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ZERO CYCLES ON SURFACES WITH $p_g = 0$*

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Let X be a nonsingular projective algebraic surface over the complex numbers. We use the standard notations

K_X = Canonical bundle of X
$P_g = \dim \Gamma(X, K_X)$

κ = Kodaira dimension $X = \dim \operatorname{Proj} \bigoplus_{n>0} \Gamma(X, K_X^{\otimes n})$.

$\operatorname{Alb}(X)$ = (group of C-points of) the Albanese variety of X.

In addition, we denote by $T(X)$ the group of zero cycles of degree zero on X which map to zero in $\operatorname{Alb}(X)$, modulo those rationally equivalent to zero. Bloch's conjecture [7] asserts that $p_g = 0$ implies $T(X) = 0$. Mumford [3] has shown $p_g \neq 0$ implies $T \neq 0$. See also Roitman [4]. We verify the conjecture under the assumption that the Kodaira dimension, κ, satisfies $\kappa < 2$. If X is an Enriques surface we find that any two points are rationally equivalent, contradicting Severi's assertion that this condition should imply $H^1(X, \boldsymbol{Z}) = 0$. The conjecture is clearly birational in X.

Note that if $\kappa < 0$ then X is either rational, or is birational to $C \times \mathbb{P}^1$ where C is nonsingular and is the image of $X \to \operatorname{Alb}(X)$. In the former case every degree zero cycle on X is rationally equivalent to zero. In the latter case every zero cycle on $C \times \mathbb{P}^1$ is rationally equivalent to a cycle on $C \times \{0\}$ and the rational equivalence class (on C) of this cycle is completely determined by its degree and its image in $\operatorname{Alb}(X) = \operatorname{Alb}(C)$.

Hence we may assume $0 \leq \kappa < 2$. A rapid glance through the classification of surfaces will reveal that in this range the hypothesis

* Appendix: Zero cycles on abelian surfaces by S. Bloch.

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$p_g = 0$ implies that X is in fact elliptic. We verify below that the associated Jacobian fibration J also satisfies $p_g = 0$ and that the conjecture for X is implied by that for J. The irregularity q is necessarily 0 or 1. In the former case we show that J is rational and in the latter case we employ the classification of surfaces with $p_g = 0$, $q = 1$ to complete the proof. These surfaces all provide counter examples to assertions made in [4b] and [8]. The authors acknowledge helpful conversations with A. Roitman, T. Suwa, P. Murthy, R. Swan, and D. Mumford. The results in this paper have been independently obtained by A. Roitman. The results for the Enriques surface were sketched to us by M. Artin. His remarks form an integral part of the present argument.

REMARK 1: In general our proofs show that given X there exists an integer $N \neq 0$ such that $N \cdot T = 0$. The result $T = 0$ then follows from:

PROPOSITION 1: *The group $T(X)$ is divisible.*

(Hence given $N \neq 0$ such that $N \cdot Z = 0$ for all $Z \in T$, then since any $W \in T$ is of the form $N \cdot Z$, $Z \in T$ it follows $W = N \cdot Z = 0$.)

PROOF: Let $\mathfrak{z}(X)$ denote the cycles of degree zero modulo rational equivalence. Note that $\mathfrak{z}(X)$ is divisible, indeed given $W \in \mathfrak{z}(X)$ there exists a curve C and a correspondence Y on $C \times X$ such that W lies in the image of $Y : \mathrm{Pic}_0(C) \to \mathfrak{z}(X)$ (e.g. take C to be a nonsingular curve on X passing through all points of W and Y to be the graph of the inclusion.) Noting that $\mathrm{Pic}_0(C)$ is divisible, one can "divide" W, in $Y(\mathrm{Pic}_0(C))$, hence in $\mathfrak{z}(X)$.

