 7.7). (every neighborhood of the identity contains an open (compact) subgroup).

FROM NOW ON  $G$  IS A LOCALLY COMPACT 0-DIMENSIONAL TOPOLOGICAL GROUP,  $K$  IS A NON-ARCHIMEDEAN COMPLETE VALUED FIELD, WHOSE RESIDUE CLASS FIELD IS DENOTED BY  $k$ .

1.2 LEMMA: Let  $G$  be  $K$ -amenable. Then

- (i) Every open subgroup of  $G$  is  $K$ -amenable.
- (ii) For a closed normal subgroup  $S$ ,  $G/S$  is  $K$ -amenable.

PROOF: (i) Let  $S$  be an open subgroup. For each right coset  $Sx$ , choose an element  $\tilde{x} \in Sx$ . The map  $\sigma : x \mapsto x\tilde{x}^{-1}$  is a surjection of  $G$  onto  $S$  and  $\sigma(sx) = s\sigma(x)$  for all  $s \in S, x \in G$ . If  $M$  is a l.i.m. on  $BC(G \rightarrow K)$ , define  $N(f) = M(f \circ \sigma)(f \in BC(S \rightarrow K))$ . This  $N$  is a l.i.m. on  $BC(S \rightarrow K)$ , which can be verified easily.

(ii) Let  $\pi : G \rightarrow G/S$  be the canonical homomorphism and let  $M$  be a l.i.m. on  $BC(G \rightarrow K)$ . Define  $N(f) = M(f \circ \pi)(f \in BC(G/S \rightarrow K))$ . This  $N$  is a l.i.m. on  $BC(G/S \rightarrow K)$ .

1.3 DEFINITION: Let  $p$  be a prime number. We call  $G$   $p$ -free if for every pair of open subgroups  $S_1 \supset S_2$  the number  $[S_1 : S_2]$  (whenever finite) is not divisible by  $p$ . By definition, every  $G$  is 0-free.

1.4 THEOREM: Let  $G$  be compact. Then  $G$  is  $K$ -amenable if and only if  $G$  is  $\chi(k)$ -free, and a l.i.m. on  $BC(G \rightarrow K)$  is unique.

PROOF: Let  $G$  be  $K$ -amenable, and let  $S_1 \supset S_2$  be open subgroups. Then by Lemma 1.2.(i),  $S_1$  is  $K$ -amenable, let  $M$  be a l.i.m. on  $BC(S_1 \rightarrow K)$ .

By invariance,  $M(\xi_{S_2}) = [S_1 : S_2]^{-1}$ , so

$$|[S_1 : S_2]|^{-1} = |M(\xi_{S_2})| \leq \|\xi_{S_2}\| = 1.$$

Hence  $|[S_1 : S_2]| = 1$  so  $[S_1 : S_2]$  is not divisible by  $\chi(k)$  (in case  $\chi(k) \neq 0$ ). Conversely, if  $G$  is  $\chi(k)$ -free, by [3], 2.2.7 there exists a  $K$ -valued left Haar integral  $m$  on  $BC(G \rightarrow K)$ , for which  $\|m\| = 1$ . Then  $M = m(\xi_G)^{-1}$ .  $m$  is a l.i.m. on  $BC(G \rightarrow K)$ , which is unique because of [3], 2.2.3 (i).

For the locally compact case we can say the following:

**1.5 THEOREM:** *Let  $G$  be  $K$ -amenable. Then  $G$  is  $\chi(k)$ -free and there exists a Haar integral  $m$  on  $C_\infty(G \rightarrow K)$  ( $= \{f \in BC(G \rightarrow K)$  vanishing at infinity}), such that  $|m(\xi_S)| = 1$  for all compact open subgroups  $S$ .*

**PROOF:** That  $G$  is  $\chi(k)$ -free can be shown as in 1.4. The rest follows from [3], 2.2.7.

We refer to [3] or [2] for properties of the convolution algebra  $L(G \rightarrow K)$ . This non-archimedean counterpart of  $L^1(G)$ , as a Banach space, equals  $C_\infty(G \rightarrow K)$ , but it has convolution as multiplication).

