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**ON DETERMINING THE QUADRATIC SUBFIELDS OF
 \mathbb{Z}_2 -EXTENSIONS OF COMPLEX QUADRATIC FIELDS**

Joseph E. Carroll

Abstract

If F is a complex quadratic field there is normal extension L/F with Galois group topologically isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ where \mathbb{Z}_2 is the additive group of 2-adic integers. $F(\sqrt{2})$ always lies in L . In this paper we attempt to determine what the other quadratic subextensions of L/F are. We show how this can be done under a hypothesis which is implied by but does not imply that the 2-primary part of the ideal class group of F has exponent 2.

1. Let F be a complex quadratic field, $F = \mathbb{Q}(\sqrt{-d})$. Let S be the set of primes of F lying above 2. For \mathfrak{p} , a prime of F , let $U_{\mathfrak{p}}$ denote the group of units in the completion, $F_{\mathfrak{p}}$, of F at \mathfrak{p} . Let

$$J^S = \prod_{\mathfrak{q} \in S} \{1\} \times \prod_{\mathfrak{p} \notin S} U_{\mathfrak{p}},$$

a subgroup of the idèle group, J , of F . By class field theory, $\overline{F^* J^S}$ corresponds to the maximal abelian 2-ramified (i.e., unramified at all primes outside S) extension of F . We can write canonically, $J/\overline{F^* J^S} = G \times G'$, where G is a pro-2 group and G' is the product of pro- p groups for odd primes p . If M is the fixed field of G' , then M contains L , the composite of all \mathbb{Z}_2 -extensions of F . Since Leopoldt's Conjecture is valid for F , $\text{Gal}(L/F) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$.

PROPOSITION (1): G is a finitely generated \mathbb{Z}_2 -module.

PROOF: It is sufficient to show that G/G^2 is finite [4, §6], but G/G^2 is the Galois group of the composite of all 2-ramified quadratic extensions of F . Such an extension is of the form $F(\sqrt{\beta})$ where the primes outside S divide β to an even power. Let A be the subgroup of all such β in F^* . Let C_S be the quotient of the ideal class group, C , of F by the subgroup

generated by classes of primes in S ; let U_S be the subgroup of elements of F^* divisible only by primes in S . Then we have an exact sequence,

$$(1) \quad 0 \rightarrow U_S/U_S^2 \rightarrow A/F^{*2} \xrightarrow{f} (C_S)_2 \rightarrow 0$$

where $(C_S)_2$ is the subgroup of elements of C_S of exponent 2 and $f(\beta)$ is the class of the ideal whose square is (β) up to primes of S . But C_S is finite and U_S/U_S^2 is finite by the S -unit theorem, so A/F^{*2} is finite and we are done.

2. Let T be the torsion subgroup of G . Then $G \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \times T$, since G is a finitely generated module over a P.I.D., and L is the fixed field of T . We must know more about T in order to find the quadratic subextensions of L . Let U denote the unit group of F , and let $B(2)$ be the 2 power torsion part of B for any abelian group B . The natural continuous map $J/F^* \rightarrow C$ induces an exact sequence

$$0 \rightarrow \left(\prod_{q \in S} U_q \right) / \bar{U} \rightarrow J/\overline{F^* J^S} \rightarrow C \rightarrow 0$$

and taking 2 power torsion parts we get another exact sequence

$$(2) \quad 0 \rightarrow \left(\left(\prod_{q \in S} U_q \right) / \bar{U} \right) (2) \rightarrow T \rightarrow C(2)$$

PROPOSITION (2): Let $H = \left(\left(\prod_{q \in S} U_q \right) / \bar{U} \right) (2)$. If $d \equiv \pm 1(8)$ and $d \neq 1$, then $H \approx \mathbb{Z}/2\mathbb{Z}$ and the sequence

$$(2') \quad 0 \rightarrow H \rightarrow T \rightarrow \text{im } T \rightarrow 0$$

splits if and only if $d \equiv -1(8)$. If $d \not\equiv \pm 1(8)$ or $d = 1$, then H is trivial.