In view of the exact sequence

$$0 \to T \to \mathfrak{z}(X) \to \mathrm{Alb}\,(X) \to 0$$

and the divisibility of $\mathfrak{z}$ and Alb, the asserted divisibility of T is equivalent to the assertion that N-torsion in $\mathfrak{z}(X)$ maps onto N-torsion in Alb (X) for all N. Now let $C \subseteq X$ be a smooth hyperplane section of X. Note that for any compact Kahler manifold M, one has a canonical isomorphism $\mathrm{Alb}\,(M) \xrightarrow{\sim} H_1(M, \mathbb{R}/\boldsymbol{Z})$, (by the usual identification of Alb (M) as the cokernel of $H_1(M, \boldsymbol{Z}) \to H_1(M, \mathbb{R})$.) The N-torsion in Alb is therefore identifiable with the image of $H_1(M, \boldsymbol{Z}/N\boldsymbol{Z}) \to H_1(M, \mathbb{R}/\boldsymbol{Z})$ (induced by the exact sequence $0 \to \boldsymbol{Z}/N\boldsymbol{Z} \to \mathbb{R}/\boldsymbol{Z} \xrightarrow{N} \mathbb{R}/\boldsymbol{Z} \to 0$.) The natural diagram

$$\begin{array}{ccc} H_1(C, \mathbf{Z}/N\mathbf{Z}) & \to & H_1(X, \mathbf{Z}/N\mathbf{Z}) \\ \downarrow & & \downarrow \\ H_1(C, \mathbb{R}/\mathbf{Z}) & \to & H_1(X, \mathbb{R}/\mathbf{Z}) \end{array}$$

has surjective rows (Lefschetz Theorem), hence the N-torsion in Alb (C) maps onto that in Alb (X). Our required surjectivity now follows immediately from the commutative diagram

$$\begin{array}{ccc} \mathfrak{z}(C) & \to & \mathfrak{z}(X) \\ \downarrow & & \downarrow \\ \mathrm{Alb}\,(C) & \to & \mathrm{Alb}\,(X) \end{array}$$

REMARK 2: The above proposition is valid for X of arbitrary dimension. The argument also shows that the group of codimension p cycles algebraically equivalent to zero modulo those $\sim$ equivalent to zero is divisible where $\sim$ denotes any adequate equivalence relation (in the sense of Samuel [6]). Indeed, all such cycles come via correspondences from jacobians of curves, and the jacobians are divisible. We assume in the sequel that $p_g = 0$, $0 \leq \kappa < 2$ and employ the standard notations: K = canonical divisor, q = irregularity; $P_r = h^0(rK)$; $\chi = 2 - 4q + h^{1,1}$ denotes the topological Euler characteristic.

PROPOSITION 2: *If* $0 \leq \kappa < 2$ *then* $K^2 = 0$.

PROOF: Indeed if $K^2 < 0$ then $\kappa < 0$, ([5], VIII, §1) while if $K^2 > 0$ and $\kappa \geq 0$ then the Riemann-Roch estimate shows P_r grows quadratically in r whence $\kappa = 2$.

The Noether formula $K^2 + \chi = 12(1 - q + P_g)$ becomes

$$\chi = 12(1 - q)$$

which combines with the definition of χ to yield

$$10 = 8q + h^{1,1}$$

whence either (a) $q = 1, h^{1,1} = 2, \chi = 0$ or (b) $q = 0, h^{1,1} = 10, \chi = 12$.

One has the well known

PROPOSITION 3: *If* $\kappa = p_g = q = 0$ *then* $2K = 0$ *and* X *is an Enriques surface.*

PROOF: See, for example, [5], VIII, §1.

The Enriques surface is known to be elliptic, (Kodaira [2] refers for this result to [5], Chapter X, although the result is not explicitly stated there). On the other hand all surfaces with $\kappa = 1$ are known to be elliptic. Moreover, if $q = 1$ then since $\chi = 0$ the Albanese mapping $X \to E$ either is a (locally analytically trivial) fibration of X with fibre F a curve of genus > 1 (in which case $\kappa = 1$), or has general fibre a curve of genus 1 and nonreduced special fibres, (c.f. [5], IV, §7). Thus in every case X admits an elliptic fibration $\pi : X \to B$. Let $J \to B$ denote the associated Jacobian fibration, ([5], VII, §5, or [1], II).

