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A NUMERICAL CRITERION FOR THE PERMISSIBILITY OF A BLOWING-UP

Balwant Singh

Introduction

Let $\mathcal{O}$ be a noetherian local ring and $\mathfrak{p}$ a proper ideal of $\mathcal{O}$. The concept of the permissibility of $\mathfrak{p}$ in $\mathcal{O}$ (more precisely, of Spec $(\mathcal{O}/\mathfrak{p})$ in Spec $\mathcal{O}$ at the closed point) as a center for blowing-up was introduced by Hironaka in his paper [3] on the resolution of singularities. If the center of a blowing-up $\mathcal{O} \to \mathcal{O}'$ is permissible in $\mathcal{O}$ then the singularity of $\mathcal{O}'$ is no worse than that of $\mathcal{O}$. Here, as a measure of singularity, we may take either the characters ν^*, τ^* defined by Hironaka in [3] in case $\mathcal{O}$ is given as the quotient of a regular local ring, or the Hilbert functions of $\mathcal{O}$ and $\mathcal{O}'$ (See [1], [4], [6]). In this note we give a numerical criterion for the permissibility of a blowing-up, i.e. of its center (Theorem 1) and study the effect of an arbitrary blowing-up on the Hilbert function of a local ring (Theorems 2 and 3). As a corollary to Theorem 1, we get yet another criterion for the permissibility of a blowing-up (Corollary (1.4)). The criterion in Theorem 1 leads to the definition of a numerical function $D_{\mathfrak{p}}$ such that $\mathfrak{p}$ is permissible in $\mathcal{O}$ if and only if $D_{\mathfrak{p}} = 0$. (See Remark 2.) A significance of this function $D_{\mathfrak{p}}$ is that it appears explicitly in a comparison between the Hilbert functions of $\mathcal{O}$ and $\mathcal{O}'$, where $\mathcal{O} \to \mathcal{O}'$ is a blowing-up of $\mathcal{O}$ with center $\mathfrak{p}$. (See Theorems 2 and 3.) In Remark 3 below we indicate how the criterion in Theorem 1 compares with a numerical criterion for normal flatness given by Bennett [1].

In order to state our results more precisely, we need some notation. By a *numerical function* H we mean a map $H: \mathbb{Z}^+ \to \mathbb{Z}^+$. If H is a numerical function, we get from H a sequence $\{H^{(r)}\}_{r \geqq 0}$ of numerical functions by successive 'integration' as follows: $H^{(0)} = H$ and, for $r \geqq 1$,

$$H^{(r)}(n) = \sum_{i=0}^{n} H^{(r-1)}(i).$$

If H_1, H_2 are numerical functions, then by $H_1 \geqq H_2$ we shall always mean the total order inequality, i.e. $H_1(n) \geqq H_2(n)$ for every $n \in \mathbb{Z}^+$.

Let $\mathcal{O}$ be a noetherian local ring. For a *proper* ideal $\mathfrak{p}$ of $\mathcal{O}$ we define a numerical function $H_\mathfrak{p}$ by

$$H_\mathfrak{p}(n) = \dim_{\mathcal{O}/\mathfrak{m}} \mathfrak{p}^n/\mathfrak{m}\mathfrak{p}^n,$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. This gives us a sequence $\{H_\mathfrak{p}^{(r)}\}_{r \geqq 0}$ of numerical functions. We write $H_\mathcal{O}^{(r)}$ for $H_\mathfrak{m}^{(r)}$, so that $\{H_\mathcal{O}^{(r)}\}_{r \geqq 0}$ is the usual sequence of the Hilbert functions of $\mathcal{O}$.

We denote by dim $\mathcal{O}$ the Krull dimension of $\mathcal{O}$ and by emdim $\mathcal{O}$ the embedding dimension of $\mathcal{O}$, i.e. emdim $\mathcal{O} = H_\mathcal{O}^{(0)}(1)$.

Recall that a proper ideal $\mathfrak{p}$ of $\mathcal{O}$ is said to be *permissible* in $\mathcal{O}$ (as a center for a blowing-up) if the following two conditions are satisfied:

(i) *regularity*: $\mathcal{O}/\mathfrak{p}$ is regular
(ii) *normal flatness*: $\mathcal{O}$ is normally flat along $\mathfrak{p}$, i.e. the graded $\mathcal{O}/\mathfrak{p}$-algebra $\mathrm{gr}_\mathfrak{p}(\mathcal{O}) = \oplus_{n \geqq 0} \mathfrak{p}^n/\mathfrak{p}^{n+1}$ is $\mathcal{O}/\mathfrak{p}$-flat.

THEOREM 1: *Let $\mathcal{O}$ be a noetherian local ring and $\mathfrak{p}$ a proper ideal of $\mathcal{O}$. Let $d =$ dim $\mathcal{O}/\mathfrak{p}$ and $e =$ emdim $\mathcal{O}/\mathfrak{p}$. Then we have $H_\mathcal{O}^{(0)} \leqq H_\mathfrak{p}^{(e)}$. Further, the following three conditions are equivalent*:

(i) $\mathfrak{p}$ *is permissible in* $\mathcal{O}$
(ii) $\mathcal{O}/\mathfrak{p}$ *is regular and* $H_\mathcal{O}^{(0)} = H_\mathfrak{p}^{(d)}$
(iii) $H_\mathcal{O}^{(0)} = H_\mathfrak{p}^{(e)}$.

We prove this theorem in § 1.

REMARK 1: For the implication (i) ⇒ (ii), cf. [3, Chapter II, Proposition 1].

REMARK 2: For a proper ideal $\mathfrak{p}$ of $\mathcal{O}$, let us define $D_\mathfrak{p} = H_\mathfrak{p}^{(e)} - H_\mathcal{O}^{(0)}$, where $e =$ emdim $\mathcal{O}$. Theorem 1 shows that $D_\mathfrak{p}$ is a numerical function, and $\mathfrak{p}$ is permissible in $\mathcal{O}$ if and only if $D_\mathfrak{p} = 0$. We may therefore call $D_\mathfrak{p}$ the *permissibility defect* of $\mathfrak{p}$. Another justification for the use of this term is provided by Theorem 2, which states, roughly, that if $\mathcal{O} \to \mathcal{O}'$ is a blowing-up of $\mathcal{O}$ with center $\mathfrak{p}$, then $H_\mathcal{O}^{(0)} - H_{\mathcal{O}'}^{(\delta)} \geqq -D_\mathfrak{p}$, where δ is the residue transcendence degree of $\mathcal{O}'$ over $\mathcal{O}$. In the case when $\mathfrak{p}$ is permissible in $\mathcal{O}$, the inequality $H_\mathcal{O}^{(0)} - H_{\mathcal{O}'}^{(\delta)} \geqq 0$ is already known [6]. One can thus say that under a blowing-up the singularity of $\mathcal{O}$ can become worse only to the extent that the blowing-up is non-permissible, this non-permissibility being measured by the numerical function $D_\mathfrak{p}$.