PROOF: Since F is complex quadratic, U is finite and so $U = \bar{U}$. (In fact $F^* J^S$ is closed also). Thus, if $\mu_{q,2}$ denotes the group of 2-power roots of 1 in F_q , $H = \left(\prod_{q \in S} \mu_{q,2} \right) / \{\pm 1\}$ (if $d = 1$ we get $\{\pm i, \pm 1\}$ in the denominator). If $d \not\equiv 1(8)$, then $\mu_{q,2} = \{\pm 1\}$ for $q \in S$ and if $d \not\equiv -1(8)$, then $|S| = 1$. Thus H is generated by i if $d \equiv 1(8)$ and by $(-1, 1)$ if $d \equiv -1(8)$; otherwise H is trivial. Let

$$(\cdots, \underset{p_1}{x_{p_1}}, \cdots, \underset{p_2}{x_{p_2}}, \cdots, \underset{p_r}{x_{p_r}}, \cdots)$$

denote the idèle of F which has components x_{p_i} in the p_i^{th} slot and 1 elsewhere. If $d \equiv 1(8)$ and $q|2$, then

$$(1 - i, \dots)_q^2 = (-2i, \dots)_q \equiv (i, \dots)_q \pmod{F^* J^S}$$

so the sequence (2') does not split in this case. To complete the proof, it is enough to show that if $d \equiv -1(8)$ and, $q, q' \nmid 2$ then

$$(-1, 1, \dots)_{q, q'} \notin T^2, \quad \text{for then } (-1, 1, \dots)_{q, q'}$$

would generate a pure subgroup of T and (2') would split. Suppose that there is an idèle (x_p) such that

$$(x_p)^2 = (-1, 1, \dots)_{q, q'}(\alpha)(u_p), \quad \text{where } \alpha \in F^*, (u_p) \in J^S.$$

Then the principal ideal, (α) , is a square in D , the ideal group of F . Since F is complex quadratic $N_{F/\mathbb{Q}}\alpha = m^2$, $m \in \mathbb{Q}$. The equation above now yields $x_q^2 x_{q'}^2 = -N_{F/\mathbb{Q}} \alpha = -m^2$, implying the contradiction that $-1 \in \mathbb{Q}_2^{*2}$.

COROLLARY (3): *If $C_2 = C(2)$ then $T = T_2$ unless $1 \neq d \equiv 1(8)$. If $1 = d \equiv 1(8)$ and $C_2 = C(2)$ then $|T/T_2| = 2$ and $(1 - \frac{1}{q}i, \dots)$ generates T/T_2 .*

PROOF: This is immediate from sequence (2) and Proposition 2.

In the sequence (2), T does not necessarily map onto $C(2)$. We can, however, compute the number of cyclic factors of T .

PROPOSITION (4): *Let $\varepsilon = 0$ if $d \equiv 3(8)$ or if all odd primes dividing d are congruent to $\pm 1(8)$ and let $\varepsilon = 1$ otherwise. Then $|T_2| = 2^{|S| - \varepsilon - 1} |C_2|$.*

PROOF: Since $G \approx T \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $|T/T^2| = \frac{1}{4}|G/G^2|$. But $|G/G^2| = |A/F^{*2}|$ (recall the proof of Proposition 1), and by the sequence (1) and the S -unit theorem, $|A/F^{*2}| = 2^{|S|+1} |(C_S)_2|$. Since T is finite, $|T_2| = |T/T^2|$, so we shall be done upon proving

LEMMA (5): $|C_2| = 2^\varepsilon |(C_S)_2|$ where ε is as in the statement of Proposition 4.

PROOF: Let $q \nmid 2$. We have the exact sequence

$$0 \rightarrow \tilde{q}C^2/C^2 \rightarrow C/C^2 \rightarrow C_S/C_S^2 \rightarrow 0$$

where \tilde{q} denotes the class of q in C . This sequence tells us that we must show that $\tilde{q} \in C^2$ if and only if $\varepsilon = 0$. If $d \equiv 3(8)$, then $\tilde{q} = (\tilde{2})$ is trivial in C . In general, if \mathcal{D} is the discriminant of F , there is an isomorphism

$$C/C^2 \simeq \prod_{p|\mathcal{D}}' \{\pm 1\}, \quad \mathfrak{A} \mapsto (\cdots, (N_{F/Q}\mathfrak{A}, \mathcal{D})_p, \cdots)$$

where \prod' means the subgroup of elements (\cdots, η_p, \cdots) of $\prod_{p|\mathcal{D}} \{\pm 1\}$ such that $\prod_{p|\mathcal{D}} \eta_p = 1$, and $(\cdot, \cdot)_p$ denotes the rational Hilbert 2-symbol at p [(3, §26, 29)]. But if $d \not\equiv 3(8)$, then

$$(N_{F/Q}q, \mathcal{D})_p = (2, -d)_p = \left(\frac{2}{p}\right) \quad \text{for } p \text{ odd.}$$

(For properties of $(\cdot, \cdot)_p$ see [5, Ch. 14]). But $(2/p) = 1$ if and only if $p \equiv \pm 1(8)$.

With this information we can find a set of generators for T_2 . Let d' be the odd part of d . For any odd integer m , let $m^* = (-1)^{(m-1)/2}m$. We denote by q, q' primes in S , and by p the prime dividing $p|d'$.