Fixing a "multisection" $Y \subseteq X$ for the map $X \to B$ (i.e. a divisor on X mapping finitely to B) one may construct a rational dominant map $f : X \to J$ as follows. Let $Y \cap \pi^{-1}(b) = \Sigma_{i=1}^n p_i(b)$ and map $X \times_B X \times_B \cdots \times_B X \to J$ by

$$q = (q_1, \ldots, q_n) \to \sum_{i=1}^n q_i - \sum_{i=1}^n p_i(\pi(q)),$$

where the q_i are points of a single general fibre X_b, $b = \pi(q_i)$. Define f by composing with the multidiagonal. Since f is dominant we see that $P_r(J) \leq P_r(X)$, in particular $p_g(J) = 0$.

PROPOSITION 4: *If* $T(J) = 0$ *then* $T(X) = 0$.

PROOF: Note that the map f has a "quasi inverse." Namely given any point $\alpha \in J$ lying over $b \in B$ there is a unique point $q_i(b)$ on X such that $q_i(b) - p_i(b)$ represents $\alpha \in \mathrm{Pic}_0(\pi^{-1}(b))$. The correspondence $\lambda : \alpha \to \Sigma_{i=1}^n\ q_i$ satisfies

$$f_*(\lambda(\alpha)) = \sum_{i=1}^n \left(n(q_i) - \sum_{j=1}^n (p_j)\right) = n\left(\sum_{i=1}^n (q_i) - \sum_{i=1}^n (p_i)\right) = n^2(\alpha).$$

Now given any degree zero cycle Z on X notice that the cycle $n^2 \cdot Z - \lambda(f_*(Z))$ is carried on a finite number of fibres of π and the portion of the cycle carried on a single fibre is rationally equivalent to zero on that fibre (Abel's Theorem).

Indeed, if we fix a fibre $\pi^{-1}(b) = F \subset X$ and let $E \subset J$ be the corresponding elliptic curve, F is a principal homogeneous space under E. We have $p_1, \ldots, p_n \in F$, and the correspondences f_* and λ are given for $q \in F$, $\alpha \in E$, by

$$f_*(q) = \sum_{\text{in } E} (q - p_i) \qquad \lambda(\alpha) = \sum_{\text{in } \mathfrak{z}(F)} (p_i + \alpha).$$

Thus

$$n^2(q) - \lambda f_*(q) = n^2(q) - \sum_{\text{in } \mathfrak{z}(F)} \Big(p_i + \sum_{\text{in } E} (q - p_i)\Big) = 0,$$

the last equality being Abel's Theorem on the genus 1 curve F (i.e. $\mathfrak{z}(F) \cong E$). Hence $n^2 Z - \lambda(f_*(Z)) \sim 0$. If Z vanishes in Alb(X) then $f_*(Z)$ vanishes in Alb(J), and if $T(J) = 0$ we conclude $f_*(Z) \sim 0$, and $\lambda(f_*(Z)) \sim 0$. Thus $n^2 Z \sim 0$. Consequently $n^2 \cdot T(X) = 0$ and we apply remark 1.

To conclude the analysis we study separately the cases $q = 0$ and $q = 1$.

$q = 0$: "Recall the criterion of Castelnuovo that a surface is rational if $P_2 = q = 0$."

