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Compositio Mathematica, tome 33, n° 3 (1976), p. 261-288

<http://www.numdam.org/item?id=CM_1976__33_3_261_0>
SOME LINEAR TOPOLOGICAL PROPERTIES
OF THE HARDY SPACES $H^p$

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Abstract

The classical Hardy classes $H^p$ ($1 \leq p < \infty$) regarded as Banach spaces are investigated. It is proved: (1) Every reflexive subspace of $L^1$ is isomorphic to a subspace of $H^1$. (2) A complemented reflexive subspace of $H^1$ is isomorphic to a Hilbert space. (3) Every infinite dimensional subspace of $H^1$ which is isomorphic to a Hilbert space contains an infinite dimensional subspace which is complemented in $H^1$. The last result is a quantitative generalization of a result of Paley that a sequence of characters satisfying the Hadamard lacunary condition spans in $H^1$ a complemented subspace which is isomorphic to a Hilbert space.

Introduction

The purpose of the present paper is to investigate some linear topological and metric properties of the Banach spaces $H^p$, $1 \leq p < \infty$ consisting of analytic functions whose boundary values are $p$-absolutely integrable. The study of $H^p$ spaces seems to be interesting for a couple of instances: (1) it requires a new technique which combines classical facts on analytic functions with recent deep results on $L^p$-spaces; several classical results on the Hardy classes seem to have natural Banach-space interpretation. (2) The spaces $H^p$ and the Sobolev spaces are the most natural examples of “$L_p$-scales” essentially different from the scale $L^p$.

*Research of the second named author was partially supported by NSF Grant MPS 74-07509-A-02.
Boas [4] has observed that, for $1 \leq p \leq \infty$, the Banach space $H^p$ is isomorphic to $L^p$. The situation in the “limit case” of $H^1$ is quite different. For instance $H^1$ is not isomorphic to any complemented subspace of $L^1$, more generally—to any $L_1$-space (cf. [16], Proposition 6.1); $H^1$ is a dual of a separable Banach space (cf. [14]) while $L^1$ is not embeddable in any separable, dual cf. [23]; in contrast with $L^1$, by a result of Paley (cf. [21], [31], [7] p. 104), $H^1$ has complemented hilbertian subspaces hence it fails to have the Dunford-Pettis property.

On the other hand in Section 2 of the present paper we show that every reflexive subspace of $L^1$ is isomorphic to a subspace of $H^1$. Furthermore an analogue of the profound result of H. P. Rosenthal [27] on the nature of an embedding of a reflexive space in $L^1$ is also true for $H^1$. This implies that a complemented reflexive subspace of $H^1$ is necessarily isomorphic to a Hilbert space. In Section 3 we study hilbertian (= isomorphic to a Hilbert space) subspaces of $H^1$. We show that $H^1$ contains “very many” complemented hilbertian subspaces. Precisely: every subspace of $H^1$ which is isomorphic to $\ell^2$ contains an infinite dimensional subspace which is complemented in $H^1$. This fact is a quantitative generalization of a result of Paley, mentioned above, on the boundedness in $H^1$ of the orthogonal projection from $H^1$ onto the closed linear subspace generated by a lacunary sequence of characters.

Section 4 contains some open problems and some results on the behaviour of the Banach-Mazur distance $d(H^p, L^p)$ as $p \to 1$ and as $p \to \infty$.

1. Preliminaries

Let $0 < p \leq \infty$. By $L^p$ (resp. $L^p_0$) we denote the space of $2\pi$-periodic complex-valued (resp. real-valued) measurable functions on the real line which are $p$-absolutely integrable with respect to the Lebesgue measure on $[0, 2\pi]$ for $0 < p < \infty$, and essentially bounded for $p = \infty$. $C_{2\pi}$ stands for the space of all continuous $2\pi$-periodic complex-valued functions. We admit

$$
\|f\|_p = \frac{1}{2\pi} \int_{0}^{2\pi} |f(t)|^p dt \quad \text{for } 0 < p < 1,
$$

$$
\|f\|_p = \left(\frac{1}{2\pi} \int_{0}^{2\pi} |f(t)|^p dt\right)^{1/p} \quad \text{for } 1 \leq p < \infty,
$$

$$
\|f\|_\infty = \text{ess sup}_{0 \leq t \leq 2\pi} |f(t)|.
$$
The $n$-th character $\chi_n$ is defined by

$$\chi_n(t) = e^{int} \quad (-\infty < t < +\infty; \; n = 0, \pm 1, \pm 2, \ldots).$$

Given $f \in L^1$ we put

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int}dt \quad (n = 0, \pm 1, \pm 2, \ldots)$$

$$f_0 = f - \hat{f}(0) \cdot \chi_0.$$

If $0 < p < \infty$, then $H^p$ is the closed linear subspace of $L^p$ which is generated by the non-negative characters, $\{\chi_n: n \geq 0\}$. We define

$$H^\infty = \{f \in L^\infty: \hat{f}(n) = 0 \text{ for all } n < 0\}.$$

By $A$ we denote the closed linear subspace of $H^\infty$ generated by the non-negative characters. We put $H^p_0 = \{f \in H^p: \hat{f}(0) = 0\}$ and $A_0 = \{f \in A: \hat{f}(0) = 0\}$.