PROPOSITION (6): *Let $d' \equiv \pm 3(8)$. For $p|d'$, define the idèle x_p by:*

$$x_p = (\sqrt{p^*}_q, \cdots, \sqrt{-d}_p, \cdots) \quad \text{if } p \equiv \pm 1(8)$$

$$x_p = (\sqrt{(-d)p^*/d'^*}_q, \cdots, \sqrt{-d}_p, \cdots) \quad \text{if } p \equiv \pm 3(8),$$

then T_2 is generated by $\{x_p \mid p|d'\}$.

PROOF: If $p \equiv \pm 1(8)$, then $x_p^2 \equiv (p^*)(\cdots, -d/p p^*, \cdots) \pmod{J^S}$; if $p \equiv \pm 3(8)$, then $x_p^2 \equiv (-d \cdot p^*/d'^*)(\cdots, d'^*/p p^*, \cdots) \pmod{J^S}$. Thus $x_p \in T_2$ for all $p|d'$. Furthermore, in the sequence (2), $x_p \mapsto \tilde{q}\tilde{p}$ if $p \equiv \pm 3(8)$ and $2|d$, and $x_p \mapsto \tilde{p}$ otherwise. Thus since \tilde{q} and the images of the x_p generate C_2 , we have $|C_2|/|\langle x_p \mid p|d' \rangle| \leq 2$ and this quotient is 1 if $d \equiv 3(8)$. Proposition 4 completes the proof.

PROPOSITION 7: *Let $d \equiv \pm 1(8)$. If there are any, let p_0 be a fixed prime, $p_0|d'$, $p_0 \equiv \pm 3(8)$. Define for $p|d'$ the idèle x_p :*

$$x_p = (\sqrt{p^*}_{q|2}, \cdots, \sqrt{-d}_p, \cdots) \quad \text{if } p \equiv \pm 1(8)$$

$$x_p = (\sqrt{p^*p_0^*}_{q|2}, \cdots, \sqrt{-d}_{p_0}, \cdots, \sqrt{-d}_p, \cdots) \quad \text{if } p \equiv \pm 3(8)$$

(if 2 splits in F , $q|2$ refers to two idèle components both of which are taken $\equiv 1(4)$). Then $\{x_p \mid p|d'\}$, along with

$$\left(-1, 1, \dots\right)_{\mathfrak{q} \quad \mathfrak{q}'}$$

if 2 splits in F , is a set of generators for T_2 .

PROOF: If $p \equiv \pm 1(8)$, $x_p^2 \in F^*J^S$ as in the proof of Proposition 6; if $p \equiv \pm 3(8)$, then

$$x_p^2 \equiv (p^*p_0^*)(\dots, -d/p^*p_0^*, \dots, -d/p^*p_0^*, \dots) \pmod{J^S},$$

so again all $x_p \in T_2$. In the sequence (2), $x_p \rightarrow \tilde{p}$ if $p \equiv \pm 1(8)$ and $x_p \rightarrow \tilde{p}p_0$ if $p \equiv \pm 3(8)$. If $d \not\equiv 1(8)$, \tilde{p}_0 and the images of the x_p generate C_2 so $(C_2 : \text{im} \langle \{x_p \mid p|d'\} \rangle) \leq 2^\epsilon$ where ϵ is as in Proposition 4. Proposition 4 completes the proof in this case after noting that

$$\left(-1, 1, \dots\right)_{\mathfrak{q} \quad \mathfrak{q}}$$

is a nontrivial element of the kernel in the sequence (2) for $d \equiv -1(8)$. If $d \equiv 1(8)$, reasoning analogous to that above gives

$$(C_2 : \text{im} \langle \{x_p \mid p|d'\} \rangle) \leq 2^{\epsilon+1}.$$

Also the number, m , of $p \equiv \pm 3(8)$ is even, and

$$\begin{aligned} \prod_{\substack{p|d' \\ p \neq p_0}} x_p &\equiv (\sqrt{-d} \cdot p_0^{*(m-2)/2})(\dots, -d/p_0, \dots)^{(m-2)/2} (i, \dots)_{\mathfrak{q}} \\ &\equiv (i, \dots)_{\mathfrak{q}} \pmod{F^*J^S} \end{aligned}$$

Thus $\langle \{x_p \mid p|d'\} \rangle$ contains the kernel in the sequence (2) and $|C_2|/|\langle \{x_p \mid p|d'\} \rangle| \leq 2^\epsilon$. Now apply Proposition 4.