## 2. $K$ -amenability for non-spherically complete $K$

**2.1 THEOREM:** *Let  $K$  be not spherically complete. Then  $G$  is  $K$ -amenable if and only if  $G$  is a  $\chi(k)$ -free compact group.*

**PROOF:** We prove: if  $G$  is  $K$ -amenable then  $G$  is compact. (The rest follows from 1.4).

Assume that  $G$  is  $\sigma$ -compact. According to [2], 2.7  $G$  is  $IN$ -compact and hence every element, including any l.i.m.  $M$ , of the dual space of  $BC(G \rightarrow K)$  is tight ([2], 7.20). So there exists a compact (open)  $Y \subset G$  such that  $|M(f)| \leq \max(\sup_{x \in Y} |f(x)|, \frac{1}{2}\|f\|)$  for all  $f \in BC(G \rightarrow K)$ . If  $G$  were not compact then there would be an  $s \in G$  with  $sY \cap Y = \emptyset$ . Now  $|M(\xi_Y)| = |M(\xi_{sY})| \leq \frac{1}{2}$ . But also  $|M(\xi_Y)| = |M(1) - M(\xi_{G \setminus Y})| = 1$ .

Contradiction. The general case follows from 1.2. (i) and the following lemma.

**2.2 LEMMA:** *A non-compact  $G$  contains an open non-compact,  $\sigma$ -compact subgroup  $S$ .*

**PROOF:** Choose any compact open subgroup  $T_0$ . Since  $G$  is not compact we can find  $x_1 \in G \setminus T_0$ . If the group  $T_1$ , generated by  $T_0$  and  $\{x_1\}$ , is not

compact, put  $S = T_1$ . Otherwise, choose  $x_2 \in G \setminus T_1$  and consider the group  $T_2$ , generated by  $T_1$  and  $\{x_2\}$ . If  $T_2$  is not compact put  $S = T_2$ , etc. We have: either  $S = T_n$  for some  $n$ , or all  $T_n$  are compact. In this last case, define  $S = \bigcup_{n=1}^{\infty} T_n$ .

**3. K-amenability for spherically complete K**

Let us denote by  $H$  the closed linear span of

$$\{f \cdot -f : f \in BC(G \rightarrow K), s \in G\}.$$

Then we have:

3.1 THEOREM: *Let  $K$  be spherically complete. Then  $G$  is  $K$ -amenable if and only if  $\inf \{\|1 - h\| : h \in H\} = 1$  (notation  $1 \perp H$ ).*

PROOF: If  $1 \perp H$  then define  $\phi : K \cdot 1 + H \rightarrow K$  via  $\phi(\lambda \cdot 1 + h) = \lambda(\lambda \in K, h \in H)$ . Then  $|\phi(\lambda \cdot 1 + h)| = |\lambda| \leq \|\lambda \cdot 1 + h\|$  and  $\phi(1) = 1$ .

By Ingleton's theorem (which is also valid for trivially valued fields) we can extend  $\phi$  to an  $M \in BC(G \rightarrow K)$  such that  $|M(f)| \leq \|f\|$  for all  $f \in BC(G \rightarrow K)$ . This  $M$  is a l.i.m. If  $\|1 - h\| < 1$  for some  $h \in H$  and  $M$  were a l.i.m. on  $BC(G \rightarrow K)$ , then  $1 > |M(1 - h)| = |1 - M(h)| = 1$  (since  $M = 0$  on  $H$ ) which is a contradiction.

3.2 DEFINITION:  *$G$  is called torsional if every finite subset of  $G$  is contained in a compact (open) subgroup of  $G$ .* (See also 3.7).

3.3 LEMMA: *If  $G$  is torsional and  $\chi(k)$ -free then  $G$  is  $K$ -amenable.*

PROOF: Suppose  $G$  is not  $K$ -amenable. Then, by 3.1, there exist  $f^{(1)}, \dots, f^{(n)} \in BC(G \rightarrow K)$  and  $s_1, \dots, s_n \in G$  such that

$$\|1 - \sum (f_{s_i}^{(i)} - f^{(i)})\| < 1.$$

Let  $S$  be a compact open subgroup, containing  $s_1, \dots, s_n$ . Being  $\chi(k)$ -free and compact  $S$  is  $K$ -amenable (1.4). But we also have

$$\|1 - \sum (g_{s_i}^{(i)} - g^{(i)})\| < 1$$

where  $g^{(i)} = f^{(i)}|_S \in BC(S \rightarrow K)$ , from which follows via 3.1 that  $S$  is not  $K$ -amenable. Contradiction.