Let $f \in H^p$. We denote by $\tilde{f}$ a unique analytic function on the unit disc $\{z: |z| < 1\}$ such that

$$\lim_{r \to 1} \tilde{f}(re^{it}) = f(t) \quad \text{for almost all } t.$$

For $u \in L^1_k$ we define $H(u) = v$ to be the unique real $2\pi$-periodic function such that for $f = u + iv$ there exists an $\tilde{f}$ analytic on the unit disc satisfying (1.1) and such that $\tilde{f}(0) = 2\pi^{-1} \int_0^{2\pi} u(t)dt$. Recall (cf. [33], Chap. VII and Chap. XII).

**Proposition 1.1:** (i) $H$ is a linear operator of weak type $(1, 1)$.

(ii) For every $p \in (0, 1)$ there exists a constant $\rho_p$ such that

$$\|H(u)\|_p \leq \rho_p \|u\|_p \quad \text{for } u \in L^1_k$$

(iii) For every $p \in (1, \infty)$ there exists a constant $\rho_p \leq C \max(p, pl(p-1))$, where $C$ is an absolute constant, such that

$$\|H(u)\|_p \leq \rho_p \|u\|_p \quad \text{for } u \in L^p.$$

Next, for $f \in L^1$, we define $B(f)$ to be the unique function in $\cap_{0 < p < 1} H^p$ such that

$$B(f) = \sum_{n=0}^{\infty} \hat{f}(n) \tilde{\chi}_{2n} + \sum_{n<0} \hat{f}(n) \tilde{\chi}_{-2n-1}.$$

Let $H(f) = H(\text{Re } f) + iH(\text{Im } f)$ for $f \in L^1$. Then

$$B(f)(t) = \frac{1}{2}[f_0(2t) + iH(f_0)(2t) + [f_0(-2t) - iH(f_0)(-2t)]e^{-it}] + \hat{f}(0) \quad (-\infty < t < +\infty)$$
Clearly $B$ is a one to one operator and if $g = B(f)$, then
\[
f(t) = \frac{1}{2} \left[ g \left( \frac{t}{2} \right) + g \left( \frac{t}{2} + \pi \right) + (\chi_1 g) \left( -\frac{t}{2} \right) + (\chi_1 g) \left( -\frac{t}{2} + \pi \right) \right]
\]
\[(-\infty < t < +\infty).\]

Combining Proposition 1.1 with the above formulae we get (cf. Boas [4]).

**PROPOSITION 1.2:** (i) $B$ is a linear operator of weak type $(1, 1)$ from $L^1$ into $\cap_{0 < p < 1} H^p$

(ii) For every $p \in (0, 1)$ there exists a constant $\beta_p$ such that
\[
\|B(f)\|_p \leq \beta_p \|f\|_p^p
\]

(iii) For every $p \in (1, \infty)$ $B$ maps isomorphically $L^p$ onto $H^p$; there exists a constant $\beta_p \leq 2\rho_p + 3$ such that
\[
2^{-1}\|f\|_p \leq \|B(f)\|_p \leq \beta_p \|f\|_p.
\]

A relative of $B$ is the orthogonal projection $\mathcal{Q}$ defined by
\[
\mathcal{Q}(f)(t) = 2^{-1} [B(f) + (B(f))^*] \left( \frac{t}{2} \right) \quad \text{for } f \in L^1,
\]
\[-\infty < t < +\infty\]

where $g^*(t) = g(t + \pi)$. Clearly, by Proposition 1.2, $\mathcal{Q}(L^1) \subset \cap_{0 < p < 1} H^p$

and, for $1 < p < \infty$, $\mathcal{Q}$ regarded as an operator from $L^p$ is a projection onto $H^p$ with $\|\mathcal{Q}\|_p \leq \|B\|_p$. In fact we have
\[
\mathcal{Q}(f) = \sum_{n=0}^\infty \hat{f}(n) \chi_n \quad \text{for } f \in L^p, \ 1 < p < \infty.
\]

2. Reflexive subspaces of $H^1$

**PROPOSITION 2.1:** A reflexive Banach space is isomorphic to a subspace of $H^1$ if (and only if) it is isomorphic to a subspace of $L^1$.