3. We now have explicit generators for T if $T^2 = 1$ or $T^2 \approx \mathbb{Z}/2\mathbb{Z}$ and $d \equiv 1(8)$. Whenever we have explicit generators for T we can determine the quadratic sub-extensions of L . To do this we use the Kummer pairing, $A/F^{*2} \times G/G^2 \rightarrow \{\pm 1\}$ (recall again the proof of Proposition 1). If we consider T/T^2 as a subgroup of G/G^2 , then the subgroup of A/F^{*2} orthogonal to T/T^2 is the set of elements of A/F^{*2} whose square roots are fixed by T , i.e., lie in L . If we identify G/G^2 with $J/\overline{F^*J^S J^2}$, the pairing translates by class field theory into the pairing,

$$A/F^{*2} \times J/\overline{F^*J^*J^2} \rightarrow \{\pm 1\}, (a, (x_p)) \rightarrow \prod_p (a, x_p)_p$$

where $(\cdot)_p$ denotes the Hilbert 2-symbol on F_p . This is because if x_p corresponds by local class field theory to $\sigma_p \in \text{Gal}(F_p(\sqrt{a})/F_p)$ which we identify with the decomposition group of \mathfrak{p} in $\text{Gal}(F(\sqrt{a})/F)$, then (x_p) corresponds to $\prod_p \sigma_p$ in global class field theory [2, Ch. 7, §10]. But $(a, x_p)_p = \sigma_p(\sqrt{a})/\sqrt{a}$ and $\text{Gal}(F(\sqrt{a})/F)$ is abelian. To work with this Kummer pairing we need a set of generators for A/F^{*2} . The proof of Lemma 5 tells us that if for all $p|d'$, $p \equiv \pm 1(8)$, then $\tilde{q} \in C^2$, for all $q|2$. In this case we pick $q|2$, $\mathfrak{A} \in D$ such that $q\mathfrak{A}^2$ is principal and define $\alpha \in F$ by $(\alpha) = q\mathfrak{A}^2$. We have only determined α up to units of F for the moment.

PROPOSITION (8): *Let $d \neq 1, 2$. The set consisting of $-1, 2$, all but one $p|d'$ and, if all $p|d'$ are congruent to $\pm 1(8)$, α , is an independent set of generators of A/F^{*2} .*

PROOF: First, we show that this set is independent. It is clear, since one $p|d'$ is missing from the set, that $-1, 2$ and the other $p|d'$ are independent mod F^{*2} . Now suppose that for all $p|d'$, $p \equiv \pm 1(8)$ and

$$(-1)^{\varepsilon-1} 2^{\varepsilon 2} \left(\prod_{p|d'} p^{\varepsilon p} \right) \alpha \in F^{*2},$$

where the ε 's are 0 or 1. Then this number has even valuation at all primes in S . But by looking at the prime decomposition of (2) and (α) , we see that this cannot be the case. Thus, our set is independent. By Lemma 5 and the proof of Proposition 4, $|A/F^{*2}| = 2^{|S|+1-\varepsilon}|C_2|$. The subgroup of A/F^{*2} generated by all but one $p|d'$ and 2 has order $2|C_2|$ if $d \equiv 3(4)$ and $|C_2|$ otherwise. Therefore, throwing in -1 gives us $4|C_2|$ elements if $d \equiv 3(4)$ and $2|C_2|$ otherwise. This is the correct number unless all $p \equiv \pm 1(8)$ and then α fills out the group.

We now explicitly compute the Kummer pairing with elements of T_2 . We shall be using the fact that if E_2/E_1 is an extension of local fields, if $(\cdot)_{E_i}$ denotes the Hilbert 2-symbol on E_i , and if $\beta \in E_2$, $c \in E_1$, then $(\beta, c)_{E_2} = (N_{E_2/E_1} \beta, c)_{E_1}$ [1].

PROPOSITION (9): *Let $a \in \mathbb{Q} \cap A$, $p|d'$. Then, if (\cdot) denotes the Kummer pairing, we have*

- (i) $x_p = (\sqrt[q|2]{p^*}, \dots, \sqrt[p]{-d}, \dots) \Rightarrow (a, x_p) = (a, d)_p$
- (ii) $x_p = (\sqrt[q]{(-d)p^*/d'^*}, \dots, \sqrt[p]{-d}, \dots) \Rightarrow (a, x_p) = (a, d)_2(a, d)_p$
- (iii) $x_p = (\sqrt[q|2]{p^*p_0^*}, \dots, \sqrt[p_0]{-d}, \dots, \sqrt[p]{-d}, \dots) \Rightarrow (a, x_p) = (a, d)_{p_0}(a, d)_p.$

PROOF: For (i),

$$\begin{aligned} (a, x_p) &= \left(\prod_{q|2} (a, \sqrt{p^*})_q \right) \cdot (a, \sqrt{-d})_p \\ &= (a, \sqrt{p^*})_2^2 (a, d)_p = (a, d)_p. \end{aligned}$$

For (ii)

$$\begin{aligned} (a, x_p) &= (a, \sqrt{-d})_q (a, \sqrt{p^*/d'^*})_q (a, \sqrt{-d})_p \\ &= (a, d)_2 (a, \sqrt{p^*/d'^*})_2^2 (a, d)_p = (a, d)_2 (a, d)_p. \end{aligned}$$

Case (iii) is similar.

