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SOME FINITENESS PROPERTIES OF THE FUNDAMENTAL GROUP OF A SMOOTH VARIETY

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In this paper we prove that for any smooth variety X over an algebraically closed field of characteristic $p \neq 2, 3, 5$ the group $\prod_1^{(p)}(X)$ is a finitely presented pro-(p)-group. We recall that $\prod_1^{(p)}(X)$ denotes the maximal quotient of $\prod_1(X)$ of order prime to p. In [8] Exposé II this result is demonstrated for smooth X provided there exists a projective smooth compactification $\bar{X}$ of X such that $\bar{X}\backslash X$ is a divisor with normal crossings on $\bar{X}$ and for all X provided we assume strong resolution of singularities for all varieties of dimension $\leqq n$. Thus the result was previously known for X of dimension $\leqq 2$.

The essential new step is Lemma 1 which allows us to reduce to the case of dimension 2. The proof of this lemma uses Abhyankar's work on resolution of singularities [1] together with the technique of fibering by curves. We follow the notation of [7] Exposé XIII and [8] Exposé II.

Let us now state our proposition.

PROPOSITION 1: *Let X/k be a connected smooth variety over the algebraically closed field k of characteristic $p \neq 2, 3, 5$. Then $\prod_1^{(p)}(X)$ is a finitely presented pro-(p)-group.*

PROOF: By [7] Exposé IX it is sufficient to prove the result for the elements of a Zariski covering of X. Thus the result follows by induction on dimension from the result in dimension 2, [8] Exposé II Theorem 2.3.1, and the following lemma:

LEMMA 1: *Let X be a smooth variety of dimension $n \geqq 3$ over the algebraically closed ground field k and x a point of X. Then x has a Zariski neighborhood U such that there exists an algebraically closed extension Ω/k and a smooth variety V over Ω of dimension $n-1$ and a morphism $f: V \to U$ such that f induces a surjection $\prod_1(V) \to \prod_1(U)$ and an isomorphism $\prod_1^{(p)}(V) \to \prod_1^{(p)}(U)$.*

PROOF OF LEMMA 1: We proceed by induction on the dimension of X.

Let U be an affine neighborhood of x. By [1] *Birational Resolution* there exists a smooth projective model of the function field $k(U)$. Let $\bar{U}$ be a projective compactification of U. By [1] *Dominance* there exists a smooth projective variety X' together with a birational morphism $X' \to \bar{U}$. By [1] *Global Resolution* there exists a smooth projective variety X'' together with a birational morphism $X'' \to \bar{U}$ and such that the inverse image of $\bar{U}\backslash U$ is a divisor with normal crossings on X''. Let U'' be the complement of this divisor. Then the map $g : U'' \to U$ is a proper birational mapping of smooth varieties. The subvariety of points of U where g is not an isomorphism is of codimension $\leqq 2$. Thus by the Purity Theorem [7] Exposé X.3, g induces an isomorphism

$$\prod_1 (U'') \to \prod_1 (U).$$

By [9], [5], or [10], a general hyperplane section of U'', call it V, gives a smooth surface in U'' such that

$$\prod_1^{(p)} (V) \simeq \prod_1^{(p)} (U'') \simeq \prod_1^{(p)} (U).$$

Thus the lemma is proved for $n = 3$.

Now assume $n > 3$. By [4] Exposé XI, x has a Zariski neighborhood W which admits an elementary fibration $g : W \to W'$ with W' smooth of dimension $n-1$. Moreover, by [6] Proposition 2.8 we may assume that g admits a finite etale multisection i.e. there exists a finite etale map $s : S \to W'$ together with a closed immersion $i : S \to W$ such that $gi = s$. Let $y = g(x)$. By induction y admits a Zariski neighborhood U' in W' such that there exists a smooth variety V' of dimension $n-2$ and a morphism $f' : V' \to U'$ such that f' induces an isomorphism of the (p)-completions of the fundamental groups of V' and U'. Let $U = g^{-1}(U')$ and $V = V' \times_{U'} U$ with projections $f : V \to U$ and $g' : V \to V'$. Then g' is an elementary fibration admitting an etale multisection. Letting C be a geometric fiber of g', we have, by [7] Exposé XIII Proposition 4.3, exact sequences

$$\begin{array}{ccccccccc}
e & \to & \prod_1^{(p)}(C) & \to & \prod_1'(V) & \to & \prod_1(V') & \to & e \\
 & & \| & & \downarrow & & \downarrow & & \\
e & \to & \prod_1^{(p)}(C) & \to & \prod_1'(U) & \to & \prod_1(U') & \to & e.
\end{array}$$

Let K be the kernel of the homomorphism $\prod_1'(U) \to \prod_1'(V)$ and K' the kernel of $\prod_1(V') \to \prod_1(U')$. Then the natural map $K \to K'$ is an isomorphism. Moreover, by hypothesis K' is contained in the closed normal subgroup of $\prod_1(V')$ generated by the Sylow p subgroups of $\prod_1(V')$. Since any Sylow p subgroup of $\prod_1(V')$ is the image of a Sylow p subgroup of $\prod_1'(V)$, K is also contained in the subgroup generated by the conjugates of the Sylow p subgroups. Thus K is contained in the kernel of $\prod_1(V) \to \prod_1^{(p)}(V)$. Therefore the homomorphism

$$\prod_1^{(p)}(V) \to \prod_1^{(p)}(U)$$

is injective and, by the five lemma, it is surjective. Thus the lemma and proposition are proved.