We prove J is rational by showing $P_2 = 0$, employing an argument of Kodaira [1], III. We have $\pi : J \to B = \mathbb{P}^1$ together with a section $\sigma : B \to J$. The fact that $B = \mathbb{P}^1$ follows, e.g. because Alb(J) maps onto Alb(B). Noting that the canonical class K_J is an integral combination of fibres of π [5], VII, §3, one has $K_J = \pi^*(K_J \cdot \sigma(B))$. By the adjunction formula, $K_J \cdot \sigma(B) = K_B - N$, where N is the class of the normal bundle of B in J. Thus $h^0(J, r \cdot K_J) = h^0(J, \pi^*(r(K_B - N))) = h^0(B, r(K_B - N))$. However, since $p_g = 0$ we see $h^0(B, (K_B - N)) = 0$. Since $B = \mathbb{P}^1$, necessarily $K_B - N$ is negative. Therefore $h^0(B, r(K_B - N)) = 0$ if $r > 0$ and $P_r(J) = 0$.

$q = 1$: Let $E = $ Alb(X). If the genus of the fibres $\pi : X \to E$ exceeds 1 then, as noted earlier, π is smooth and in fact $X = F \times E/G$ where G is a finite group of translations on E acting on $F \times E$ by $g(f, e) = (\varphi(g)f, e + g)$ for a suitable homomorphism $\varphi : G \to$ Aut(F). If the genus of the fibres is 1 then replacing X by its Jacobian fibration we may assume that $\pi : X \to E$ has a section, and hence is necessarily smooth and of the form $F \times E/G$ as above, (c.f. [5], VII, §9).

Since G acts without fixed points, the global 2-forms on X will be the G invariant global 2-forms on $F \times E$, i.e. the tensors of $\varphi(G)$ invariant 1-forms on F with global 1-forms on E. Since $\varphi(G)$ invariant 1-forms are precisely 1-forms on $F/\varphi(G)$, we see that $p_g(X) = 0$ is equivalent to $\mathbb{P}^1 = F/\varphi(G)$.

Now let Z be any degree zero cycle in $T(X)$. Lifting Z back to $\hat{Z}$ on $F \times E$ we have

$$\hat{Z} = \sum_{g \in G} \sum_i r_i(\varphi(g) \cdot q_i, p_i + g)$$

where $\Sigma_i r_i = 0$ and $\Sigma_i r_i p_i = 0$ in E. Let $n = |G|$. Note that

$n(\varphi(g)\cdot q_i, p_i+g) \sim n\cdot(\varphi(g)\cdot q_i, p_i)$, (Abel) hence

$$n\cdot\hat{Z}\sim n\cdot\sum_i r_i \sum_{g\in G}(\varphi(g)q_i, p_i).$$

However since F/G is *rational* the equivalence class Y of $\Sigma_{g\in G}\,\varphi(g)\cdot q_i$ is independent of i. Hence $n\hat{Z}\sim\Sigma_i\, r_i(Y\times p_i)$. Finally since $\Sigma\, r_i p_i = 0$ we have $n\hat{Z}\sim 0$. However $n^2\cdot Z$ is the image of $n\hat{Z}$ under $F\times E\to X$. Thus $n^2\cdot Z\sim 0$ and $n^2\cdot T(X)=0$, as required.

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Appendix—Zero cycles on an abelian surface

Spencer Bloch

In contrast to the above, it is amusing to see what can be said about the structure of 0-cycles on surfaces with $P_g\neq 0$. In this appendix, I consider the case of abelian and Kummer surfaces.

THEOREM (A.1): *Let A be an abelian surface over an algebraically closed field k of characteristic 0. Let a, b, $c\in A(k)$ be points, and write (a) for the rational equivalence class of a. Then the zero cycle*

$$(a+b+c)-(a+b)-(a+c)-(b+c)+(a)+(b)+(c)-(0)$$

is rationally equivalent to zero. In particular, the map

$$A\times A\to T(A),\ (a,b)\mapsto(a+b)-(a)-(b)+(0)$$

is bilinear, and defines a surjection

$$A \underset{Z}{\otimes} A \twoheadrightarrow T(A).$$

THEOREM (A.2): *With hypotheses as above, intersection of divisors defines a surjective map*

$$\mu_0 : \mathrm{Pic}_0(A) \underset{Z}{\otimes} \mathrm{Pic}_0(A) \twoheadrightarrow T(A).$$