PROPOSITION (10): Suppose $p \equiv \pm 1(8)$ for all $p|d'$. Let $\alpha = a + b\sqrt{-d}$ with $(\alpha) = q\mathfrak{A}^2$ for some $q|2$. If $N_{F/Q}\alpha = 2s^2$ and $m = a + s$, then $(\alpha, x_p) = (-1)^{(p^*-1)/8} (a, d)_p = (m, d)_p$ for all $p|d'$.

PROOF: We may assume that \mathfrak{A} is integral and divisible by no rational prime since altering \mathfrak{A} to be so only changes α, a, s , and m by rational squares. Therefore, no odd prime divides two of a, bd , and s .

$$\begin{aligned} (\alpha, x_p) &= \left(\prod_{q|2} (\alpha, \sqrt{p^*}) \right) \cdot (\alpha, \sqrt{-d})_p = (2s^2, \sqrt{p^*})_2 (\alpha, \sqrt{-d})_p \\ &= (-1)^{(p^*-1)/8} (\alpha, \sqrt{-d})_p. \end{aligned}$$

Now,

$$\begin{aligned} (a + b\sqrt{-d}, \sqrt{-d})_p &= (a, \sqrt{-d})_p (1 + b\sqrt{-d}/a, \sqrt{-d})_p \\ &= (a, d)_p (1 + b\sqrt{-d}/a, -b\sqrt{-d}/a)_p (1 + b\sqrt{-d}/a, -a/b)_p \\ &= (a, d)_p (2s^2/a^2, -a/b)_p = (a, d)_p. \end{aligned}$$

We have proved the first equality for (α, x_p) . It remains to show that

$$(m/a, d)_p = (-1)^{(p^*-1)/8}.$$

Now $p \nmid a$, and if $p|m$, we would have $p|a^2 - s^2 = s^2 - b^2d$, so $p|s$, which is not the case. Thus $(m/a, d)_p = ((m/a)/p)$,

$$\left(\frac{m/a}{p} \right) = \left(\frac{2m/a}{p} \right) = \left(\frac{2(a+s)/a}{p} \right) = \left(\frac{2+2s/a}{p} \right)$$

and $a^2 + b^2d = 2s^2$ implies that $(s/a)^2 \equiv \frac{1}{2}(p)$. Thus we shall be done if we prove the following

LEMMA (11): *Let $p \equiv \pm 1(8)$. Then $2 + \sqrt{2}$ is a square in \mathbb{F}_p if and only if $p \equiv \pm 1(16)$.*

PROOF: Note first that the choice of $\sqrt{2}$ is unimportant since $(2 + \sqrt{2})(2 - \sqrt{2}) = 2 \in F_p^{*2}$. Since $p^2 \equiv 1(16)$, \mathbb{F}_{p^2} contains the sixteenth roots of 1. Let ζ be a primitive eight root of 1. Then

$$(\zeta + \zeta^{-1})^2 = \zeta^2 + \zeta^{-2} + 2 = 2.$$

Let $\eta^2 = \zeta$. Then

$$(\eta + \eta^{-1})^2 = \zeta + \zeta^{-1} + 2 = 2 + \sqrt{2}.$$

We wish to know when $\eta + \eta^{-1} \in \mathbb{F}_p$. But by Galois theory, $\eta + \eta^{-1} \in \mathbb{F}_p$ if and only if $(\eta + \eta^{-1})^p = \eta + \eta^{-1}$. And $(\eta + \eta^{-1})^p = \eta^p + \eta^{-p} = \eta + \eta^{-1}$ if $p \equiv \pm 1(16)$ and $-(\eta + \eta^{-1})$ if $p \equiv \pm 9(16)$. This completes the proof.