REMARK 3: Bennett has given a numerical criterion for the permissibility of $\mathfrak{p}$ in $\mathcal{O}$ in the case when $\mathcal{O}/\mathfrak{p}$ is regular [1, Theorem (3) and 0(2.1.2)]. He has shown that if $\mathcal{O}/\mathfrak{p}$ is regular of dimension d then $\mathfrak{p}$ is permissible in $\mathcal{O}$ if and only if $H^{(0)}_{\mathcal{O}} = H^{(d)}_{\mathcal{O}_\mathfrak{p}}$. Let us compare this criterion with the one given in Theorem 1 above. Suppose that $\mathcal{O}$ is excellent. Then we have $H^{(d)}_{\mathcal{O}_\mathfrak{p}} \leqq H^{(0)}_{\mathcal{O}}$, where $d = \dim \mathcal{O}/\mathfrak{p}$. (See [1, Theorem (2)] and [6, page 202].) In this case, therefore, the difference $D'_\mathfrak{p} = H^{(0)}_{\mathcal{O}} - H^{(d)}_{\mathcal{O}_\mathfrak{p}}$ is a numerical function, and $\mathfrak{p}$ is permissible in $\mathcal{O}$ if and only if $D'_\mathfrak{p} = 0$. However, the definition of this measure $D'_\mathfrak{p}$ of the deviation of $\mathfrak{p}$ from being permissible requires, in the first place, that $\mathfrak{p}$ be a prime ideal. Even then it is apparently defined (i.e. is non-negative) only for $\mathcal{O}$ excellent, it being not known whether the inequality $H^{(d)}_{\mathcal{O}_\mathfrak{p}} \leqq H^{(0)}_{\mathcal{O}}$ holds for non-excellent $\mathcal{O}$. Moreover, in order that $D'_\mathfrak{p} = 0$ imply the permissibility of $\mathfrak{p}$ in $\mathcal{O}$, we have to assume already that $\mathcal{O}/\mathfrak{p}$ is regular. Finally, $D'_\mathfrak{p}$ does not seem to intervene directly in a formula for the difference $H^{(0)}_{\mathcal{O}} - H^{(\delta)}_{\mathcal{O}'}$ as $D_\mathfrak{p}$ does. (Here $\mathcal{O} \to \mathcal{O}'$ is a blowing-up as in Remark 2.) It is interesting, however, to note that if $\mathcal{O}$ is excellent and $\mathcal{O}/\mathfrak{p}$ is regular of dimension d then we have

$$(*) \qquad H^{(d)}_{\mathcal{O}_\mathfrak{p}} \leqq H^{(0)}_{\mathcal{O}} \leqq H^{(d)}_{\mathfrak{p}}$$

and one of these inequalities is an equality if and only if the other is. One may therefore ask: What is the relationship, in this case, between $D_\mathfrak{p} = H^{(d)}_\mathfrak{p} - H^{(0)}_{\mathcal{O}}$ and $D'_\mathfrak{p} = H^{(0)}_{\mathcal{O}} - H^{(d)}_{\mathcal{O}_\mathfrak{p}}$?

REMARK 4: The inequalities (*) of Remark 3 yield another interesting criterion for the permissibility of $\mathfrak{p}$ in $\mathcal{O}$. (See Corollary (1.4) in § 1.)

REMARK 5: With the notation of Theorem 1, we do not, in general, have the inequality $H^{(0)}_{\mathcal{O}} \leqq H^{(d)}_\mathfrak{p}$. *Example*: Let $\mathcal{O}$ be a non-regular Cohen-Macaulay local ring of dimension 1 (e.g., $\mathcal{O} = k[[X, Y]]/(Y^2 - X^3)$, where k is a field). Choose any non-zero divisor x in the maximal ideal of $\mathcal{O}$, and let $\mathfrak{p} = \mathcal{O}x$. Then $d = 0$, $H^{(0)}_\mathfrak{p}(n) = 1$ for every n, but $H^{(0)}_{\mathcal{O}}(1) \geqq 2$.

REMARK 6: With the notation of Theorem 1, the equality $H^{(0)}_{\mathcal{O}} = H^{(d)}_\mathfrak{p}$ alone does not imply that $\mathfrak{p}$ is permissible in $\mathcal{O}$. *Example*: Let $\mathcal{O}$ be a regular local ring of dimension 1. Let x be any non-zero element in the square of the maximal ideal of $\mathcal{O}$ and let $\mathfrak{p} = x\mathcal{O}$.

We now proceed to state Theorems 2 and 3. Let $\mathcal{O} \to \mathcal{O}'$ be a blowing-up of $\mathcal{O}$ with center a proper ideal $\mathfrak{p}$ of $\mathcal{O}$. Let $e = \operatorname{emdim} \mathcal{O}/\mathfrak{p}$. Choose $t_1, \ldots, t_e$ in the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ such that $\mathfrak{m} = \mathfrak{p} + \sum_{i=1}^{e} t_i \mathcal{O}$. Let

$t_0 \in \mathfrak{p}$ be such that $\mathfrak{p}\mathcal{O}' = t_0\mathcal{O}'$. For such a choice of $t = (t_0, t_1, \ldots, t_e)$ we define, for every i, $0 \leqq i \leqq e$, a sequence $\{\mathfrak{a}_{t,i}(n)\}_{n \geqq 0}$ of ideals of $\mathcal{O}'$ as follows:

$$\mathfrak{a}_{t,i}(n) = \{f \in \mathcal{O}' | t_i f \in \mathfrak{m}'^{n+1} + \sum_{j=0}^{i-1} t_j \mathcal{O}'\},$$

where $\mathfrak{m}'$ is the maximal ideal of $\mathcal{O}'$. Clearly, $\mathfrak{a}_{t,i}(n) \supset \mathfrak{m}'^n + \sum_{j=0}^{i-1} t_j \mathcal{O}'$ for every i and n. Let $L_{t,i}$, $0 \leqq i \leqq e$, be the numerical functions defined by

$$L_{t,i}(n) = \text{length}_{\mathcal{O}'} \mathfrak{a}_{t,i}(n)/(\mathfrak{m}'^n + \sum_{j=0}^{i-1} t_j \mathcal{O}').$$

THEOREM 2: *Let* $\mathfrak{p}$ *be a proper ideal of a noetherian local ring* $\mathcal{O}$ *and let* $e = \text{emdim}\, \mathcal{O}/\mathfrak{p}$. *Let* $\mathcal{O} \to \mathcal{O}'$ *be a blowing-up of* $\mathcal{O}$ *with center* $\mathfrak{p}$ *and let* δ *be the residue transcendence degree of* $\mathcal{O}'$ *over* $\mathcal{O}$. *Then, for any choice of* $t = (t_0, t_1, \ldots, t_e)$ *as above, we have*

$$H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(\delta)} \geqq \sum_{i=0}^{e} L_{t,i}^{(i+\delta)} - D_{\mathfrak{p}} \geqq -D_{\mathfrak{p}}.$$