THEOREM (A.3): *Let X be the Kummer surface associated to an abelian surface A. Then $\mathfrak{z}(X)$ is isogenous to $T(A)$.*

These theorems give some indication what sort of structure one can expect $T(X)$ to have (in particular, what sort of boundedness conditions one can hope for) when $P_g > 0$. A general reference for the results abelian varieties needed is [9]. For example the rigidity lemma used in (A.4) is found on page 43. For $X \subset A$ ample divisor, the fact, used in (A.5) that $a \to X_a - X$ defines an isogeny $A \to \mathrm{Pic}_0(A)$ follows from Corollary 4, page 59, and theorem 1, page 77. The fact that any abelian variety is isogenous to a principally polarized abelian variety is essentially the Corollary on page 231.

Some lemmas

It will be convenient to write $CH(X)$ for the group of zero cycles on a surface X modulo rational equivalence. We have exact sequences

$$0 \to Z(X) \to CH(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

$$0 \to T(X) \to Z(X) \to \mathrm{Alb}(X) \to 0.$$

Let $\mu : \mathrm{Pic}(X) \underset{Z}{\otimes} \mathrm{Pic}(X) \to CH(X)$ denote the intersection map, and let μ_0 be the restriction to $\mathrm{Pic}_0(X) \underset{Z}{\otimes} \mathrm{Pic}_0(X)$.

LEMMA (A.4): *Image $\mu_0 \subseteq T(X)$.*

PROOF: Clearly Image $\mu_0 \subseteq Z(X)$. Composing with the map $Z(X) \to \mathrm{Alb}(X)$ we obtain a family of maps $i_\eta : \mathrm{Pic}_0(X) \to \mathrm{Alb}(X)$ $i_\eta(\xi) = \eta \cdot \xi$ parameterized by $\eta \in \mathrm{Pic}_0(X)$. By rigidity, this family is constant, hence equal to $i_0 = 0$-map. Q.E.D.

LEMMA (A.5): *Let $f: A' \to A$ be an isogeny of abelian surfaces. Assume Image $\mu_{0,A'} = T(A')$. Then Image $\mu_{0,A} = T(A)$.*

PROOF: Let $d = \deg f$ so $f_* f^* =$ multiplication by d on $CH(A)$. The functor $T(\cdot)$ is covariant and also contravariant for finite maps. Since $T(A)$ is divisible (proposition 1 of the paper) we get a surjection $f_*: T(A') \twoheadrightarrow T(A)$.

Let X be a non-degenerate (ample) divisor on A, $X' = f^*(X)$ on A'. Then X' is non-degenerate, so every element in $\mathrm{Pic}_0(A')$ can be written in the form

$$X'_{a'} - X' = f^*(X_{f(a')} - X).$$

For any $y' \in \mathrm{Pic}_0(A')$ we have by the projection formula

$$f_* \mu_{0,A'}([X'_{a'} - X'] \otimes y') = \mu_{0,A}([X_{f(a')} - X] \otimes f_*(y'))$$

so f_* Image $\mu_{0,A'} \subseteq$ Image $\mu_{0,A}$. Thus we get

$$\begin{array}{ccc} \text{Image } \mu_{0,A'} & = & T(A') \\ {\scriptstyle f_*}\downarrow & & \downarrow\!\!\downarrow {\scriptstyle f_*} \\ \text{Image } \mu_{0,A} & \subseteq & T(A) \end{array}$$

So Image $\mu_{0,A} = T(A)$. Q.E.D.

LEMMA (A.6): *Let A be a principally polarized abelian surface, $p \in A$ a point. Let*

$$\mu: \mathrm{Pic}(A) \underset{Z}{\otimes} \mathrm{Pic}(A) \to CH(A)$$

be the intersection map. Then $2(p) \in$ Image μ.