**PROOF:** By a result of Rosenthal (cf. [27]) every reflexive subspace of $L^1$ is isomorphic to a reflexive subspace of $L^r$ for some $r$ with $1 < r \leq 2$. Therefore it is enough to prove that, for every $r$ with $1 < r \leq 2$, the space $L^r$ is isomorphic to a subspace of $H^1$. It is well known (cf. e.g. [27], p. 354) that, for $r \in [1, 2]$, there exists in $\cap_{0 < p < r} L^p$ a subspace $E_r$ which, for every fixed $p \in (0, r)$, regarded as a subspace of $L^p$ is isometrically isomorphic to $L^r$. Moreover (if $r > 1$), for every $p_1$ and $p_2$ with $1 \leq p_1 < p_2 < r$, there exists a constant
Now fix $p_1$ and $p_2$ with $1 < p_1 < p_2 < r$. By Proposition 1.2(iii), the operator $B$ embeds isomorphically $E_r$ regarded as a subspace of $L^{p_1}$ into $H^{p_1}$. Clearly we have the set theoretical inclusion $H^{p_1} \subset H^1$. Thus it suffices to prove that the norm $\| \cdot \|_1$ and $\| \cdot \|_{p_1}$ are equivalent on $B(E_r)$. By (1.2) and (2.1), for every $g \in B(E_r)$ we have $\|g\|_{p_2} \leq k\|g\|_{p_1}$ where $k = \gamma_{p_1,p_2} \cdot 2\beta_{p_1}$. Letting $s = (p_1 - 1)(p_2 - 1)^{-1}$, in view of the logarithmic convexity of the function $p \to \|g\|_p$, we have

$$\|g\|_{p_1}^{p_1} \leq \|g\|_{p_2}^{p_2} \|g\|_{p_1}^{-s} \leq k^{p_2} \|g\|_{p_2}^{p_2} \|g\|^{1-s}$$

whence

$$\|g\|_1 \leq \|g\|_{p_1} \leq k^{p_2} t^{-s} \|g\|.$$

This completes the proof.

**REMARK:** Using the technique of [15] (cf. also [19]) instead of the logarithmic convexity of the function $p \to \|g\|_p$ one can show that on $B(E_r)$ all the norms $\| \cdot \|_p$ are equivalent for $0 < p < r$ (in fact equivalent to the topology of convergence in measure). Hence if $0 < p \leq 1$, then $H^p$ contains isomorphically every reflexive subspace of $L^1$. We do not know any satisfactory description of all Banach subspaces of $H^p$ for $0 < p < 1$.

Our next result provides more information on isomorphic embeddings of reflexive spaces into $H^1$. It is a complete analogue of Rosenthal’s Theorem on reflexive subspaces of $L^1$ (cf. [27]).

**Proposition 2.2:** Let $X$ be a reflexive subspace of $H^1$. Then there exists a $p > 1$ such that for every $r$ with $p > r > 1$ the natural embedding $j : X \to H^1$ factors through $H^r$, i.e. there are bounded linear operators $U : X \to H^r$ and $V : H^r \to H^1$ with $VU = j$. Moreover $U$ and $V$ can be chosen to be operators of multiplication by analytic functions.

**Proof:** By a result of Rosenthal ([27], Theorem 5 and Theorem 9), there exists a $p > 1$ such that for every $r$ with $p > r > 1$ there exist a $K > 0$ and a non-negative function $\varphi$ with $1/2\pi \int_0^{2\pi} \varphi(t)dt = 1$ such that

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |x(t)|^r [\varphi(t)]^1 \cdot dt\right)^{1/r} \leq K\|x\|, \quad \text{for } x \in X.$$
(In this formula we admit $0/0 = 0$). Let us set $\psi = \max (\varphi, 1)$. Let $g$ be the outer function defined by
\[
\tilde{g}(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + 2}{e^{it} - 2} \log \left[ \psi(t) \frac{r-1}{r} \right] dt \quad \text{for} |z| < 1
\]
and let
\[
g(t) = \lim_{\rho \to 1} \tilde{g}(\rho e^{it}) \quad \text{for} \quad t \in [0, 2\pi]
\]

Then (cf. [7], Chap. 2) $g \in \mathcal{H}^{r/(r-1)}$, $|g(t)| = \psi(t)^{r/(r-1)}$ for $t$ a.e., $|\tilde{g}(z)| \geq 1$ for $|z| < 1$ and $g^{-1} \in \mathcal{H}^\infty$.

Let us set $U(x) = x/g$ for $x \in X$ and $V(f) = g \cdot f$ for $f \in \mathcal{H}^r$. Since $\|g\|_{r/(r-1)} \leq 2^{r-1/r}$, $V$ maps $\mathcal{H}^r$ into $\mathcal{H}^1$ and $\|V\| \leq 2^{1-1/r}$. Finally, for every $x \in X$, we have
\[
\|U(x)\|^r = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{x(t)}{g(t)} \right|^r dt = \frac{1}{2\pi} \int_0^{2\pi} |x(t)|^r |[\psi(t)]^{1-r} dt 
\]
\[
\leq \frac{1}{2\pi} \int_0^{2\pi} |x(t)|^r |[\varphi(t)]^{1-r} dt \leq K'\|x\|_r^r.
\]
Thus $U(x) \in \mathcal{L}^r$. Therefore $U(x) \in \mathcal{H}^1$ because $U(x) \in \mathcal{H}^1$ being a product of an $x \in \mathcal{H}^1$ by $g^{-1} \in \mathcal{H}^\infty$.