In particular, if $\mathfrak{p}$ *is permissible in* $\mathcal{O}$, *then*

$$H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(\delta)} \geqq \sum_{i=0}^{e} L_{t,i}^{(i+\delta)} \geqq 0.$$

In the case when $\mathcal{O} \to \mathcal{O}'$ is residually rational, we can give a more precise formula for the difference $H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(0)}$. As above, let $t_0 \in \mathfrak{p}$ be such that $\mathfrak{p}\mathcal{O}' = t_0\mathcal{O}'$. Then $\mathcal{O}'$ is obtained as a localization of the subring $\{f/t_0^n | n \geqq 0, f \in \mathfrak{p}^n\}$ of $\mathcal{O}_{t_0}$. We define a sequence $\{\mathfrak{b}_{t_0}(n)\}_{n \geqq 0}$ of ideals of $\mathcal{O}$ by

$$\mathfrak{b}_{t_0}(n) = \{f \in \mathfrak{p}^n | f/t_0^n \in \mathfrak{m}'^{n+1} + \mathfrak{m}\mathcal{O}'\},$$

where $\mathfrak{m}, \mathfrak{m}'$ are the maximal ideals of $\mathcal{O}, \mathcal{O}'$, respectively. Clearly, $\mathfrak{b}_{t_0}(n) \supset \mathfrak{m}\mathfrak{p}^n$ for every n. Let L_{t_0} be the numerical function defined by

$$L_{t_0}(n) = \text{length}_{\mathcal{O}}\, \mathfrak{b}_{t_0}(n)/\mathfrak{m}\mathfrak{p}^n.$$

THEOREM 3: *Let the notation be as in Theorem* 2. *Assume, moreover, that* $\mathcal{O} \to \mathcal{O}'$ *is residually rational. Then for any choice of* $t = (t_0, t_1, \ldots, t_e)$ *as above, we have*

$$H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(0)} = L_{t_0}^{(e)} + \sum_{i=0}^{e} L_{t,i}^{(i)} - D_{\mathfrak{p}}.$$

In particular, if $\mathfrak{p}$ *is permissible in* $\mathcal{O}$, *then*

$$H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(0)} = L_{t_0}^{(e)} + \sum_{i=0}^{e} L_{t,i}^{(i)}.$$

Theorems 2 and 3 are proved in § 2.

1. Proof of Theorem 1

(1.1) Let $\mathcal{O}$ be a noetherian local ring with maximal ideal $\mathfrak{m}$. For any ideal $\mathfrak{p}$ of $\mathcal{O}$ we define

$$\mu(\mathfrak{p}) = \dim_{\mathcal{O}/\mathfrak{m}} \mathfrak{p}/\mathfrak{m}\mathfrak{p},$$

so that $\mu(\mathfrak{p})$ is the cardinality of a minimal set of generators of $\mathfrak{p}$. Note that, if $\mathfrak{p}$ is a proper ideal of $\mathcal{O}$, then $\mu(\mathfrak{p}^n) = H_{\mathfrak{p}}^{(0)}(n)$ for every n.

(1.2) Lemma:

(1) *Let* $\mathfrak{a}_i$, $1 \leqq i \leqq r$, *be ideals of* $\mathcal{O}$ *such that* $\mu(\sum_i \mathfrak{a}_i) = \sum_i \mu(\mathfrak{a}_i)$. *If* S_i *is a minimal set of generators of* $\mathfrak{a}_i$, *then* $\bigcup_i S_i$ *is a minimal set of generators of* $\sum_i \mathfrak{a}_i$. *In particular, for every* j, $1 \leqq j \leqq r$, *we have*

$$S_j \cap (\mathfrak{m}(\sum_i \mathfrak{a}_i) + \sum_{i \neq j} \mathfrak{a}_i) = \emptyset.$$

(2) *Let* $\mathfrak{p}, \mathfrak{q}$ *be proper ideals of* $\mathcal{O}$ *and let* $\mathfrak{a} = \mathfrak{p} + \mathfrak{q}$. *Let* $e = \mu(\mathfrak{q})$. *Then* $H_{\mathfrak{a}}^{(0)} \leqq H_{\mathfrak{p}}^{(e)}$.

(3) *With the notation of* (2), *suppose that* $H_{\mathfrak{a}}^{(0)} = H_{\mathfrak{p}}^{(e)}$. *Then, for every* $m, n \geqq 0$, *we have*

(a) $\mu(\mathfrak{q}^n) = \binom{n+e-1}{e-1}$

(b) $\mu(\mathfrak{p}^m \mathfrak{q}^n) = \mu(\mathfrak{p}^m)\mu(\mathfrak{q}^n)$

(c) $\mu(\mathfrak{a}^{n+1}) = \mu(\mathfrak{q}^{n+1}) + \mu(\mathfrak{a}^n \mathfrak{p})$.

Proof: (1) is immediate. To prove (2) and (3), we have only to observe the following easily verified facts:

(i) $\mu(\mathfrak{a}^n) \leqq \sum_{i=0}^{n} \mu(\mathfrak{p}^{n-i}\mathfrak{q}^i) \leqq \sum_{i=0}^{n} \mu(\mathfrak{p}^{n-i})\mu(\mathfrak{q}^i)$.

(ii) $\mu(\mathfrak{q}^n) \leqq \binom{n+e-1}{e-1}$.

(iii) For any numerical function $H = H^{(0)}$ we have

$$H^{(e)}(n) = \sum_{i=0}^{n} \binom{i+e-1}{e-1} H^{(0)}(n-i).$$

(1.3) LEMMA: (Bennett). *Let $\mathcal{O}$ be a noetherian local ring and $\mathfrak{p}$ an ideal of $\mathcal{O}$ such that $\mathcal{O}/\mathfrak{p}$ is regular. Let $d = \dim \mathcal{O}/\mathfrak{p}$. Then $\mathfrak{p}$ is permissible in $\mathcal{O}$ if and only if $H^{(0)}_{\mathcal{O}} = H^{(d)}_{\mathcal{O}_\mathfrak{p}}$.*

For a proof of this lemma, see [1, Theorem (3) and 0(2.1.2)].