PROOF: A principally polarized abelian surface is either a product of two elliptic curves or the jacobian of a genus 2 curve. In the former case the assertion is clear. In fact if $A = A_1 \times A_2$ and $p = (p_1, p_2)$ then $(p) = \mu[(A_1 \times \{p_2\}) \otimes (\{p_1\} \times A_2)]$.

Suppose now $A = J(C)$ for C a genus 2 curve. The Image of μ is stable under translation by points $a \in A$, so we may assume $p \in C \hookrightarrow J(C)$. Let $(p)_C \in \mathrm{Pic}(C)$ denote the class of p as a divisor on C. Since $\mathrm{Pic}_0(J(C)) \xrightarrow{\approx} \mathrm{Pic}_0(C)$ and $(C \cdot C) = 2$, we get $2(p)_C \in$ Image $(\mathrm{Pic}(J(C)) \to \mathrm{Pic}(C))$ so $2(p) = C \cdot$ (something in $\mathrm{Pic}(J(C))$). Q.E.D.

LEMMA (A.7): *Let A be an abelian surface. Then $T(A)$ is generated by cycles $(a+b) - (a) - (b) + (0)$.*

PROOF: This is clear if one thinks of $T(A) = \operatorname{Ker}(Z(A) \to \operatorname{Alb}(A))$. Q.E.D.

LEMMA (A.8): *Let A be an abelian surface, and let $F \in$ Image ($\mu : \operatorname{Pic}(A) \otimes \operatorname{Pic}(A) \to CH(A)$). Let $a, b \in A$ and write $F_a = T_a(F)$ for the translation of F by a. Then*
(i) *$F_{a+b} - F_a - F_b + F \in$ Image ($\mu_0 : \operatorname{Pic}_0(A) \otimes \operatorname{Pic}_0(A) \to T(A)$)*
(ii) *The map $A \times A \to T(A)$, $(a, b) \mapsto F_{a+b} - F_a - F_b + F$ is bilinear.*

PROOF: Both assertions are linear in F so we may assume $F = D \cdot E$ for divisors D and E. Note for example that $D_b - D$ is square equivalent to zero so $T_a(D_b - D)$ is rationally equivalent to $D_b - D$. Thus we have

$$\begin{aligned} F_{a+b} - F_a = T_a(F_b - F) &= T_a[D(E_b - E) + E_b(D_b - D)] \\ &= D_a(E_b - E) + E_{a+b}(D_b - D) \\ F - F_b &= -D(E_b - E) - E_b(D_b - D) \end{aligned}$$

so

$$\begin{aligned} F_{a+b} - F_a - F_b + F &= (D_a - D)(E_b - E) + (E_{a+b} - E_b)(D_b - D) \\ &= (D_a - D)(E_b - E) + (E_a - E)(D_b - D) \\ &\in \text{Image } \mu_0. \end{aligned}$$

Since e.g. $D_a - D$ is linear in a (Theorem of the square) we get (ii) as well. Q.E.D.

The theorems

THEOREM (A.2): *Let A be an abelian surface over an algebraically closed field k of characteristic 0. Then the map $\mu_0 : \operatorname{Pic}_0(A) \otimes \operatorname{Pic}_0(A) \to T(A)$ is surjective.*

PROOF: By (A.5) we may assume A is principally polarized. By (A.6) we have $2(0) \in$ Image μ, and by (A.8) (i) we get $2[(a + b) - (a) - (b) + (0)]$ Image μ_0. By (A.7) we conclude that the quotient

$$T(A)/\text{Image } \mu_0$$

is killed by multiplication by 2. Since $T(A)$ is divisible (proposition 1 of the paper) we get Image $\mu_0 = T(A)$. Q.E.D.