For a proof of the converse of lemma 3.3 we first reduce it to the case where  $K$  is trivially valued. Indeed, if  $G$  is  $K$ -amenable then  $G$  is  $K_0$ -amenable, where  $K_0$  is the closure of the prime field. This follows from 3.1 and the fact that  $K_0$  is always spherically complete. ( $K_0$  is isomorphic to either  $\mathbb{F}_p$ ,  $\mathbb{Q}_p$  or  $\mathbb{Q}$ ). It is also an easy matter to show directly that  $\mathbb{Q}_p$ -amenable groups are also  $\mathbb{F}_p$ -amenable. So we have to deal only with  $\mathbb{F}_p$  and  $\mathbb{Q}$  (both trivially valued).

For  $x \in G$ , let  $\delta_x \in BC(G \rightarrow K)$  be the evaluation map  $f \mapsto f(x)$ . Let  $D(G)$  be the  $K$ -linear span of  $\{\delta_x : x \in G\}$  and let  $P(G) = \{\mu \in D(G) : \mu(1) \neq 0\}$ . For

$$\mu = \sum_{i=1}^n \lambda_i \delta_{x_i} \in D(G)$$

and  $f \in L(G \rightarrow K)$  define

$$(\mu * f)(x) = \sum_{i=1}^n \lambda_i f(x_i^{-1}x) \quad (x \in G)$$

$$(f * \mu)(x) = \sum_{i=1}^n \lambda_i f(xx_i^{-1})\Delta(x_i^{-1}) \quad (x \in G)$$

$$\mu' = \sum \lambda_i \delta_{x_i^{-1}}$$

$$f'(x) = f(x^{-1})\Delta(x^{-1}) \quad (x \in G)$$

$$f^s(x) = f(xs^{-1}) \quad (s, x \in G)$$

where  $\Delta$  is the  $K$ -valued modular function ([3], 2.4). Clearly, both  $\mu * f$  and  $f * \mu$  are in  $L(G \rightarrow K)$  and  $(\mu * f)' = f' * \mu'$ ,  $f_s * \mu = (f * \mu)_s$ ,  $\mu * f^s = (\mu * f)^s$ .

The space  $D(G)$  becomes a  $K$ -algebra under convolution: for  $f \in BC(G \rightarrow K)$ , let

$$(\mu * \nu)(f) = \sum_{i,j} \lambda_i \tau_j f(x_i y_j) \quad (\mu = \sum \lambda_i \delta_{x_i}, \nu = \sum \tau_j \delta_{y_j}).$$

$P(G)$  is a multiplicatively closed subset of  $D(G)$ . We have the usual relations:

$$\left. \begin{aligned} (\mu * \nu) * f &= \mu * (\nu * f) \\ f * (\mu * \nu) &= (f * \mu) * \nu \end{aligned} \right\} (\mu, \nu \in D(G), f \in L(G \rightarrow K))$$

Let us denote the  $K$ -valued Haar integral on  $L(G \rightarrow K)$  by  $m$ .

3.4 LEMMA: *Let  $G$  be  $K$ -amenable where  $K$  is trivially valued. For  $f \in L(G \rightarrow K)$  with  $m(f) = 0$  there is  $\mu \in P(G)$  with  $f * \mu = 0$ .*

PROOF: It suffices to show that  $v * f = 0$  for some  $v \in P(G)$ . (If  $m(f) = 0$ , then  $m(f') = 0$ . Then  $v * f' = 0$  implies  $f * v' = 0$ ). If  $\mu * f \neq 0$  for all  $\mu \in P(G)$ , define a map  $\phi \in L(G \rightarrow K)$  by extending the map  $\mu * f \mapsto \mu(1)$ , defined on  $D(G) * L(G \rightarrow K)$ . (The definition makes sense: if  $\mu * f = v * f$  then  $(\mu - v) * f = 0$  so  $\mu - v \notin P(G)$  which means  $(\mu - v)(1) = 0$ ). Let  $M$  be a l.i.m. on  $BC(G \rightarrow K)$  and define  $\psi \in L(G \rightarrow K)$  by