Before coming to the proof of Theorem 1, we note the following corollary to Theorem 1:

(1.4) COROLLARY: *Suppose $\mathcal{O}$ is excellent*[1] *and $\mathcal{O}/\mathfrak{p}$ is regular. Then $\mathfrak{p}$ is permissible in $\mathcal{O}$ if and only if $\mu(\mathfrak{p}^n) = \mu(\mathfrak{p}^n\mathcal{O}_\mathfrak{p})$ for every $n \geqq 0$.*

PROOF: As mentioned in Remark 3 of the Introduction, we have

$$H^{(d)}_{\mathcal{O}_\mathfrak{p}} \leqq H^{(0)}_{\mathcal{O}} \leqq H^{(d)}_{\mathfrak{p}}.$$

(The second inequality follows from Theorem 1 and the first from [1, Theorem (2)] and [6, page 202].) By Theorem 1, $H^{(0)}_{\mathcal{O}} = H^{(d)}_{\mathfrak{p}}$ if and only if $\mathfrak{p}$ is permissible in $\mathcal{O}$. By Lemma (1.3), $\mathfrak{p}$ is permissible in $\mathcal{O}$ if and only if $H^{(d)}_{\mathcal{O}_\mathfrak{p}} = H^{(0)}_{\mathcal{O}}$. Therefore, $\mathfrak{p}$ is permissible in $\mathcal{O}$ if and only if $H^{(d)}_{\mathcal{O}_\mathfrak{p}} = H^{(d)}_{\mathfrak{p}}$. Now, clearly, $H^{(d)}_{\mathcal{O}_\mathfrak{p}} = H^{(d)}_{\mathfrak{p}}$ if and only if $H^{(0)}_{\mathcal{O}_\mathfrak{p}} = H^{(0)}_{\mathfrak{p}}$. This proves the corollary, since $\mu(\mathfrak{p}^n) = H^{(0)}_{\mathfrak{p}}(n)$ and $\mu(\mathfrak{p}^n\mathcal{O}_\mathfrak{p}) = H^{(0)}_{\mathcal{O}_\mathfrak{p}}(n)$.

PROOF OF THEOREM 1: Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$ and let $k = \mathcal{O}/\mathfrak{m}$. Since $e = \operatorname{emdim} \mathcal{O}/\mathfrak{p}$, there exists an ideal $\mathfrak{q}$ of $\mathcal{O}$ such that $\mathfrak{m} = \mathfrak{p}+\mathfrak{q}$ and $\mu(\mathfrak{q}) = e$. Therefore, the inequality $H^{(0)}_{\mathcal{O}} \leqq H^{(e)}_{\mathfrak{p}}$ follows from Lemma (1.2)(2).

[1] It was pointed out by W. Vogel that the proof of this corollary goes through also for non-excellent $\mathcal{O}$. For it follows, from Lemma 1 of [A. Ljungström, "An inequality between Hilbert functions of certain prime ideals one of which is immediately included in the other", Preprint, University of Stockholm, 1975] that $H^{(d)}_{\mathcal{O}_\mathfrak{p}} \leqq H^{(0)}_{\mathcal{O}}$ for arbitrary $\mathcal{O}$ if $\mathcal{O}/\mathfrak{p}$ is regular of dimension d. It was precisely for this inequality that we assumed the excellence of $\mathcal{O}$. For a more direct proof of this corollary, see [R. Achilles, P. Schenzel and W. Vogel, "Einige Anwendungen der normalen Flachheit", Preprint, Martin-Luther-Universität, 1975].

We now proceed to show that conditions (i), (ii) and (iii) of Theorem 1 are equivalent.

(i) ⇒ *(ii)*. Since $\mathfrak{p}$ is permissible in $\mathcal{O}$, we have $d = e$, and for every $n \geqq 0$, $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is $\mathcal{O}/\mathfrak{p}$-flat, hence $\mathcal{O}/\mathfrak{p}$-free. Therefore, we have

$$\begin{aligned} H_{\mathfrak{p}}^{(0)}(n) &= \dim_k \mathfrak{p}^n/\mathfrak{m}\mathfrak{p}^n \\ &= \dim_k \mathfrak{p}^n/\mathfrak{p}^{n+1} \otimes_{\mathcal{O}/\mathfrak{p}} k \\ &= \operatorname{rank}_{\mathcal{O}/\mathfrak{p}} \mathfrak{p}^n/\mathfrak{p}^{n+1} \\ &= \dim_{\mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p}} \mathfrak{p}^n\mathcal{O}_\mathfrak{p}/\mathfrak{p}^{n+1}\mathcal{O}_\mathfrak{p} \\ &= H_{\mathcal{O}_\mathfrak{p}}^{(0)}(n). \end{aligned}$$

Thus $H_{\mathfrak{p}}^{(0)} = H_{\mathcal{O}_\mathfrak{p}}^{(0)}$, so that $H_{\mathfrak{p}}^{(d)} = H_{\mathcal{O}_\mathfrak{p}}^{(d)} = H_{\mathcal{O}}^{(0)}$, the last equality by Lemma (1.3).

(ii) ⇒ *(iii)*. Since $\mathcal{O}/\mathfrak{p}$ is regular, we have $d = e$.

(iii) ⇒ *(ii)*. We have only to show that $\mathcal{O}/\mathfrak{p}$ is regular. Choose $t_1, \ldots, t_e \in \mathfrak{m}$ such that their canonical images $\bar{t}_1, \ldots, \bar{t}_e$ in $\bar{\mathcal{O}} = \mathcal{O}/\mathfrak{p}$ form a (necessarily minimal) set of generators of $\bar{\mathfrak{m}} = \mathfrak{m}/\mathfrak{p}$. Let $\mathfrak{q} = \sum_{i=1}^{e} t_i\mathcal{O}$. Then $\mathfrak{m} = \mathfrak{p}+\mathfrak{q}$ and $e = \mu(\mathfrak{q})$. Therefore, the assumption $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{p}}^{(e)}$ implies, by Lemma (1.2)(3), that we have

$$(*) \qquad \begin{aligned} \mu(\mathfrak{q}^n) &= \binom{n+e-1}{e-1} \\ \mu(\mathfrak{m}^{n+1}) &= \mu(\mathfrak{q}^{n+1}) + \mu(\mathfrak{m}^n\mathfrak{p}) \end{aligned}$$

for every $n \geqq 0$. Let $S_n = \{t^\alpha \mid |\alpha| = n\}$. (Here we have used the standard notation: $t^\alpha = t_1^{\alpha_1} \ldots t_e^{\alpha_e}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_e$ for $\alpha = (\alpha_1, \ldots, \alpha_e) \in (\mathbb{Z}^+)^e$.) It follows from (*) and Lemma (1.2)(1) that the following two statements are true for every $n \geqq 0$.

$(1)_n$ S_n is a minimal set of generators of $\mathfrak{q}^n$.

$(2)_n$ If T_n is any minimal set of generators of $\mathfrak{m}^n\mathfrak{p}$, then $T_n \cup S_{n+1}$ is a minimal set of generators of $\mathfrak{m}^{n+1}$.