**Corollary 2.1:** A complemented reflexive subspace of $\mathcal{H}^1$ is isomorphic to a Hilbert space.

**Proof:** Let $X$ be a complemented reflexive subspace of $\mathcal{H}^1$. Then, by Proposition 2.2, there exists a $p > 1$ such that for every $r$ with $p > r > 1$ there are bounded linear operators $U$ and $V$ such that the following diagram is commutative
\[
\begin{array}{ccc}
\mathcal{H}^r & \xrightarrow{V} & \mathcal{H}^r \\
\downarrow & & \downarrow \\
X & \xrightarrow{U} & X
\end{array}
\]
where $j : X \rightarrow \mathcal{H}^1$ is the natural inclusion and $P : \mathcal{H}^1 \xrightarrow{onto} X$ is a projection. Thus, for every $r \in (1, p)$, $Pj = \text{the identity operator on } X$ admits a factorization through $\mathcal{H}^r$. Therefore $X$ is isomorphic to a complemented subspace of $\mathcal{L}^r$ because, by Proposition 1.2(iii), $\mathcal{H}^r$ is isomorphic to $\mathcal{L}^r$. Since this holds for at least two different $r \in (1, p)$, we infer that $X$ is isomorphic to a Hilbert space (cf. [16] and [18]).

**Remarks:** (1) The following result has been kindly communicated to us by Joel Shapiro.

If $0 < p < 1$ and if a Banach space $X$ is isomorphic to a complemented subspace of $\mathcal{H}^p$, then either $X$ is isomorphic to $\ell^1$ or $X$ is finite dimensional.
The proof (due to J. Shapiro) uses the result of Duren, Romberg and Shields [8], sections 2 and 3:

(D.R.S) the adjoint of the natural embedding \( g \rightarrow \tilde{g} \) of \( H^p \) into the space \( B^p \) is an isomorphism between conjugate spaces. Here \( B^p \) denotes the Banach space of holomorphic functions on the open unit disc with the norm

\[
\|f\|_{B^p} = \int_{x^2+y^2=1} |f(x+iy)|(1-(x^2+y^2)^{1/2})^{(1/p)-2} \, dx \, dy.
\]

It follows from (D.R.S) that a complemented Banach subspace of \( H^p \) (\( 0 < p < 1 \)) is isomorphic to a complemented subspace of \( B^p \). Next using technique similar to that of [17], Theorem 6.2 (cf. also [31]) one can show that \( B^p \) is isomorphic to \( \ell^1 \). Now the desired conclusion follows from [22], Theorem 1.

Problem (J. Shapiro). Does \( H^p \) (\( 0 < p < 1 \)) actually contain a complemented subspace isomorphic to \( \ell^1 \)?

(2) Slightly modifying the proof of Proposition 2.2 one can show the following

**PROPOSITION 2.2a:** Let \( 1 \leq p_0 < 2 \). Let \( X \) be a subspace of \( H^{p_0} \) which does not contain any subspace isomorphic to \( \ell^{p_0} \). Then there exists a \( p \in (p_0, 2) \) such that, for every \( r \) with \( p_0 < r < p \) there exists an outer \( g \in H^{p_0(r-r^{-1})} \) with \( g \neq 0 \) such that \( j = VU \) where \( U : X \rightarrow H^r \) and \( V : H^r \rightarrow H^{p_0} \) are operators of multiplication by \( 1/|g| \) and \( g \) respectively and \( j : X \rightarrow H^{p_0} \) denotes the natural inclusion.

The proof imitates the proof of Proposition 2.2; instead of Rosenthal’s result we use its generalization due to Maurey (cf. [19], Théorème 8 and Proposition 97).

Our next result is in fact a quantitative version of Proposition 2.2a for hilbertian subspaces.

**PROPOSITION 2.3:** Let \( K \geq 1 \) and let \( 1 \leq p \leq 2 \). Let \( X \) be a subspace of \( H^p \) and let \( T : \ell^2 \xrightarrow{\text{onto}} X \) be an isomorphism with \( \|T\|\|T^{-1}\| \leq K \). Then there exists an outer \( \varphi \in H^1 \) such that

\[
|\varphi(z)| \geq 1 \text{ for every } z \text{ with } |z| < 1
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} |\varphi(t)| \, dt = 1
\]

\[
\left( \int_0^{2\pi} |f(t)|^2 |\varphi(t)|^{-2(p-1)} \, dt \right)^{1/2} \leq \gamma K \|f\|_p \text{ for every } f \in X
\]

where \( \gamma \) is an absolute constant, in fact \( \gamma \leq 4/\sqrt{\pi} \).
PROOF: A result of Maurey ([19] Théorème 8, 50a, cf. also [20]), applied for the identity inclusion $X \rightarrow L^p$, yields the existence of a $g \in L'$ where $1/r = 1/p - 1/2$ such that $\|g\|_r = 1$ and

$$
(\frac{1}{\pi} \int_0^{2\pi} \left| \frac{f(t)}{g(t)} \right|^2 \, dt)^{1/2} \leq C \|f\|_p \quad \text{for every } f \in X
$$

where $C$ is the smallest constant such that

$$
(\frac{1}{2\pi} \int_0^{2\pi} \left( \sum_j |f_j(t)|^2 \right)^{p/2} \, dt)^{1/p} \leq C \left( \sum_j \|f_j\|_p^2 \right)^{1/2}
$$