$$\psi(g) = M(x \mapsto \phi(g_x)) \quad (g \in L(G \rightarrow K)).$$

$\psi$  is left invariant and since  $\phi(f_x) = \phi(\delta_{x^{-1}} * f) = 1$ , we obtain  $\psi(f) = 1$ . By the uniqueness of the Haar integral, we have  $\psi = cm$  for some  $c \neq 0$ . But  $1 = \psi(f) = cm(f) = 0$ . Contradiction.

3.5 LEMMA: ('Property  $P$ ' of Reiter). *Let  $G$  be  $K$ -amenable, where  $K$  is trivially valued. Then for every compact set  $C \subset G$  there exists a non zero  $f \in L(G \rightarrow K)$  such that  $f_x = f$  for all  $x \in C$ .*

PROOF: Choose a compact open subgroup  $S$  of  $G$ . Then  $C$  is covered by, say  $Sa_1^{-1}, \dots, Sa_n^{-1}$ . Inductively, we define  $\mu_1, \mu_2, \dots, \mu_n \in P(G)$  such that

$$(\xi_S - \xi_{a_k S}) * \mu_1 * \mu_2 * \dots * \mu_k = 0. \quad (k = 1, \dots, n)$$

(for any  $v \in P(G) : m((\xi_S - \xi_{a_k S}) * v) = m(\xi_S - \xi_{a_k S})v(1) = 0$ , then use 3.4). Define  $f = \xi_S * \mu_1 * \dots * \mu_n$ . Then  $f \neq 0$  since  $m(f) = m(\xi_S) \neq 0$ . Any  $x \in C$  can be written as  $sa_i^{-1}$  for some  $s \in S$  and  $i$ .

$$f_x = f_{sa_i^{-1}} = (\xi_S)_{sa_i^{-1}} * \mu_1 * \dots * \mu_n = \xi_{a_i S} * \mu_1 * \dots * \mu_n = f.$$

3.6 THEOREM: *Let  $K$  be spherically complete. Then  $G$  is  $K$ -amenable if and only if  $G$  is torsional and  $\chi(k)$ -free.*

PROOF: We prove:  $G$   $K$ -amenable  $\Rightarrow G$  is torsional. (1.5 and 3.3 take care of the rest). We assume  $K$  to have trivial valuation (that this is without loss of generality follows from the remark following 3.3). Let  $C \subset G$  be compact. By 3.5 there is  $f \in L(G \rightarrow K)$  such that  $f \neq 0$  and  $f_x = f$  for all  $x \in C$ . But it is easy to see that  $\{x : f_x = f\}$  is an open compact subgroup  $S$  of  $G$ . Hence any compact set is contained in a compact subgroup:  $G$  is torsional.

3.7 COROLLARY: *For a locally compact 0-dimensional group  $G$  the following conditions are equivalent:*

- (1)  $G$  is torsional
- (2) Every compact set is contained in a compact subgroup.

PROOF: Use the proof of 3.5 for  $K = \mathbb{Q}$  (every  $G$  is 0-free).

*Note:*  $K$ -amenability for some non-archimedean  $K$  implies ‘amenability’ in the ordinary (real) sense. ( $G$  is torsional, hence inductive limit of compact (amenable) groups, so  $G$  itself is amenable).

#### 4. Uniqueness of invariant means

We show here that, unless  $G$  is compact (see 1.4), a l.i.m. is *never* unique for  $K$ -amenable  $G$ .