Suppose now that $\mathcal{O}/\mathfrak{p}$ is not regular. Then there exists $r \in \mathbb{Z}^+$ and $\alpha = (\alpha_1, \ldots, \alpha_e) \in (\mathbb{Z}^+)^e$ with $|\alpha| = r$ such that

$$\bar{t}^\alpha \in \sum_{\substack{|\beta| = r \\ \beta \neq \alpha}} \bar{t}^\beta \bar{\mathcal{O}} + \bar{\mathfrak{m}}^{r+1}.$$

This means that

$$t^\alpha \in \sum_{x \in S_r - \{t^\alpha\}} x\mathcal{O} + \mathfrak{m}^{r+1} + \mathfrak{p}.$$

We can therefore write $t^\alpha = y + p$ with $p \in \mathfrak{p}$ *and*

$$y \in \sum_{x \in S_r - \{t^\alpha\}} x\mathcal{O} + \mathfrak{m}^{r+1}.$$

If $p \neq 0$, let $s \in \mathbb{Z}^+$ be such that $p \in \mathfrak{m}^s\mathfrak{p} - \mathfrak{m}^{s+1}\mathfrak{p}$. Then there exists a minimal set T of generators of $\mathfrak{m}^s\mathfrak{p}$ such that $p \in T$. If $p = 0$, we put $s = \infty$. Now consider the three cases $s+1 < r$, $s+1 = r$ and $s+1 > r$.

Case (1). $s+1 < r$. Then $p = t^\alpha - y \in \mathfrak{m}^r \subset \mathfrak{m}^{s+2}$. This contradicts $(2)_s$, since we may take $T_s = T$, so that $p \in T_s$.

Case (2). $s+1 = r$. In this case we have

$$t^\alpha = y + p \in \sum_{x \in S_r - \{t^\alpha\}} x\mathcal{O} + p\mathcal{O} + \mathfrak{m}^{s+2},$$

which again contradicts $(2)_s$, by taking $T_s = T$.

Case (3). $s+1 > r$. In this case $p \in \mathfrak{m}^s\mathfrak{p} \subset \mathfrak{m}^{r+1}$, so that we have

$$t^\alpha = y + p \in \sum_{x \in S_r - \{t^\alpha\}} x\mathcal{O} + \mathfrak{m}^{r+1},$$

which contradicts $(2)_{r-1}$.

This shows that $\mathcal{O}/\mathfrak{p}$ is regular and $d = e$, which proves (ii).

(ii) $\Rightarrow$ *(i)*. We prove this implication by induction on d. The case $d = 0$ is trivial. We shall now prove:

(A) *The implication* (ii) $\Rightarrow$ (i) *for* $d = 1$.

(B) *The inductive step from* $d-1$ *to* d, *assuming* (A).

We first prove (B). Let $d \geqq 1$ and let $t_1, \ldots, t_d \in \mathfrak{m}$ be such that $\mathfrak{m} = \mathfrak{p} + \sum_{i=1}^d t_i\mathcal{O}$. Let $\mathfrak{n} = \mathfrak{p} + \sum_{i=1}^{d-1} t_i\mathcal{O}$. Then $\mathfrak{m} = \mathfrak{n} + t_e\mathcal{O}$. Therefore $H_{\mathcal{O}}^{(0)} \leqq H_{\mathfrak{n}}^{(1)}$, by Lemma (1.2)(2). Also $H_{\mathfrak{n}}^{(0)} \leqq H_{\mathfrak{p}}^{(d-1)}$, by Lemma (1.2)(2). Therefore $H_{\mathcal{O}}^{(0)} \leqq H_{\mathfrak{n}}^{(1)} \leqq H_{\mathfrak{p}}^{(d)}$. Since $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{p}}^{(d)}$, we get $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{n}}^{(1)}$. Now $\mathcal{O}/\mathfrak{n}$ is regular of dimension 1. Therefore, by (A), $H_{\mathcal{O}}^{(0)} = H_{\mathfrak{n}}^{(1)}$ implies that $\mathfrak{n}$ is permissible in $\mathcal{O}$. Hence

$$(*) \qquad H_{\mathcal{O}}^{(0)} = H_{\mathcal{O}_\mathfrak{n}}^{(1)}$$

by Lemma (1.3). Thus $H_{\mathcal{O}_\mathfrak{n}}^{(1)} = H_{\mathfrak{p}}^{(d)}$, which gives $H_{\mathcal{O}_\mathfrak{n}}^{(0)} = H_{\mathfrak{p}}^{(d-1)}$. This implies that $H_{\mathcal{O}_\mathfrak{n}}^{(0)} \geqq H_{\mathfrak{p}\mathcal{O}_\mathfrak{n}}^{(d-1)}$, since $\mu(\mathfrak{p}^n\mathcal{O}_\mathfrak{n}) \leqq \mu(\mathfrak{p}^n)$ for every n. On the other hand, by Lemma (1.2)(2), we have $H_{\mathcal{O}_\mathfrak{n}}^{(0)} \leqq H_{\mathfrak{p}\mathcal{O}_\mathfrak{n}}^{(d-1)}$, since

$$\mathfrak{n}\mathcal{O}_\mathfrak{n} = \mathfrak{p}\mathcal{O}_\mathfrak{n} + \sum_{i=1}^{d-1} t_i\mathcal{O}_\mathfrak{n}.$$

Thus $H^{(0)}_{\mathcal{O}_\mathfrak{n}} = H^{(d-1)}_{\mathfrak{p}\mathcal{O}_\mathfrak{n}}$. This implies, by induction hypothesis, that $\mathfrak{p}\mathcal{O}_\mathfrak{n}$ is permissible in $\mathcal{O}_\mathfrak{n}$, since $\mathcal{O}_\mathfrak{n}/\mathfrak{p}\mathcal{O}_\mathfrak{n}$ is regular of dimension $d-1$. Therefore $H^{(d-1)}_{\mathcal{O}_\mathfrak{p}} = H^{(0)}_{\mathcal{O}_\mathfrak{n}}$, by Lemma (1.3). This gives $H^{(d)}_{\mathcal{O}_\mathfrak{p}} = H^{(1)}_{\mathcal{O}_\mathfrak{n}} = H^{(0)}_{\mathcal{O}}$, by (*). Therefore, by Lemma (1.3), $\mathfrak{p}$ is permissible in $\mathcal{O}$, and (B) is proved.

We now turn to the proof of (A). We are given that $\mathcal{O}/\mathfrak{p}$ is a discrete valuation ring and $H^{(0)}_{\mathcal{O}} = H^{(1)}_{\mathfrak{p}}$. We have to show that $\mathrm{gr}_\mathfrak{p}(\mathcal{O})$ is $\mathcal{O}/\mathfrak{p}$-flat or, equivalently, that $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is $\mathcal{O}/\mathfrak{p}$-free for every $n \geqq 0$. Choose $t \in \mathfrak{m}$ such that its image $\bar{t}$ in $\mathcal{O}/\mathfrak{p}$ is a uniformising parameter for $\mathcal{O}/\mathfrak{p}$. It is then enough to show that $\bar{t}$ is a non-zero divisor in $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ for every $n \geqq 0$.

By the choice of t, we have $\mathfrak{m} = \mathfrak{p} + t\mathcal{O}$. Therefore the equality $H^{(0)}_{\mathcal{O}} = H^{(1)}_{\mathfrak{p}}$ implies, by Lemma (1.2)(3), that $\mu(t^m\mathfrak{p}^n) = \mu(\mathfrak{p}^n)$ for all $m, n \geqq 0$, so that $\mu(\mathfrak{m}^n) = \sum_{i=0}^{n} \mu(t^i\mathfrak{p}^{n-i})$.