4. Because $(G/G^2 : T/T^2) = 4$, the kernel on the left in the pairing $A/F^{*2} \times T/T^2 \rightarrow \pm 1$ has order 4. It is this kernel whose elements have square roots lying in L . We already know one, however: $F(\sqrt{2})$ begins the cyclotomic Z_2 -extension of F . Thus we have a pairing $A/\langle 2 \rangle F^{*2} \times T/T^2 \rightarrow \pm 1$, and we wish to compute the kernel on the left. We choose a particular set of generators for $A/\langle 2 \rangle F^{*2}$, namely the p^* for all but one $p|d'$, -2 , and if all $p|d'$ are congruent to $\pm 1(8)$, α . Further, if $d \equiv -1(8)$, we choose α so that $\alpha \equiv 1(4)$ in $F_{q'} \approx \mathbb{Q}_2$. In this case, the p^* and α generate the subgroup of $A/\langle 2 \rangle F^{*2}$ orthogonal to

$$(-1, 1, \dots)_{\substack{q \\ q'}}$$

THEOREM (12): *Suppose $d \neq 1, 2$. Let B be the subgroup of F^* generated by the p^* for all but one $p|d'$, -2 if $d \not\equiv -1(8)$, and, if all $p|d'$ are congruent to $\pm 1(8)$, α , with the sign of α chosen so that $\alpha \equiv 1(4)$ in $F_{q'}$ if $d \equiv -1(8)$. If $d' \equiv \pm 1(8)$ but not all $p|d'$ are congruent to $\pm 1(8)$, let $p_0|d'$ be fixed, $p_0 \equiv \pm 3(8)$. Define a homomorphism $\theta : B/B^2 \rightarrow \prod_{p|d'} \{\pm 1\}$ as follows. Let π_p be projection onto the p factor. If $y \in \mathbb{Q} \cap B$,*

$$\pi_p \circ \theta(y) = (y, d)_p \quad \text{for } p \equiv \pm 1(8) \text{ and all } p \text{ if } d \equiv 3(8)$$

$$\pi_p \circ \theta(y) = (y, d)_2(y, d)_p \quad \text{for } p \equiv \pm 3(8) \text{ when } d' \equiv \pm 3(8) \text{ and } d \not\equiv 3(8)$$

$$\pi_p \circ \theta(y) = (y, d)_{p_0}(y, d)_p \quad \text{for } p \equiv \pm 3(8) \text{ when } d' \equiv \pm 1(8)$$

and if $\alpha = a + b\sqrt{-d}$, $N_{F/\mathbb{Q}}\alpha = 2s^2$, $m = a + s$

$$\pi_p \circ \theta(\alpha) = (m, d)_p.$$

Then $|\ker \theta| = 2$ if and only if $T^2 = 1$, and, in this case, if $\ker \theta = \langle x \rangle$, then $F(\sqrt{x})$ is a quadratic subextension of L . Also, if $d \equiv 1(8)$, then $T^2 \approx \mathbb{Z}/2\mathbb{Z}$ if and only if (a), $|\ker \theta| = 4$, (b), $\ker \theta$ contains only one rational integer, x , with odd part congruent to $\pm 1(8)$, and, (c), $d \equiv 9(16)$ if all $p|d'$ are congruent to $\pm 1(8)$. In this case $F(\sqrt{x})$ is a quadratic subextension of L .

PROOF: Propositions 9 and 10 tell us that $\pi_p \circ \theta(y) = (y, x_p)$ except for $d \equiv 3(8)$. But when $d \equiv 3(8)$, $(-2, d)_2 = (p^*, d)_2 = 1$. If $d \equiv -1(8)$, B generates the subgroup of $A/\langle 2 \rangle F^{*2}$ orthogonal to

$$\underset{q}{(-1, 1, \dots)}.$$

Thus $\ker \theta$ can be considered the subgroup of $A/\langle 2 \rangle F^{*2}$ orthogonal to T_2 . Since the subgroup orthogonal to all of T has order 2, $|\ker \theta| = 2$ if and only if $T = T_2 \cdot T^2$, i.e., $T = T_2$. If $d \equiv 1(8)$, $T^2 \approx \mathbb{Z}/2\mathbb{Z}$ if and only if

$$\underset{q}{(1 - i, \dots)}$$

generates T/T^2 , and this can happen if and only if $|\ker \theta| = 4$ and the pairing $\ker \theta \times \langle (1 - i, \dots) \rangle \rightarrow \pm 1$ has kernel on the left of order 2. Now if $y \in \mathbb{Q}$, then

$$\underset{q}{(y, (1 - i, \dots))} = (y, 1 - i)_q = (y, 2)_2.$$

But $(y, 2)_2 = 1$ if and only if the odd part of y is congruent to $\pm 1(8)$. If all $p|d$ are congruent to $\pm 1(8)$, then

$$\underset{q}{(y, (1 - i, \dots))} = 1$$

for $y \in B \cap \mathbb{Q}$ since such y have odd part congruent to $\pm 1(8)$. We are done if we show that