4.1 THEOREM: *Let  $G$  be not compact and  $K$ -amenable. Then*

- (1) *There exists a l.i.m. on  $BC(G \rightarrow K)$ , which is an extension of the Haar integral on  $C_\infty(G \rightarrow K)$ .*
- (2) *There exists a l.i.m. on  $BC(G \rightarrow K)$ , which is 0 on  $C_\infty(G \rightarrow K)$ .*

PROOF: By 2.1  $K$  is spherically complete, by 3.6  $G$  is torsional and  $\chi(k)$ -free. Let  $S$  be any compact open subgroup of  $G$ . We show that for any  $\lambda \in K$  and  $h \in H$

$$\|1 + \lambda \xi_S + h\| \geq \max(1, |\lambda|)$$

First, if  $\|1 + \lambda \xi_S + h\|$  were  $< 1$ , then there is  $h' = \sum_{i=1}^n (h_{x_i}^{(i)} - h^{(i)})$  such that

$$\|1 + \lambda \xi_S + h'\| < 1$$

There is a compact open subgroup  $T$  such that  $S \subset T$  and  $\{x_1, \dots, x_n\} \subset T$ . Since  $G$  is not compact there is  $a \in G$  with  $Ta \cap T = \emptyset$ . Then we may write

$$\|1 + \lambda \xi_S^a + (h')^a\| < 1.$$

Restricted to  $T$ , this expression comes down to

$$\|1 + \lambda \xi_{Sa \cap T} + h''\| = \|1 + h''\| < 1.$$

where  $h' \in BC(T \rightarrow K)$  is of the form  $\sum_{i=1}^n (t_{x_i}^{(i)} - t^{(i)})$  for some  $t^{(i)} \in BC(T \rightarrow K)$ . But this implies that  $T$  is not amenable, a contradiction.

Next, we show that  $\|\xi_S + h\| \geq 1$ . (Then we are done, since

$$|\lambda| \leq \|\lambda \xi_S + h\| \leq \max(\|1 + \lambda \xi_S + h\|, \|-1\|) = \|1 + \lambda \xi_S + h\|.$$

Again, suppose

$$\|\xi_S + \sum_{i=1}^n (h_{x_i}^{(i)} - h^{(i)})\| < 1.$$

Let  $T$  be a compact open subgroup containing  $S$  and  $\{x_1, \dots, x_n\}$ . Restricted to  $T$  the above expression yields an inequality for elements of  $BC(T \rightarrow K)$ :

$$\|\xi_S + \sum (t_{x_i}^{(i)} - t^{(i)})\| < 1.$$

Since  $T$  is amenable, there is a Haar integral  $m$  on  $BC(T \rightarrow K)$  with  $|m(\xi_S)| = 1, \|m\| = 1$ . But

$$1 > \|\xi_S + \sum (t_{x_i}^{(i)} - t^{(i)})\| \geq |m(\xi_S + \sum (t_{x_i}^{(i)} - t^{(i)}))| = |m(\xi_S)| = 1,$$

again a contradiction.

The map

$$M : \xi \cdot 1 + \eta \xi_S + h \mapsto \xi + \eta m(\xi_S)$$

is well-defined on  $K \cdot 1 + K \xi_S + H$ .  $M(1) = 1$  and  $\|M\| \leq 1$ , and it can be extended to a l.i.m. by Ingleton's theorem. Clearly, its restriction to  $C_\infty(G \rightarrow K)$  is a Haar integral. And by carrying out the same thing for the map

$$N : \xi \cdot 1 + \eta \xi_S + h \mapsto \xi$$

we find a l.i.m. that is 0 on  $C_\infty(G \rightarrow K)$ .

### 5. Invariant means on $PC(G \rightarrow K)$

Let  $PC(G \rightarrow K) = \{f \in BC(G \rightarrow K) : f(G) \text{ has compact closure in } K\}$ . Then  $PC(G \rightarrow K)$  is a closed subspace of  $BC(G \rightarrow K)$ . If  $f \in PC(G \rightarrow K)$  and  $s \in G$  then  $f_s$  and  $f^s$  are in  $PC(G \rightarrow K)$ . Clearly  $1 \in PC(G \rightarrow K)$ .

If every closed and bounded subset of  $K$  is compact, then  $PC(G \rightarrow K) = BC(G \rightarrow K)$ . The latter is also true if  $G$  is compact.

5.1 DEFINITION: A left invariant mean on  $PC(G \rightarrow K)$  is a  $K$ -linear function  $M : PC(G \rightarrow K) \rightarrow K$  satisfying

- (1)  $M(1) = 1$
- (2)  $|M(f)| \leq \|f\|$  for all  $f \in PC(G \rightarrow K)$
- (3)  $M(f_s) = M(f)$  for all  $f \in PC(G \rightarrow K)$ ,  $BC(G \rightarrow K)$  and  $s \in G$ .

