MICHAEL P. ANDERSON

Some finiteness properties of the fundamental group of a smooth variety

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In this paper we prove that for any smooth variety $X$ over an algebraically closed field of characteristic $p \neq 2, 3, 5$ the group $\prod_1^{p}(X)$ is a finitely presented pro-$(p)$-group. We recall that $\prod_1^{p}(X)$ denotes the maximal quotient of $\prod_1(X)$ of order prime to $p$. In [8] Exposé II this result is demonstrated for smooth $X$ provided there exists a projective smooth compactification $\tilde{X}$ of $X$ such that $\tilde{X}\backslash X$ is a divisor with normal crossings on $\tilde{X}$ and for all $X$ provided we assume strong resolution of singularities for all varieties of dimension $\leq n$. Thus the result was previously known for $X$ of dimension $\leq 2$.

The essential new step is Lemma 1 which allows us to reduce to the case of dimension 2. The proof of this lemma uses Abhyankar’s work on resolution of singularities [1] together with the technique of fibering by curves. We follow the notation of [7] Exposé XIII and [8] Exposé II.

Let us now state our proposition.

**Proposition 1:** Let $X/k$ be a connected smooth variety over the algebraically closed field $k$ of characteristic $p \neq 2, 3, 5$. Then $\prod_1^{p}(X)$ is a finitely presented pro-$(p)$-group.

**Proof:** By [7] Exposé IX it is sufficient to prove the result for the elements of a Zariski covering of $X$. Thus the result follows by induction on dimension from the result in dimension 2, [8] Exposé II Theorem 2.3.1, and the following lemma:

**Lemma 1:** Let $X$ be a smooth variety of dimension $n \geq 3$ over the algebraically closed ground field $k$ and $x$ a point of $X$. Then $x$ has a Zariski neighborhood $U$ such that there exists an algebraically closed extension $\Omega/k$ and a smooth variety $V$ over $\Omega$ of dimension $n-1$ and a morphism $f: V \to U$ such that $f$ induces a surjection $\prod_1(V) \to \prod_1(U)$ and an isomorphism $\prod_1^{p}(V) \to \prod_1^{p}(U)$. 

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PROOF OF LEMMA 1: We proceed by induction on the dimension of $X$.

Let $U$ be an affine neighborhood of $x$. By [1] Birational Resolution there exists a smooth projective model of the function field $k(U)$. Let $\bar{U}$ be a projective compactification of $U$. By [1] Dominance there exists a smooth projective variety $X'$ together with a birational morphism $X' \to \bar{U}$. By [1] Global Resolution there exists a smooth projective variety $X''$ together with a birational morphism $X'' \to \bar{U}$ and such that the inverse image of $\bar{U}\setminus U$ is a divisor with normal crossings on $X''$. Let $U''$ be the complement of this divisor. Then the map $g : U'' \to U$ is a proper birational mapping of smooth varieties. The subvariety of points of $U$ where $g$ is not an isomorphism is of codimension $\leq 2$. Thus by the Purity Theorem [7] Exposé X.3, $g$ induces an isomorphism

$$\prod_1 (U'') \to \prod_1 (U).$$

By [9], [5], or [10], a general hyperplane section of $U''$, call it $V$, gives a smooth surface in $U''$ such that

$$\prod_1 (V) \simeq \prod_1 (U'') \simeq \prod_1 (U).$$

Thus the lemma is proved for $n = 3$.

Now assume $n > 3$. By [4] Exposé XI, $x$ has a Zariski neighborhood $W$ which admits an elementary fibration $g : W \to W'$ with $W'$ smooth of dimension $n - 1$. Moreover, by [6] Proposition 2.8 we may assume that $g$ admits a finite etale multisection i.e. there exists a finite etale map $s : S \to W'$ together with a closed immersion $i : S \to W$ such that $gi = s$. Let $y = g(x)$. By induction $y$ admits a Zariski neighborhood $U'$ in $W'$ such that there exists a smooth variety $V'$ of dimension $n - 2$ and a morphism $f' : V' \to U'$ such that $f'$ induces an isomorphism of the $(p)$-completions of the fundamental groups of $V'$ and $U'$. Let $U = g^{-1}(U')$ and $V = V' \times_{U'} U$ with projections $f : V \to U$ and $g' : V \to V'$. Then $g'$ is an elementary fibration admitting an etale multisection. Letting $C$ be a geometric fiber of $g'$, we have, by [7] Exposé XIII Proposition 4.3, exact sequences

$$e \to \prod_1 (C) \to \prod_1 (V) \to \prod_1 (V') \to e$$

$$e \to \prod_1 (U) \to \prod_1 (U) \to \prod_1 (U) \to e.$$
Let $K$ be the kernel of the homomorphism $\prod_1'(U) \to \prod_1'(V)$ and $K'$ the kernel of $\prod_1(V') \to \prod_1(U')$. Then the natural map $K \to K'$ is an isomorphism. Moreover, by hypothesis $K'$ is contained in the closed normal subgroup of $\prod_1(V')$ generated by the Sylow $p$ subgroups of $\prod_1(V')$. Since any Sylow $p$ subgroup of $\prod_1(V')$ is the image of a Sylow $p$ subgroup of $\prod_1(V)$, $K$ is also contained in the subgroup generated by the conjugates of the Sylow $p$ subgroups. Thus $K$ is contained in the kernel of $\prod_1(V) \to \prod_1^{(p)}(V)$. Therefore the homomorphism

$$\prod_1^{(p)}(V) \to \prod_1^{(p)}(U)$$

is injective and, by the five lemma, it is surjective. Thus the lemma and proposition are proved.