for every finite sequence $(f_j)$ in $X$. A standard application of the integration against the independent standard complex Gaussian variables $\xi_i$ gives

$$
\sum_j \|f_j\|_p^2 = \|T^{-1}\|^{-2} \sum_j \|T^{-1}(f_j)\|^2
$$

$$
= \|T^{-1}\|^2 \int_0^\pi \left\| \sum_j T^{-1}(f_j) \xi_j(s) \right\|^2 \, ds
$$

$$
\leq \|T^{-1}\|^2 \int_0^\pi \left\| \sum_j f_j \xi_j(s) \right\|_p^2 \, ds
$$

$$
\leq K^{-2} \left( \int_0^{2\pi} \left( \sum_j f_j(t) \xi_j(s) \right)^p \, dt ds \right)^{2/p}
$$

$$
= K^{-2} k_p^2 \left( \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_j |f_j(t)|^2 \right)^{p/2} \, dt \right)^{2/p}
$$

where $k_p = (1/\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2)^{p/2} e^{-(x^2+y^2)} \, dx \, dy)^{1/p}$. Since $k_p \geq k_1 = \sqrt{\pi}/2$, one can replace $C$ in (2.5) and in (2.6) by $K/k_1 = 2K/\sqrt{\pi}$.

Now, by [14], p. 53, there exists an outer function $\varphi \in H^1$ satisfying (2.2), (2.3) and such that

$$
|\varphi(t)| = \max \left( \frac{|g(t)|^r}{\frac{1}{2\pi} \int_0^{2\pi} \max \left( |g(t)|^r, 1 \right) \, dt}^{1/r} \right) \quad \text{for almost all } t
$$

It can be easily checked that (2.7) and (2.5) imply (2.4) with $\gamma = 2/k_1$.

Our last result in this section gives some information on reflexive subspaces of the quotient $L^1/H_0^1$.

PROPOSITION 2.4: Let $X$ be a reflexive subspace of $L^1$ such that $\hat{f}(k) = 0$ for $k > 0$, $f \in X$. Then the sum $X + H_0^1$ is closed, equivalently the restriction of the quotient map $L^1 \rightarrow L^1/H_0^1$ to $X$ is an isomorphic embedding.

PROOF: Let $\mathcal{P}(f) = f - 2(f)$ for $f \in L^1$ where $\mathcal{P}$ is the projection...
defined, by \((1.3)\). It follows from Proposition 1.2(ii) that there exists a constant \(a > 0\) such that
\[ \|P(f)\|_{1/2} \leq a\|f\|_{1/2} \quad \text{for } f \in L^1. \]
On the other hand if \(X\) is a reflexive subspace of \(L^1\), then \(X\) contains no subspace isomorphic to \(\ell^1\). Hence (cf. [15], [19]) the norm topology in \(X\) coincides with the topology of convergence in measure, in particular
\[ \|f_n\|_1 \to 0 \text{ iff } \|f_n\|_{1/2} \to 0 \quad \text{for every sequence } (f_n) \subset X. \]
Thus there exists a constant \(b_X = b > 0\) such that
\[ \|f\|_1 \leq b\|f\|_{1/2} \quad \text{for } f \in X. \]

Now fix \(f \in X\) and \(g \in H^0\). Then \(P(g) = 0\), and \(P(f) = f\) because \(\hat{f}(k) = 0\) for \(k > 0\). Hence
\[ \|f + g\|_1 \geq a^2\|P(f + g)\|_{1/2} = a^2\|P(f)\|_{1/2} = a^2\|f\|_{1/2} \geq \frac{a^2}{b}\|f\|_1. \]
Thus the sum \(X + H^0\) is closed.

**Remark:** Proposition 2.4 yields, in particular, the following “classical” result.
If \((n_k)\) is a sequence of negative integers such that the space
\[ \mathcal{X} = \{f \in L^1 : \hat{f}(n) = 0 \quad \text{for } n \neq n_k \,(k = 1, 2, \ldots)\} \]
is isomorphic to \(\ell^2\) (in particular if \(\lim (n_{k+1}/n_k) > 1\)) then the space \(\mathcal{X} + H^1\) is closed or equivalently in the “dual language” the operator \(A \to \ell^2\) defined by \(f \to (\hat{f}(-n_k))\) is a surjection.

### 3. Hilbertian subspaces of \(H^1\)

The existence of infinite-dimensional complemented hilbertian subspaces of \(H^1\) follows from the classical result of R.E.A.C. Paley (cf. [21], [29], [7] p. 104, [33], Chap. XII, Theorem 7.8) which yields (P). If \(\lim (n_{k+1}/n_k) > 1\), then the closed linear subspace of \(H^1\) spanned by the sequence of characters \((\chi_{n_k})_{1 \leq k < \infty}\) is isomorphic to \(\ell^2\) and complemented in \(H^1\).