Suppose now that there exists $n \geqq 0$ such that $\bar{t}$ is a zero-divisor in $\mathfrak{p}^n/\mathfrak{p}^{n+1}$. Then there exists $p \in \mathfrak{p}^n - \mathfrak{p}^{n+1}$ such that $tp \in \mathfrak{p}^{n+1}$. We consider the two cases $p \notin \mathfrak{m}\mathfrak{p}^n$ and $p \in \mathfrak{m}\mathfrak{p}^n$.

Case (1). $p \notin \mathfrak{m}\mathfrak{p}^n$. In this case p can be completed to a minimal set, say S, of generators of $\mathfrak{p}^n$. Then $tS = \{tx | x \in S\}$ is a minimal set of generators of $t\mathfrak{p}^n$, since $\mu(t\mathfrak{p}^n) = \mu(\mathfrak{p}^n)$, as noted above. But this is a contradiction, by Lemma (1.2)(1), of the equality

$$\mu(\mathfrak{m}^{n+1}) = \sum_{i=0}^{n+1} \mu(t^i\mathfrak{p}^{n+1-i}),$$

since $tp \in tS \cap \mathfrak{p}^{n+1}$.

Case (2)[2] $p \in \mathfrak{m}\mathfrak{p}^n$. Since $\mathfrak{m}\mathfrak{p}^n = (\mathfrak{p} + t\mathcal{O})\mathfrak{p}^n = \mathfrak{p}^{n+1} + t\mathfrak{p}^n$, we can write $p = q'_{n+1} + t^{\alpha_0 - 1}p_n$ with $q'_{n+1} \in \mathfrak{p}^{n+1}$, $p_n \in \mathfrak{p}^n$ and α_0 an integer $\geqq 2$. Since $p \notin \mathfrak{p}^{n+1}$, we may choose q'_{n+1}, α_0 and p_n to be such that $p_n \in \mathfrak{p}^n - \mathfrak{m}\mathfrak{p}^n$. Now $tp = tq'_{n+1} + t^{\alpha_0}p_n$. Put $q_{n+1} = t^{\alpha_0}p_n = tp - tq'_{n+1}$. Then $q_{n+1} \in \mathfrak{p}^{n+1}$. Suppose $q_{n+1} \in \mathfrak{m}\mathfrak{p}^{n+1} = \mathfrak{p}^{n+2} + t\mathfrak{p}^{n+1}$. Then we can write $q_{n+1} = q_{n+2} - t^{\alpha_1}p_{n+1}$ with $q_{n+2} \in \mathfrak{p}^{n+2}$, $\alpha_1 \geqq 1$ and $p_{n+1} \in \mathfrak{p}^{n+1}$. Now, if $q_{n+1} \notin \mathfrak{p}^{n+2}$, we may assume (by choosing $q_{n+2}, \alpha_1, p_{n+1}$ suitably) that $p_{n+1} \in \mathfrak{p}^{n+1} - \mathfrak{m}\mathfrak{p}^{n+1}$. If $q_{n+1} \in \mathfrak{p}^{n+2}$, then we put $q_{n+2} = q_{n+1}$, $p_{n+1} = 0$ and $\alpha_1 = \alpha_0 + 1$. We get $q_{n+2} = t^{\alpha_0}p_n + t^{\alpha_1}p_{n+1}$. Proceeding thus, we write

$$(**) \qquad q_{n+r+1} = t^{\alpha_0}p_n + t^{\alpha_1}p_{n+1} + \dots + t^{\alpha_r}p_{n+r},$$

[2] The author wishes to express his thanks to the referee for pointing out a correction in the proof of this case.

where $q_{n+r+1} \in \mathfrak{p}^{n+r+1}$ and for every $i, 0 \leqq i \leqq r$, either $p_{n+i} \in \mathfrak{p}^{n+i} - \mathfrak{m}\mathfrak{p}^{n+i}$ and $\alpha_i \geqq 1$ or $p_{n+i} = 0$ and $\alpha_i = \alpha_0 + 1$. Now suppose we have obtained q_{n+r+1} for a given $r \geqq 0$. For this r, let

$$\alpha = \inf\{\alpha_0, \alpha_1 + 1, \ldots, \alpha_r + r\}$$

and let

$$J = \{j | 0 \leqq j \leqq r \text{ and } \alpha = \alpha_j + j\}.$$

Then J is not empty, $\alpha_j = \alpha - j$ for every j in J and from (**) we get

(***) $$q_{n+r+1} \equiv \sum_{j \in J} t^{\alpha - j} p_{n+j} (\text{mod } \mathfrak{m}^{n+\alpha+1}).$$

Now, since $p_{n+j} \in \mathfrak{p}^{n+j} - \mathfrak{m}\mathfrak{p}^{n+j}$ for every $j \in J$, we can complete p_{n+j} to a minimal set of generators of $\mathfrak{p}^{n+j}$. Therefore, since we have

$$\mu(\mathfrak{m}^{n+\alpha}) = \sum_{i=0}^{n+\alpha} \mu(t^{n+\alpha-i}\mathfrak{p}^i),$$

we see by Lemma (1.2) that the set $\{t^{\alpha-j}p_{n+j} | j \in J\}$ can be completed to a minimal set of generators of $\mathfrak{m}^{n+\alpha}$. In particular, $\sum_{j \in J} t^{\alpha-j} p_{n+j}$ is not in $\mathfrak{m}^{n+\alpha+1}$, since J is non-empty. Therefore, by (***), q_{n+r+1} is not in $\mathfrak{m}^{n+\alpha+1}$. Therefore, since $q_{n+r+1} \in \mathfrak{p}^{n+r+1}$, we conclude that $n+r+1 < n+\alpha+1$, so that $r < \alpha \leqq \alpha_0$.

This shows that the process of generating the q_{n+r+1} cannot go on indefinitely, i.e. we must eventually come to an r for which q_{n+r+1} is not in $\mathfrak{m}\mathfrak{p}^{n+r+1}$. For this r, q_{n+r+1} can be completed to a minimal set of generators of $\mathfrak{p}^{n+r+1}$ and hence of $\mathfrak{m}^{n+r+1}$ by Lemma (1.2), since by hypothesis

$$\mu(\mathfrak{m}^{n+r+1}) = \sum_{i=0}^{n+r+1} \mu(t^i \mathfrak{p}^{n+r+1-i}).$$

Now if $\alpha > r+1$ then (***) shows that $q_{n+r+1} \in m^{n+r+2}$, which is a contradiction. If $\alpha = r+1$ then, by Lemma (1.2), the set

$$\{q_{n+r+1}\} \cup \{t^{\alpha-j}p_{n+j} | j \in J\}$$

can be completed to a minimal set of generators of $\mathfrak{m}^{n+\alpha}$. This contradicts (***).

Thus (A) is proved, and the proof of the theorem is complete.