Using Proposition 1 and standard descent techniques we can weaken the resolution hypotheses required to prove finite presentation of $\prod_1^{(p)}(X)$ for arbitrary X. We shall say that a point x of a variety X admits a 'weak resolution of singularities' if there exists a Zariski neighborhood U of x in X and a morphism of effective descent for the category of etale coverings $f: U' \to U$ such that U' is a smooth variety. We have then the following:

PROPOSITION 2: *Let X be a variety over an algebraically closed field of characteristic $p \neq 2, 3, 5$. Assume that every point of X admits a weak resolution of singularities. Then $\prod_1^{(p)}(X)$ is a finitely presented pro-(p)-group.*

COROLLARY: *Let X be a variety of dimension 3 over an algebraically closed field of characteristic $p \neq 2, 3, 5$. Then $\prod_1^{(p)}(X)$ is a finitely p presented pro-(p)-group.*

PROOF: Proposition 2 is a straightforward application of [7] IX.5 together with Proposition 1. The Corollary follows from Proposition 2 and Abhyankar's results on resolution [1].

As another application of the fibering by curves method we will outline a proof of the following result:

PROPOSITION 3 (Kunneth Formula): *Let X and Y be connected varieties over the algebraically closed field k of characteristic p. Then the natural homomorphism*

$$\prod_1^{(p)}(X \times Y) \to \prod_1^{(p)}(X) \times \prod_1^{(p)}(Y)$$

is an isomorphism.

In [7] Exposé XIII this proposition is demonstrated using the hypothesis of strong resolution of singularities. We avoid the use of resolution of singularities as follows:

First we consider the case where X and Y are normal varieties. Then it is sufficient to prove the formula for some non-trivial open subsets of X and Y. Choose U in X and V in Y such that U and V admit elementary fibrations $f : U \to U'$ and $g : V \to V'$ with etale multisections. By induction on the dimensions of U and V we may assume the proposition holds for U' and V'. Let C and D be geometric fibers of f and g respectively. Since f and g are elementary fibrations admitting etale multisections we have the following exact sequences

$$e \to \prod_1^{(p)}(C) \to \prod_1{}'(U) \to \prod_1(U') \to e$$

$$e \to \prod_1^{(p)}(D) \to \prod_1{}'(V) \to \prod_1(V') \to e$$

$$e \to \prod_1^{(p)}(C \times D) \to \prod_1{}'(U \times V) \to \prod_1(U' \times V') \to e.$$

Arguing now as in the proof of Lemma 1, we see that the natural homomorphism

$$\prod_1{}'(U \times V) \to \prod_1{}'(U) \times \prod_1{}'(V)$$

induces an isomorphism on (p)-completions.

Consider now the case in which Y is assumed normal, and X is arbitrary. Let $X' \to X$ be the normalization of X, and define

$$X'' = X' \underset{X}{\times} X', \qquad X''' = X' \underset{X}{\times} X' \underset{X}{\times} X'.$$

Let X'_α, $\alpha \in \prod_0(X')$, be the connected components of X'. Then by [7] IX Theorem 5.1, $\prod_1(X)$ is the free product of the groups $\prod_1(X_\alpha)$ and the free group generated by the elements of the set $\prod_0(X'')$ modulo certain relations defined by the projections:

$$X''' \rightrightarrows X'' \rightrightarrows X' \to X.$$

Thus the same description holds for $\prod_1^{(p)}(X)$ after replacing all the groups involved by their prime to p completions. Moreover, the same result applies to $X' \times Y \to X \times Y$. This gives a description of $\prod_1^{(p)}(X \times Y)$ as the free product (in the category of pro-(p)-groups) of the groups $\prod_1^{(p)}(X_\alpha \times Y)$ and the free pro-(p)-group generated by the elements of the set $\prod_0(X'' \times Y)$ modulo relations defined by the projections:

$$X''' \times Y \rightrightarrows X'' \times Y \rightrightarrows X' \times Y \to X \times Y.$$

It is long and tedious, but straightforward, to check that, since

$$\prod_1^{(p)}(X_\alpha \times Y) = \prod_1^{(p)}(X_\alpha) \times \prod_1^{(p)}(Y) \quad \text{and} \quad \prod_0(X'' \times Y) = \prod_0(X''),$$

the above relations force

$$\prod_1^{(p)}(X \times Y) = \prod_1^{(p)}(X) \times \prod_1^{(p)}(Y).$$

Now applying the same argument as above without the assumption that Y is normal (which is valid because we just verified that

$$\prod_1^{(p)}(X_\alpha \times Y) = \prod_1^{(p)}(X_\alpha) \times \prod_1^{(p)}(Y)$$

for X_α normal and Y arbitrary) gives the result for X and Y arbitrary varieties.

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