$$\underset{q}{(\alpha, (1 - i, \dots))} = (-1)^{(d-1)/8}$$

Now,

$$(\pm \alpha \bar{\alpha}, 1 - i)_q = (\pm 2s^2, 2)_2 = 1,$$

so

$$(\alpha, 1 - i)_q = (\bar{\alpha}, 1 - i)_q = (-\alpha, 1 - i)_q = (-\bar{\alpha}, 1 - i)_q,$$

and there is no loss in assuming that if $\alpha = a + b\sqrt{-d} = a + ib\sqrt{d}$ in $F_q \approx \mathbb{Q}_2(i)$, then $a \equiv \sqrt{d} \equiv -b \equiv 1(4)$ (we may assume that $2 \nmid \alpha$ since $(2, 1 - i)_q = 1$, so s is odd). Because $a^2 + b^2d = 2s^2$, we see that 2 is a square modulo all primes dividing b , so $b \equiv -1(8)$. Since $s^2 \equiv 1(8)$, we have $2s^2 \equiv 2(16)$ and $b^2 \equiv 1(16)$ from which we extract the congruence $a^2 + d \equiv 2(16)$. Thus

$$a \equiv \sqrt{d} \equiv -b\sqrt{d} \pmod{8} \quad \text{and} \quad \alpha \equiv (1 - i)\sqrt{d} \pmod{8}, \quad \alpha/1 - i = \sqrt{d} \cdot u$$

where $u \equiv 1(q^5)$. But then $u \in F_q^{*2}$ by the theory of local fields, so

$$\begin{aligned} (\alpha, 1 - i)_q &= (\alpha/1 - i, 1 - i)_q \\ &\text{since } (1 - i, 1 - i)_q = (-1, 1 - i)_q = (i, 1 - i)_q^2 = 1 \\ &= (\sqrt{d}, 1 - i)_q \quad \text{since } u \text{ is a square} \\ &= (\sqrt{d}, 2)_2 = (-1)^{(d-1)/8}. \end{aligned}$$

This finishes the proof.

REMARK (13): It is an easy consequence of reciprocity of the rational Hilbert 2-symbols, the fact that $(d/\ell) = 1$ for odd primes $\ell|m$ (because $\ell|m \Rightarrow \ell|a^2 - s^2 = s^2 - b^2d$) and the fact, not proven here, that the odd part of m is congruent to 1(4) if $d \equiv 7(8)$ that we may replace the range group of θ by

$$\prod'_{\substack{p|\mathcal{O} \\ p \neq 2}} \{ \pm 1 \} \quad \text{if } d' \equiv \pm 3(8), \quad \text{and by } \prod_{\substack{p|\mathcal{O} \\ p \neq p_0}} \{ \pm 1 \} \quad \text{if } d' \equiv \pm 1(8),$$

letting $\pi_2 \circ \theta(y) = (y, d)_2$ for $y \in \mathbb{Q}$ and $\pi_2 \circ \theta(\alpha) = (m, d)_2$. Also, the order of these new range groups is $\frac{1}{2}|B/B^2|$, so $|\ker \theta| = 2$ if and only if θ is surjective, etc. It is this form of the map θ which shall be referred to in a later paper.

REMARK (14): The cases $d = 1, 2$ have been skipped over in some of the theorems. It is simple to work out the whole story in these cases. Namely, $T = 1$ in both cases and $F(\sqrt{1-i})$, resp. $F(\sqrt{\sqrt{-2}})$, lie in a \mathbb{Z}_2 -extension of F .

5. We illustrate with two examples.

EXAMPLE (15): Let $F = \mathbb{Q}(\sqrt{-pq})$, $p \equiv 1(4)$, $pq \equiv 3(8)$. In this case, B is generated by -2 and p .

$$\begin{aligned} \theta(-2) &= ((-2, d)_p, (-2, d)_q) = \left(\left(\frac{-2}{p} \right), \left(\frac{-2}{p} \right) \right) \\ \theta(p) &= ((p, d)_p, (p, d)_q) = \left((p, -q)_p, \left(\frac{p}{q} \right) \right) = \left(\left(\frac{q}{p} \right), \left(\frac{p}{q} \right) \right). \end{aligned}$$

It is easy to see directly or by using Remark 13 that $(-2/p) = (-2/q)$, $(-q/p) = (p/q)$. Thus we deduce, noting that T is cyclic by Proposition 2,

- (a) If $p \equiv 1(8)$ and $(p/q) = 1$ then $|T| \geq 4$.
- (b) If $p \equiv 1(8)$ and $(p/q) = -1$ then $T = T_2 \approx \mathbb{Z}/2\mathbb{Z}$ and $F(\sqrt{-2})$ lies in L
- (c) If $p \equiv 5(8)$ and $(p/q) = 1$ then $T = T_2 \approx \mathbb{Z}/2\mathbb{Z}$ and $F(\sqrt{p})$ lies in L
- (d) If $p \equiv 5(8)$ and $(p/q) = -1$ then $T = T_2 \approx \mathbb{Z}/2\mathbb{Z}$ and $F(\sqrt{-2p})$ lies in L .