On the other hand there are subspaces of \(H^1\) spanned by sequences of characters which are isomorphic to \(\ell^2\) but uncomplemented in \(H^1\) (cf. Rudin [30] and Rosenthal [26]).

In this section we shall show that, in fact, \(H^1\) contains “very many” complemented and “very many” uncomplemented hilbertian sub-
spaces not necessarily translation invariant. The situation is similar to that in $L^p$ (and therefore $H^p$, by Proposition 1.2(iii)) for $1 < p < 2$ (cf. [25], Theorem 3.1) but not in $L^1$ which contains no complemented infinite-dimensional hilbertian subspaces ([13], [22]).

If $(x_n)$ is a sequence of elements of a Banach space $X$ then $[x_n]$ denotes the closed linear subspace of $X$ generated by the $x_n$'s.

Let $1 \leq K < \infty$. Recall that a sequence $(x_n)$ of elements of a Banach space is said to be $K$-equivalent to the unit vector basis of $\ell^2$ provided there exist positive constants $a$ and $b$ with $ab = K$ such that

$$a^{-1} \left( \sum_n |t_n|^2 \right)^{1/2} \leq \left\| \sum_n t_n x_n \right\| \leq b \left( \sum_n |t_n|^2 \right)^{1/2}$$

for every finite sequence of scalars $(t_n)$.

Now we are ready to state the main result of the present section

**THEOREM 3.1:** Let $1 \leq K < \infty$. Let $(f_n)_{1 \leq n < \infty}$ be a sequence in $H^1$ which is $K$-equivalent to the unit vector basis of $\ell^2$. Then, for every $\epsilon > 0$, there exists an infinite subsequence $(n_k)$ such that the closed linear subspace $[f_{n_k}]$ spanned by the sequence $(f_{n_k})$ is complemented in $H^1$. Moreover, there exists a projection $P$ from $H^1$ onto $[f_{n_k}]$ with $\|P\| < 4K + \epsilon$.

The proof of Theorem 3.1 follows immediately from Propositions 3.1, 3.2 and 3.3 given below. We begin with the following general criterion

**PROPOSITION 3.1:** Let $X$ be a Banach space with separable conjugate $X^*$. Assume that there exists a constant $c = c_X$ such that every weakly convergent to zero sequence $(y_m)$ in $X$ contains an infinite subsequence $(y_{m_k})$ such that

$$(3.1) \quad \left\| \sum_t t_k y_{m_k} \right\| \leq c \sup_m \|y_m\| \left( \sum |t_k|^2 \right)^{1/2}$$

for every finite sequence of scalars $(t_k)$. Then, for every $K \geq 1$ and for every $\epsilon > 0$, every sequence $(x^*_n)$ in $X^*$ which is $K$-equivalent to the unit vector basis of $\ell^2$ contains an infinite subsequence $(x^*_n)$ such that the closed linear subspace $[x^*_{n_k}]$ admits a projection $P : X^* \to [x^*_{n_k}]$ with $\|P\| < 2Kc + \epsilon$.

**PROOF:** Define $V : \ell^2 \to X^*$ by $V((t_n)) = \sum_n t_n x^*_n$ for $(t_n) \in \ell^2$. Clearly $V$ is an isomorphic embedding with $\|V\| \|V^{-1}\| \leq K$ ($V^{-1}$ acts from $V(\ell^2)$ onto $\ell^2$). Since $\ell^2$ is reflexive, $V$ is weak-star continuous.
Hence there exists an operator $U : X \to \ell^2$ whose adjoint is $V$. It is easy to check that the operator $U$ is defined by $U(x) = (x^*_n(x))_{1 \leq n < \infty}$ for $x \in X$. Since $\|U^*((t_n))\| = \|V((t_n))\| \geq \|V^{-1}\|^{-1}(\sum |t_n|^2)^{1/2}$ for every $(t_n) \in \ell^2$, the operator $U$ is a surjection such that, for every $r > \|V^{-1}\|$, the set $U(\{x \in X : \|x\| \leq r\})$ contains the unit ball of $\ell^2$ (cf. [32] Chap. VII, §5). Hence there exists a sequence $(x_s)$ in $X$ such that $\sup \|x_s\| \leq r$ and $(U(x_s))$ is the unit vector basis of $\ell^2$, equivalently $x^*_n(x_s) = \delta^*_n$ for $n, s = 1, 2, \ldots$. Since $X^*$ is separable and $\sup \|x_s\| \leq r$, there exists an infinite subsequence $(x_{s_k})$ which is a weak Cauchy sequence. Let us set $y_m = x_{2m} - x_{2m-1}$ for $m = 1, 2, \ldots$. Clearly the sequence $(y_m)$ tends weakly to zero. Thus the condition imposed on $X$ yields the existence of an infinite subsequence $(y_{m_k})$ satisfying (3.1). Let us set $n_k = s_{2m_k}$ for $k = 1, 2, \ldots$ and put

$$P(x^*) = \sum_{k=1}^\infty x^*_k(y_{m_k})x^*_{n_k} \quad \text{for } x^* \in X^*.$$  