COROLLARY (A.9): *Let $n\delta : A \to A$ be the isogeny multiplication by n for some $n \in \mathbb{Z}$. Then the maps $(n\delta)^*$ and $(n\delta)_*$ on $T(A)$ are both multiplication by n^2.*

PROOF: The map $(n\delta)^*$ is multiplication by n on $\mathrm{Pic}_0(A)$, so the assertion for $(n\delta)^*$ follows from (A.8) together with the compatibility of products with pullback. In particular $(n\delta)^*$ is surjective. Since

$$(n\delta)_*(n\delta)^* = \text{multiplication by } n^4$$

we get that $(n\delta)_* =$ multiplication by n^2 also Q.E.D.

COROLLARY (A.10): *Let A be an abelian surface and let $X = A/\pm 1$ (with singularities resolved) be the Kummer surface of A. The rational map $f : A \to X$ induces a surjective isogeny*

$$f^* : T(X) \twoheadrightarrow T(A).$$

PROOF: The map $f^*f_* : CH(A) \to CH(A)$ is given by $f^*f_*((a)) = (a) + (-a)$. Thus

$$\begin{aligned} f^*f_*[(a+b)-(a)-(b)+(0)] \\ &= (a+b)-(a)-(b)+(0)+(-a-b)-(-a)-(-b)+(0) \\ &= (a+b)-(a)-(b)+(0)+(-1\delta)^*[(a+b)-(a)-(b)+(0)] \\ &= 2[(a+b)-(a)-(b)+(0)]. \end{aligned}$$

Since these cycles generate $T(A)$, it follows that f^* is surjective. $f_*f^* = 2 \Rightarrow f^*$ an isogeny.

REMARK (A.11) (i): Roitman has recently announced a result which implies $T(X)$ is torsion-free for any surface X. Using this, one gets an isomorphism between T (Kummer Surface) and $T(A)$.

(ii) Theorem (A.3) follows from (A.10) plus the fact Alb (Kummer) = (0).

THEOREM (A.1): *Let A be an abelian surface, and let a, b, c be points of A. Then the cycle*

$$(a+b+c)-(a+b)-(a+c)-(b+c)+(a)+(b)+(c)-(0)$$

is zero in $T(A)$. Equivalently, the map $\Delta : A \times A \to T(A)$, $\Delta(a,b) = (a+b)-(a)-(b)+(0)$ is bilinear and defines a surjection

$$\Delta : A \underset{Z}{\otimes} A \twoheadrightarrow T(A).$$

PROOF: If $f: A' \to A$ is an isogeny, then

$$\text{(A.1) for } A' \Rightarrow \text{(A.1) for } A,$$

so we may assume as before that A is either a product of two elliptic curves or the jacobian of a genus two curve C. It follows from (A.6) and (A.8) (ii) (applied to $F = 2(0)$) that 2Δ is bilinear. From (A.9)

$$4\Delta(a, b) = (2\delta)_* \Delta(a, b) = \Delta(2a, 2b).$$

To show, e.g. $\Delta(a, b + c) = \Delta(a, b) + \Delta(a, c)$ we choose a', b', c' with $2a' = a$, $2b' = b$, $2c' = c$ and compute

$$\begin{aligned}\Delta(a, b + c) = 4\Delta(a', b' + c') &= 4\Delta(a', b') + 4\Delta(a', c')\\ &= \Delta(a, b) + \Delta(a, c).\end{aligned}$$

Surjectivity of Δ follows from (A.7). Q.E.D.

COROLLARY (A.12): *Let X be a smooth, quasi-projective variety, and let A be an abelian surface. Let Γ be a cycle of codimension p on $A \times X$. Let $CH^p(X)$ denote the Chow group of codimension p cycles on X. Then the map of sets $\Gamma: A \to CH^p(X)$,*

$$\Gamma(a) = pr_{2*}(\Gamma \cdot (\{a\} \times X))$$

is quadratic, i.e. the expression

$$\Gamma(a, b) = \Gamma(a + b) - \Gamma(a) - \Gamma(b) + \Gamma(0)$$

is linear in a and in b.

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