Using Proposition 1 and standard descent techniques we can weaken the resolution hypotheses required to prove finite presentation of $\prod_1^{(p)}(X)$ for arbitrary $X$. We shall say that a point $x$ of a variety $X$ admits a ‘weak resolution of singularities’ if there exists a Zariski neighborhood $U$ of $x$ in $X$ and a morphism of effective descent for the category of étale coverings $f : U' \to U$ such that $U'$ is a smooth variety. We have then the following:

**Proposition 2:** Let $X$ be a variety over an algebraically closed field of characteristic $p \neq 2, 3, 5$. Assume that every point of $X$ admits a weak resolution of singularities. Then $\prod_1^{(p)}(X)$ is a finitely presented pro-(p)-group.

**Corollary:** Let $X$ be a variety of dimension 3 over an algebraically closed field of characteristic $p \neq 2, 3, 5$. Then $\prod_1^{(p)}(X)$ is a finitely $p$ presented pro-(p)-group.

**Proof:** Proposition 2 is a straightforward application of [7] IX.5 together with Proposition 1. The Corollary follows from Proposition 2 and Abhyankar’s results on resolution [1].

As another application of the fibering by curves method we will outline a proof of the following result:

**Proposition 3 (Kunneth Formula):** Let $X$ and $Y$ be connected varieties over the algebraically closed field $k$ of characteristic $p$. Then the natural homomorphism
In [7] Exposé XIII this proposition is demonstrated using the hypothesis of strong resolution of singularities. We avoid the use of resolution of singularities as follows:

First we consider the case where $X$ and $Y$ are normal varieties. Then it is sufficient to prove the formula for some non-trivial open subsets of $X$ and $Y$. Choose $U$ in $X$ and $V$ in $Y$ such that $U$ and $V$ admit elementary fibrations $f : U \to U'$ and $g : V \to V'$ with etale multisections. By induction on the dimensions of $U$ and $V$ we may assume the proposition holds for $U'$ and $V'$. Let $C$ and $D$ be geometric fibers of $f$ and $g$ respectively. Since $f$ and $g$ are elementary fibrations admitting etale multisections we have the following exact sequences

$$
eq \prod_1^p (C) \to \prod_1^p (U) \to \prod_1^p (U') \to \neq$$

$$
eq \prod_1^p (D) \to \prod_1^p (V) \to \prod_1^p (V') \to \neq$$

$$
eq \prod_1^p (C \times D) \to \prod_1^p (U \times V) \to \prod_1^p (U' \times V') \to \neq.$$

Arguing now as in the proof of Lemma 1, we see that the natural homomorphism

$$\prod_1^p (U \times V) \to \prod_1^p (U) \times \prod_1^p (V)$$

induces an isomorphism on $(p)$-completions.

Consider now the case in which $Y$ is assumed normal, and $X$ is arbitrary. Let $X' \to X$ be the normalization of $X$, and define

$$X'' = X' \times X', \quad X''' = X' \times X' \times X'.$$

Let $X'_\alpha, \alpha \in \prod_0^1 (X')$, be the connected components of $X'$. Then by [7] IX Theorem 5.1, $\prod_1^1 (X)$ is the free product of the groups $\prod_1^1 (X_\alpha)$ and the
free group generated by the elements of the set \( \prod_0(X'') \) modulo certain relations defined by the projections:

\[
X'' \cong X'' \Rightarrow X' \rightarrow X.
\]

Thus the same description holds for \( \prod_0^p(X) \) after replacing all the groups involved by their prime to \( p \) completions. Moreover, the same result applies to \( X' \times Y \rightarrow X \times Y \). This gives a description of \( \prod_0^p(X \times Y) \) as the free product (in the category of pro-(\( p \))-groups) of the groups \( \prod_0^p(X \times Y) \) and the free pro-(\( p \))-group generated by the elements of the set \( \prod_0(X'' \times Y) \) modulo relations defined by the projections:

\[
X''' \times Y \cong X'' \times Y \Rightarrow X' \times Y \rightarrow X \times Y.
\]

It is long and tedious, but straightforward, to check that, since

\[
\prod_1^{(p)}(X \times Y) = \prod_1^{(p)}(X) \times \prod_1^{(p)}(Y)
\]

and

\[
\prod_0^{(p)}(X'') = \prod_0^{(p)}(X'''),
\]

the above relations force

\[
\prod_1^{(p)}(X \times Y) = \prod_1^{(p)}(X) \times \prod_1^{(p)}(Y).
\]

Now applying the same argument as above without the assumption that \( Y \) is normal (which is valid because we just verified that

\[
\prod_1^{(p)}(X \times Y) = \prod_1^{(p)}(X) \times \prod_1^{(p)}(Y)
\]

for \( X' \) normal and \( Y \) arbitrary) gives the result for \( X \) and \( Y \) arbitrary varieties.

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Yale University
New Haven, Connecticut 06520
U.S.A.