Clearly we have

$$\|P(x^*)\| \leq \|V\| \left(\sum_{k=1}^\infty |x^*_k(y_{m_k})|^2\right)^{1/2}.$$  

Thus, by (3.1),

$$\|P(x^*)\| \leq \|V\| \sup_{\|t_k\| = 1} \left|\sum_{k=1}^\infty x^*_k(y_{m_k})t_k\right|$$

$$\leq \|V\|\|x^*\| \sup_{\|t_k\| = 1} \left|\sum_{k=1}^\infty t_ky_{m_k}\right|$$

$$c \sup \|y_m\| \|V\|\|x^*\|.$$  

Thus $P$ is a linear operator with $\|P\| \leq 2cr\|V\|$ (because $\sup \|y_{m_k}\| \leq 2\sup \|x_s\| \leq 2r$). Letting $r < \|V^{-1}\| + \epsilon(2c\|V\|)^{-1}$, we get $\|P\| < 2K + \epsilon$. Since $P(x^*) \in [x^*_{n_k}]$ for every $x^* \in X^*$ and since $P(x^*_k) = x^*_k$ for $k = 1, 2, \ldots$, we infer that $P$ is the desired projection.

**REMARK:** The assertion of Proposition 3.1 remains valid if we replace the assumption of separability of $X^*$ by the weaker assumption that $X$ does not contain subspace isomorphic to $\ell^1$. To extract a weak Cauchy subsequence from the sequence $(x_s)$ we apply the result of Rosenthal [28].

To apply Proposition 3.1 we need a description of a predual of $H^1$. Our next proposition is known. Its part (ii) is a particular case of the Caratheodory-Fejer Theorem, cf. [1].
Proposition 3.2: (i) The conjugate space of the quotient \( C_{2\pi}/A_0 \) is isometrically isomorphic to \( H^1 \).

(ii) The space \( C_{2\pi}/A_0 \) is isometrically isomorphic to a subspace of the space of compact operators on a Hilbert space.

Proof: (i) The desired isometric isomorphism assigns to each \( f \in H^1 \) the linear functional \( x_\sharp f \) defined by

\[
x_\sharp f([g + A_0]) = \frac{1}{2\pi} \int_0^{2\pi} f(t)g(t)dt \quad \text{for the coset } [g + A_0] \in C_{2\pi}/A_0.
\]

The fact that this map is onto \((C_{2\pi}/A_0)^*\) follows from the F. and M. Riesz Theorem. For details cf. [14], p. 137, the second Theorem.

(ii) To each coset \([f + A_0]\) we assign the linear operator \( T_f : H^2 \to H^2 \) defined by

\[
\langle T_f(g), h \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)g(t)h(-t)dt \quad (g, h \in H^2).
\]

Clearly the definition of \( T_f \) is independent of the choice of a representative in the coset \([f + A_0]\). Moreover, for every \( f, \in [f + A_0] \), we have

\[
\|T_f\| = \inf \{\|f^*\| : f, \in [f + A_0]\} = \|f + A_0\|.
\]

Thus \( \|T_f\| \leq \inf \{\|f^*\| : f, \in [f + A_0]\} = \|f + A_0\| \).

Conversely, it follows from part (i) and the Hahn Banach Theorem that there exists a \( \varphi \in H^1 \) with \( \|\varphi\| = 1 \) such that \( 1/2\pi \int_0^{2\pi} f(t)\varphi(t)dt = \|f + A_0\| \). By the factorization theorem (cf. [14], p. 67), we pick functions \( g \) and \( h_1 \) in \( H^2 \) with \( gh = \varphi \) and \( \|g\|_2 = \|h_1\|_2 = 1 \) (cf. [14], p. 71), and we define \( h \in H^2 \) by \( h(t) = \overline{h_1(-t)} \). Then \( \langle T_f(g), h \rangle = \|f + A_0\| = \|f + A_0\|\|g\|\|h\|_2 \). Hence \( \|T_f\| = \|f + A_0\| \). This shows that the map \( [f + A_0] \to T_f \) is an isometrically isomorphic embedding of \( C_{2\pi}/A_0 \) into the space of bounded operators on \( H^2 \). Finally observe that each operator \( T_f \) is compact because the cosets \( \{\chi_{-n} + A_0 : n = 0, 1, 2, \ldots\} \) are linearly dense in \( C_{2\pi}/A_0 \) (by the Fejer Theorem) and \( T_{\chi_{-n}} = \Sigma_{j=0}^n \langle :, \chi_j \rangle \chi_{n-j} \) is an \((n + 1)\)-dimensional operator \( (n = 0, 1, \ldots) \). This completes the proof.

To complete the proof of Theorem 3.1 it is enough to show that the space \( K(h) \) of the compact operators on an infinite-dimensional Hilbert space \( h \) (and therefore every subspace of \( K(h) \)) satisfies the assumption of Proposition 3.1. Precisely we have

Proposition 3.3: Let \( h \) be an infinite-dimensional Hilbert space. Let \( \{T_m\} \) be a weakly convergent to zero sequence in \( K(h) \). Then, for
every $\epsilon > 0$, there exists an infinite subsequence $(m_k)$ such that
\[
\left\| \sum_k t_k T_{m_k} \right\| \leq (2 + \epsilon) \sup_m \| T_m \| \left( \sum |t_k|^2 \right)^{1/2}
\]
for every finite sequence of scalars $(t_k)$.