2. Proof of Theorems 2 and 3

(2.1) The proof of Theorems 2 and 3 is contained essentially in the proof of the Main Theorem in [6]. What is needed is elaboration of certain points. We do this in the proof below, referring frequently to [6].

(2.2) We have the following situation: $\mathfrak{p}$ is a proper ideal of $\mathcal{O}$, and $\mathcal{O} \xrightarrow{h} \mathcal{O}'$ is a blowing-up of $\mathcal{O}$ with center $\mathfrak{p}$. We have $e = \operatorname{emdim} \mathcal{O}/\mathfrak{p}$ and $\delta = \operatorname{tr.deg}_k k'$, where $k \to k'$ is the residue field extension induced by h. We are given $t = (t_0, t_1, \ldots, t_e)$ with $t_0 \in \mathfrak{p}$ such that $\mathfrak{p}\mathcal{O}' = t_0\mathcal{O}'$ and $t_i \in \mathfrak{m}$, $1 \leqq i \leqq e$, such that $\mathfrak{m} = \mathfrak{p} + \sum_{i=1}^{e} t_i\mathcal{O}$. The ideals $\mathfrak{a}_{t,i}(n)$ of $\mathcal{O}'$ and $\mathfrak{b}_{t_0}(n)$ of $\mathcal{O}$ and the numerical functions $L_{t,i}$, $1 \leqq i \leqq e$, and L_{t_0} are defined as in the Introduction. Let $\mathcal{O}'' = \mathcal{O}'/\mathfrak{m}\mathcal{O}'$.

With the notation of (2.2) we shall prove the following three lemmas:

(2.3) Lemma:

$$H_{\mathcal{O}'}^{(0)} = H_{\mathcal{O}''}^{(e+1)} - \sum_{i=0}^{e} L_{t,i}^{(i)}.$$

(2.4) Lemma: *If* $k = k'$ *then* $H_{\mathfrak{p}}^{(0)} = H_{\mathcal{O}''}^{(1)} + L_{t_0}$.

(2.5) Lemma: $H_{\mathfrak{p}}^{(0)} \geqq H_{\mathcal{O}''}^{(1+\delta)}$.

Assume these three lemmas for the moment. Then we get an immediate

Proof of Theorems 2 and 3: Since $D_{\mathfrak{p}} = H_{\mathfrak{p}}^{(e)} - H_{\mathcal{O}}^{(0)}$, we have

$$\begin{aligned} H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(\delta)} &= H_{\mathfrak{p}}^{(e)} - H_{\mathcal{O}'}^{(\delta)} - D_{\mathfrak{p}} \\ &= H_{\mathfrak{p}}^{(e)} - H_{\mathcal{O}''}^{(e+1+\delta)} + \sum_{i=0}^{e} L_{t,i}^{(i+\delta)} - D_{\mathfrak{p}} && \text{(Lemma (2.3))} \\ &\geqq \sum_{i=0}^{e} L_{t,i}^{(i+\delta)} - D_{\mathfrak{p}} && \text{(Lemma (2.5)).} \end{aligned}$$

This proves Theorem 2. Now, if $k = k'$, then

$$\begin{aligned} H_{\mathcal{O}}^{(0)} - H_{\mathcal{O}'}^{(0)} &= H_{\mathfrak{p}}^{(e)} - H_{\mathcal{O}''}^{(e+1)} + \sum_{i=0}^{e} L_{t,i}^{(i)} - D_{\mathfrak{p}} \quad \text{(as above, since } \delta = 0\text{)} \\ &= L_{t_0}^{(e)} + \sum_{i=0}^{e} L_{t,i}^{(i)} - D_{\mathfrak{p}} && \text{(Lemma (2.4)).} \end{aligned}$$

This proves Theorem 3.

PROOF OF LEMMA (2.3): Since $\mathcal{O}'' = \mathcal{O}'/\sum_{i=0}^{e} t_i\mathcal{O}'$, the lemma follows from [6, Theorem 1] and a straightforward induction on e.

PROOF OF LEMMA (2.4): Let $\mathfrak{m}''$ be the maximal ideal of $\mathcal{O}''$. It is enough to show that there exists an exact sequence

$$(*) \qquad 0 \to \mathfrak{b}_{t_0}(n)/\mathfrak{m}\mathfrak{p}^n \to \mathfrak{p}^n/\mathfrak{m}\mathfrak{p}^n \xrightarrow{\varphi} \mathcal{O}''/\mathfrak{m}''^{n+1} \to 0$$

of k-vector spaces. For we have

$$H_{\mathfrak{p}}^{(0)}(n) = \dim_k \mathfrak{p}^n/\mathfrak{m}\mathfrak{p}^n, \qquad H_{\mathcal{O}''}^{(1)}(n) = \dim_k \mathcal{O}''/\mathfrak{m}''^{n+1}$$

and

$$L_{t_0}(n) = \text{length}_{\mathcal{O}}\, \mathfrak{b}_{t_0}(n)/\mathfrak{m}\mathfrak{p}^n = \dim_k \mathfrak{b}_{t_0}(n)/\mathfrak{m}\mathfrak{p}^n.$$

To show the existence of (*) we have only to define φ suitably. Since $\mathfrak{p}\mathcal{O}' = t_0\mathcal{O}'$, we can identify $\mathcal{O}'$ with a localization of the subring $\{f/t_0^n | n \geqq 0, f \in \mathfrak{p}^n\}$ of $\mathcal{O}_{t_0}$. Define $\psi: \mathfrak{p}^n \to \mathcal{O}''$ by $\psi(f) = \eta(f/t_0^n)$, where $\eta: \mathcal{O}' \to \mathcal{O}''$ is the canonical homomorphism. Then ψ induces a k-homomorphism $\bar{\psi}: \mathfrak{p}^n/\mathfrak{m}\mathfrak{p}^n \to \mathcal{O}''$. We define φ to be the composite of $\bar{\psi}$ and the canonical homomorphism $\mathcal{O}'' \to \mathcal{O}''/\mathfrak{m}''^{n+1}$. It was proved in [6, (3.3), Proof of Lemma 2] that φ is surjective if $k = k'$. Also, it is clear from the definition of $\mathfrak{b}_{t_0}(n)$ that $\ker \varphi = \mathfrak{b}_{t_0}(n)/\mathfrak{m}\mathfrak{p}^n$. Thus (*) is exact and the lemma is proved.