Case (a) is still up in the air. We consider a particular example: $p = 73$, $q = 3$. Hoping that $|T| = 4$, we compute a square root, z , of $x_{73} \pmod{F^*J^S}$. Any such z would map to a square root of \tilde{p}_{73} in C . Let $\beta = \frac{73}{2} + \frac{3}{2}\sqrt{-219}$. Since $N_{F/Q}\beta = 73 \cdot 5^2$, we have $(\beta) = \mathfrak{p}_{73} \mathfrak{p}_5^2$ for some $\mathfrak{p}_5|5$ (5 splits in F), and $\tilde{p}_{73} = \tilde{p}_5^{-2}$ in C . Thus as a first guess for z we use

$$\left(\cdots, \frac{1}{\mathfrak{p}_5}, \cdots \right).$$

Now

$$\begin{aligned} \left(\cdots, \frac{1}{\mathfrak{p}_5}, \cdots \right)^2 &\equiv (\beta) \left(\cdots, 1/5^2, \cdots \right) \equiv \left(\beta, \cdots, 73/\bar{\beta}, \cdots, \beta, \cdots \right) \\ &\equiv \left(\beta, \cdots, \sqrt{-219}, \cdots \right) \pmod{F^*J^S} \end{aligned}$$

since $\mathfrak{p}_5 \nmid \bar{\beta}$ and β and $\sqrt{-219}$ are both exactly divisible by \mathfrak{p}_{73} . Now,

$$x_{73} = \left(\sqrt{\frac{73}{q}}, \cdots, \sqrt{-219} \right),$$

so if we can find a square root, γ , of $\sqrt{73}/\beta$ in F_q , then we can take

$$z = (\gamma, \dots, \frac{1}{5}, \dots)_{\substack{q \\ p_5}}$$

In $F_q = \mathbb{Q}_2(\sqrt{-3})$ we have $\beta/\sqrt{73} = \sqrt{73}/2 + \frac{3}{2}\sqrt{-3}$.
 $3^2 \equiv 73(64)$, so $3 \equiv \sqrt{73}(32)$, $\frac{3}{2} \equiv \sqrt{73}/2(16)$, thus

$$\beta/\sqrt{73} \equiv -3(-\frac{1}{2} - \frac{1}{2}\sqrt{-3})(16)$$

and $\sqrt{\beta/\sqrt{73}} \equiv \rho\sqrt{-3}(8)$ where $\rho^3 = 1$. Now we evaluate the Kummer pairing:

$$(-2, z) = (-2, \gamma)_q(-2, \frac{1}{5})_{p_5} = (-2, 1/\rho\sqrt{-3})_q(-2, \frac{1}{5})_5$$

since $\rho\sqrt{-3}/\sqrt{\beta/\sqrt{73}} \in F_q^2$ and 5 splits. Thus

$$(-2, z) = (-2, \frac{1}{3})_2(-2, \frac{1}{5})_5 = 1 \cdot (-1) = -1.$$

It follows that z generates T (and so $|T| = 4$) because $A/\langle 2 \rangle F^{*2} \times \langle z \rangle / \langle z \rangle^2$ has kernel on the left of order 2. To finish, we observe

$$(73, z) = (73, 1/\rho\sqrt{-3})_q(73, \frac{1}{5})_{p_5} = (73, \frac{1}{3})_2(73, \frac{1}{5})_5 = 1 \cdot (-1) = -1.$$

Thus $F(\sqrt{-146})$ begins a Z_2 -extension of F .

EXAMPLE (16): Let $F = \mathbb{Q}(\sqrt{-7 \cdot 17})$. B is generated by 17 and α , where we may take $\alpha = (-9 + \sqrt{-119})/2$. Then $m = -\frac{9}{2} + 5 = \frac{1}{2}$. Thus

$$\theta(17) = ((17, 119)_7, (17, 119)_{17}) = ((\frac{17}{7}), (17, -7)_{17}) = ((\frac{17}{7}), (\frac{-7}{17})) = (-1, -1)$$

$$\theta(\alpha) = ((\frac{1}{2}, 119)_7, (\frac{1}{2}, 119)_{17}) = (1, 1).$$

Since θ has kernel of order 2 generated by α , we see that $F(\sqrt{\alpha})$ begins a Z_2 -extension of F and $T \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

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