$G$  is called weakly  $K$ -amenable if there is a l.i.m. on  $PC(G \rightarrow K)$ .

Let  $\Omega$  denote the ring of clopen subsets of  $G$ . Then  $\xi_U \in PC(G \rightarrow K)$  for all  $U \in \Omega$ .

5.2 THEOREM: The following conditions are equivalent.

- (1)  $G$  is weakly  $K$ -amenable
- (2)  $G$  is weakly  $K_0$ -amenable (where  $K_0$  is the closure of the prime field of  $K$ )
- (3) There exists an additive set function  $\mu : \Omega \rightarrow K_0$  with  $\mu(G) = 1$ ;  $\mu(sA) = \mu(A)$  and  $|\mu(A)| \leq 1$  for all  $s \in G$  and  $A \in \Omega$ .

PROOF: We prove: (1)  $\rightarrow$  (2)  $\rightarrow$  (3)  $\rightarrow$  (1). If  $M$  is a l.i.m. on  $PC(G \rightarrow K)$ , take  $\phi : K \rightarrow K_0$  with  $\phi(1) = 1$ ,  $|\phi(x)| \leq |x|$  for all  $x \in K$ ,  $\phi$  is  $K_0$ -linear. (Such  $\phi$  exists since  $K_0$  is spherically complete). Define

$$N : PC(G \rightarrow K_0) \rightarrow K_0$$

via  $N(f) = \phi(M(f))$ . This  $N$  is a l.i.m. on  $PC(G \rightarrow K_0)$ . (2)  $\rightarrow$  (3) is almost trivial (if  $M$  is a l.i.m. on  $PC(G \rightarrow K_0)$ , put  $\mu(A) = M(\xi_A)$  for  $A \in \Omega$ ). (3)  $\rightarrow$  (1): If  $f \in PC(G \rightarrow K)$  has the form  $\sum_{i=1}^n \lambda_i \xi_{U_i}$  where  $U_i \in \Omega$  are disjoint, define  $M(f) = \sum \lambda_i \mu(U_i)$ . This way  $M$  is well-defined on the set  $\mathcal{T}$  of 'simple functions' and has the properties (1), (2), (3) of 5.1. For  $f \in PC(G \rightarrow K)$  and  $\varepsilon > 0$  define  $x \sim y$  if  $|f(x) - f(y)| < \varepsilon$  ( $x, y \in G$ ).

Let  $U_1, U_2, \dots, U_n$  be the (clopen) equivalence classes. (Since  $\overline{f(G)}$  is compact the number of equivalence classes is finite). Choose  $a_i \in U_i$  for each  $i$ . Then  $g = \sum f(a_i) \xi_{U_i} \in \mathcal{T}$  and  $\|g - f\| < \varepsilon$ . Thus  $\mathcal{T}$  is dense in  $PC(G \rightarrow K)$  and the continuous extension of  $M$  is a l.i.m. on  $PC(G \rightarrow K)$ .

5.3 COROLLARY: Let  $\chi(k) \neq 0$ . Then the following conditions are equivalent.

- (1)  $G$  is weakly  $K$ -amenable.  
 (2)  $G$  is torsional and  $\chi(k)$ -free.

If  $K$  is spherically complete, then  $G$  is weakly  $K$ -amenable if and only if  $G$  is  $K$ -amenable.

PROOF: (1)  $\rightarrow$  (2): by 5.2.  $G$  is weakly  $K_0$ -amenable. Since  $\chi(k) \neq 0$  we have either  $K_0 = \mathbb{F}_p$  or  $K_0 = \mathbb{Q}_p$ , in both cases  $PC(G \rightarrow K_0) = BC(G \rightarrow K_0)$  and  $K_0$  is spherically complete. Now use 3.6. (2)  $\rightarrow$  (1): by 3.6  $G$  is  $K_0$ -amenable, hence weakly  $K_0$ -amenable. Now use 5.2. The second part is obvious (use (1)  $\rightarrow$  (2) and 3.6).

The situation is radically different if  $\chi(k) = 0$  (note that in general  $PC(G \rightarrow \mathbb{Q}) \neq BC(G \rightarrow \mathbb{Q})$ ).