PROOF OF LEMMA (2.5): By Lemma (2.4), we already have the inequality $H_{\mathfrak{p}}^{(0)} \geqq H_{\mathcal{O}''}^{(1+\delta)}$ in the case $k = k'$. The inequality in the general case can now be proved by a standard inductive procedure used in [1], [4] and [6]. What we do is the following: Choose an element $\alpha \in k' - k$. If $\delta \geqq 1$, we assume that α is transcendantal. If $\delta = 0$, we assume that α is either separable or purely inseparable. Let $\bar{f}(Z) \in k[Z]$ be the minimal monic polynomial of α over k. (If α is transcendental, we take $\bar{f}(Z) = 0$.) Let $f(Z) \in \mathcal{O}[Z]$ be a monic lift of $\bar{f}(Z)$ such that, for every $i \geqq 0$, if the coefficient of Z^i in $\bar{f}(Z)$ is 0 then the coefficient of Z^i in $f(Z)$ is also 0. Let $\tilde{\mathcal{O}}$ be the localization of $0[Z]/f(Z)\mathcal{O}[Z]$ at the prime ideal $\mathfrak{n} = (\mathfrak{m}[Z]+f(Z)\mathcal{O}[Z])/f(Z)\mathcal{O}[Z]$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. Let $\eta: \mathcal{O} \to \tilde{\mathcal{O}}$ be the canonical homomorphism. Let a be a lift of α to $\mathcal{O}'$ and let $\tilde{\mathcal{O}}'$ be the localization of $\mathcal{O}'[Z]/f(Z)\mathcal{O}'[Z]$ at the maximal ideal

$$\mathfrak{n}' = (\mathfrak{m}'[Z]+(Z-a)\mathcal{O}'[Z])/f(Z)\mathcal{O}'[Z],$$

where $\mathfrak{m}'$ is the maximal ideal of $\mathcal{O}'$. Let $\eta' : \mathcal{O}' \to \tilde{\mathcal{O}}'$ be the canonical homomorphism. Then there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{h} & \mathcal{O}' \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta'} \\ \tilde{\mathcal{O}} & \xrightarrow{\tilde{h}} & \tilde{\mathcal{O}}' \end{array}$$

such that

(i) $\tilde{h}$ is a blowing-up of $\tilde{\mathcal{O}}$ with center $\tilde{\mathfrak{p}} = \mathfrak{p}\tilde{\mathcal{O}}$;

(ii) the residue field extension induced by $\tilde{h}$ is the k-inclusion $k(\alpha) \to k'$. (See [6, (4.3), (4.6)].) Let $\tilde{\delta} = \text{tr.deg}_{k(\alpha)} k'$. If $\delta \geqq 1$, then $\tilde{\delta} = \delta - 1$. If $\delta = 0$, then $[k' : k(\alpha)] < [k' : k]$. Therefore, by an obvious induction, we may assume that $H^{(0)}_{\tilde{\mathfrak{p}}} \geqq H^{(1+\tilde{\delta})}_{\tilde{\mathcal{O}}''}$, where $\tilde{\mathcal{O}}'' = \tilde{\mathcal{O}}'/\tilde{\mathfrak{m}}\tilde{\mathcal{O}}'$, $\tilde{\mathfrak{m}}$ being the maximal ideal of $\tilde{\mathcal{O}}$. Now, in order to complete the proof of the lemma, it is clearly enough to prove the following three statements:

(1) $H^{(0)}_{\mathfrak{p}} = H^{(0)}_{\tilde{\mathfrak{p}}}$.

(2) $H^{(0)}_{\tilde{\mathcal{O}}''} \geqq H^{(0)}_{\mathcal{O}''}$ *if* $\tilde{\delta} = \delta = 0$.

(3) $H^{(0)}_{\tilde{\mathcal{O}}''} = H^{(1)}_{\mathcal{O}''}$ *if* $\tilde{\delta} = \delta - 1$.

Proof of (1): Let $\tilde{k} = k(\alpha)$ be the residue field of $\tilde{\mathcal{O}}$. For every $n \geqq 0$, we have $H^{(0)}_{\tilde{\mathfrak{p}}}(n) = \dim_{\tilde{k}} \tilde{\mathfrak{p}}^n \otimes_{\tilde{\mathcal{O}}} \tilde{k} = \dim_{\tilde{k}} \mathfrak{p}^n \otimes_{\mathcal{O}} \tilde{k}$, since, $\tilde{\mathcal{O}}$ being $\mathcal{O}$-flat, we have $\tilde{\mathfrak{p}}^n \approx \mathfrak{p}^n \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$. Now $\mathfrak{p}^n \otimes_{\mathcal{O}} \tilde{k} \approx (\mathfrak{p}^n \otimes_{\mathcal{O}} k) \otimes_k \tilde{k}$. Therefore,

$$\dim_{\tilde{k}} \mathfrak{p}^n \otimes_{\mathcal{O}} \tilde{k} = \dim_k \mathfrak{p}^n \otimes_{\mathcal{O}} k = H^{(0)}_{\mathfrak{p}}(n).$$

Proof of (2) and (3): Let $\mathfrak{m}''$ be the maximal ideal of $\mathcal{O}''$. Then $\tilde{\mathcal{O}}'' = \tilde{\mathcal{O}}'/\tilde{\mathfrak{m}}\tilde{\mathcal{O}}' = \tilde{\mathcal{O}}'/\mathfrak{m}\tilde{\mathcal{O}}' = (\mathcal{O}''[Z]/f(Z)\mathcal{O}''[Z])_{\mathfrak{n}''}$, where

$$\mathfrak{n}'' = (\mathfrak{m}''[Z] + (Z-a)\mathcal{O}''[Z])f(Z)\mathcal{O}''[Z].$$

Now, if $\tilde{\delta} = \delta - 1$, then α is transcendental and $f(Z) = 0$. Therefore the equality $H^{(0)}_{\tilde{\mathcal{O}}''} = H^{(1)}_{\mathcal{O}''}$ is clear in this case. This proves (3). If $\delta = 0$ and α is separable then $\bar{f}(Z)$ being a separable polynomial, $\mathcal{O}'' \to \tilde{\mathcal{O}}''$ is etale, so that in this case we have, in fact, $H^{(0)}_{\tilde{\mathcal{O}}''} = H^{(0)}_{\mathcal{O}''}$. Now suppose $\delta = 0$ and α is purely inseparable. Then $\bar{f}(Z) = Z^q - \beta$, where q is a power of char k and $\beta = \alpha^q \in k$. This implies that $f(Z) = Z^q - b$, where $b \in \mathcal{O}$ is some lift of β. Let $\bar{b}$ be the canonical image of b in $\mathcal{O}''$. Since $\mathcal{O}''[Z]/(Z^q - \bar{b})\mathcal{O}''[Z]$ is already a local ring, we have

$$\tilde{\mathcal{O}}'' = \mathcal{O}''[Z]/(Z^q - \bar{b})\mathcal{O}''[Z].$$

Let $\bar{a}$ be the canonical image of a in $\mathcal{O}''$ and let $t = \bar{b} - \bar{a}^q$. Then $t \in \mathfrak{m}''$. Let $Y = Z - \bar{a}$. Then $\tilde{\mathcal{O}}'' = \mathcal{O}''[Y]/(Y^q - t)\mathcal{O}''[Y]$. Now, the inequality $H^{(0)}_{\tilde{\mathcal{O}}''} \geqq H^{(0)}_{\mathcal{O}''}$ follows from [6, Lemma (4.5)]. This proves (2).

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