Let us call  $G$   $IR$ -amenable if there exists a left invariant mean on  $BC(G \rightarrow IR)$  (the 'classical' definition of amenability). We have:

5.4 THEOREM: If  $G$  is  $IR$ -amenable and  $\chi(k) = 0$  then  $G$  is weakly  $K$ -amenable.

PROOF: By 5.2 it suffices to show that there exists a l.i.m. on  $PC(G \rightarrow \mathbb{Q})$ , where  $\mathbb{Q}$  has the trivial valuation. Compact subsets of  $\mathbb{Q}$  are finite so every  $f \in PC(G \rightarrow \mathbb{Q})$  is a simple function and we have an embedding  $PC(G \rightarrow \mathbb{Q}) \rightarrow BC(G \rightarrow IR)$ . Construct a  $\mathbb{Q}$ -linear  $\phi : IR \rightarrow \mathbb{Q}$  with  $\phi(1) = 1$ . If  $M$  is a l.i.m. on  $BC(G \rightarrow IR)$  define  $N(f) = \phi(M(f))$  ( $f \in PC(G \rightarrow \mathbb{Q})$ ). This  $N$  is a l.i.m. on  $PC(G \rightarrow \mathbb{Q})$ .

It is still an open question whether the converse of 5.4 holds. As an example we show that the discrete free group on two generators  $F_2$ , the classical example of a non- $IR$ -amenable group, is also not weakly  $K$ -amenable.

5.5 LEMMA: Let  $F_2$  have generators  $a, b$  and let  $h : F_2 \rightarrow K$  (here  $K$  may be any additive group). Then there exist  $f, g : F_2 \rightarrow K$  such that

- (1)  $f - f_a + g - g_b = h$   
 (2)  $f(F_2) \subset h(F_2) \cup \{0\}; g(F_2) \subset h(F_2) \cup \{0\}$ .

PROOF: Define  $f(e) = f(a) = 0; g(e) = h(e), g(b) = 0$ . Then

$$(*) \quad f(x) - f(ax) + g(x) - g(bx) = h(x)$$

holds for  $x = e$  (all  $x$  with length  $\leq 0$ ). Suppose we have defined already  $f(x), g(x)$  for all  $x$  with length  $\leq n-1$  and  $f(y)$  for all  $y$  with length  $n$  of the form  $y = a \cdots$  and  $g(z)$  for all  $z$  of length  $n$  of the form  $z = b \cdots$  such that

(\*) holds for all words with length  $\leq n-1$ . Then we extend  $f$  and  $g$  as follows:

(1) If  $x$  has length  $n$ :

$$\begin{aligned} f(x) = h(x) & \text{ if } x = b^{\pm 1} \cdots & \text{ and } & f(x) = f(ax) & \text{ if } x = a^{-1} \cdots \\ g(x) = h(x) & \text{ if } x = a^{\pm 1} \cdots & \text{ and } & g(x) = g(bx) & \text{ if } x = b^{-1} \cdots \end{aligned}$$

(2) If  $x$  has length  $n+1$ :

$$\begin{aligned} f(x) = f(a^{-1}x) & \text{ if } x = aa \cdots \\ f(x) = 0 & \text{ if } x = ab^{\pm 1} \cdots \\ g(x) = g(b^{-1}x) & \text{ if } x = bb \cdots \\ g(x) = 0 & \text{ if } x = ba^{\pm 1} \cdots \end{aligned}$$

This way we now have defined  $f(x)$ ,  $g(x)$  for all  $x$  with length  $\leq n$ ,  $f(y)$  for all  $y$  with length  $n+1$  of the form  $y = a \cdots$ ,  $g(z)$  for all  $z$  with length  $n+1$  of the form  $z = b \cdots$ .

It is easy to check that now (\*) holds for all  $x$  with length  $\leq n$ . Inspection of the above inductive definition of  $f$  and  $g$  learns us right away that also (2) holds.

5.6 COROLLARY:  $F_2$  is not weakly  $K$ -amenable. In fact, every left invariant linear function on  $PC(F_2 \rightarrow K)$  is the zero map.

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(Oblatum 4-VII-1974 & 15-X-1974)

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