**Proof:** The assumption that the sequence $(T_m)$ converges weakly to zero in $K(h)$ means
\[
\lim_{m} \langle T_m(x), y \rangle = 0 \quad \text{for every } x, y \in h.
\]
Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $h$. Let $(e_\alpha)_{\alpha \in \mathbb{N}}$ be an orthonormal basis for $h$. Since each $T_m$ is compact, the ranges of $T_m$ and its adjoint $T_m^*$ are separable. Hence there exists a countable set $\mathcal{A}_0$ such that $\langle T_m(x), e_\alpha \rangle = \langle T_m^*(x), e_\alpha \rangle = 0$ for every $m = 1, 2, \ldots$ for every $x \in h$ and for every $\alpha \in \mathcal{A} \setminus \mathcal{A}_0$. Let $j \to \alpha(j)$ be an enumeration of the elements of $\mathcal{A}_0$. Let furthermore $P_n$ denote the orthogonal projection onto the $n$-dimensional subspace generated by the elements $e_{\alpha(1)}, e_{\alpha(2)}, \ldots, e_{\alpha(n)}$. Since $\dim P_n(h) = n$, it follows from (3.2) that
\[
\lim_{m} \| P_n T_m P_n \| = 0 \quad \text{for } n = 1, 2, \ldots
\]
Next the compactness of each $T_m$ and the definition of the set $\mathcal{A}_0$ yield
\[
\lim_{m} \| T_m - P_n T_m P_n \| = 0 \quad \text{for } m = 1, 2, \ldots
\]
Let $\epsilon > 0$ be given. Assuming that $\sup_m \| T_m \| > 0$ we fix a positive sequence $(\epsilon_k)$ with $(\sum_{k=1}^{\infty} 4\epsilon_k^2) \leq \epsilon \sup_m \| T_m \|$. Now using (3.3) and (3.4) we define inductively increasing sequences of indices $(m_k)_{k=1}^{\infty}$ and $(n_k)_{k=0}^{\infty}$ with $m_1 = 1$ and $n_0 = 0$ so that (admitting $P_0 = 0$)
\[
\| P_{n_k-1} T_{m_k} P_{n_k-1} \| \leq \epsilon_k \quad \text{for } k = 1, 2, \ldots
\]
\[
\| T_{m_k} - P_{n_k} T_{m_k} P_{n_k} \| \leq \epsilon_k \quad \text{for } k = 1, 2, \ldots
\]
Let us put, for $k = 1, 2, \ldots$,
\[
B_k = (P_{n_k} - P_{n_k-1}) T_{m_k} P_{n_k}, \quad C_k = P_{n_k-1} T_{m_k} (P_{n_k} - P_{n_k-1}).
\]
Clearly (3.5) and (3.6) yield
\[
\| T_{m_k} - B_k - C_k \| = \| T_{m_k} - P_{n_k} T_{m_k} P_{n_k} + P_{n_k-1} T_{m_k} P_{n_k-1} \| \leq 2\epsilon_k.
\]
Let \((t_k)\) be a fixed finite sequence of scalars. Since the projections \(P_{n_k} - P_{n_{k-1}}\) (\(k = 1, 2, \ldots\)) are orthogonal and mutually disjoint, for every \(x \in \mathcal{h}\), we have

\[
\left\| \sum t_k B_k(x) \right\|^2 = \left\| \sum t_k (P_{n_k} - P_{n_{k-1}})((T_m P_{n_k})(x)) \right\|^2 \\
= \left( \sum |t_k|^2 \right)^2 \left\| (P_{n_k} - P_{n_{k-1}})((T_m P_{n_k})(x)) \right\|^2 \\
\leq \left( \sum |t_k|^2 \right)^2 \left\| P_{n_k} - P_{n_{k-1}} \right\|^2 \left\| T_m \right\|^2 \left\| x \right\|^2 \\
\leq \left( \sum |t_k|^2 \right)^2 \sup_m \left\| T_m \right\|^2 \left\| x \right\|^2.
\]

Hence

\[
\left\| \sum t_k B_k \right\| \leq \left( \sum |t_k|^2 \right)^{1/2} \sup_m \left\| T_m \right\|.
\]

Similarly

\[
\left\| \sum t_k C_k \right\| = \left\| \sum \bar{t}_k C_k^* \right\| = \left\| \sum \bar{t}_k (P_{n_k} - P_{n_{k-1}})T_m^* P_{n_{k-1}} \right\| \\
\leq \left( \sum |t_k|^2 \right)^{1/2} \sup_m \left\| T_m \right\|.
\]

Thus

\[
\left\| \sum t_k T_{n_k} \right\| \leq \sum |t_k| \left\| T_{n_k} - B_k - C_k \right\| + \left\| \sum t_k B_k \right\| + \left\| \sum t_k C_k \right\| \\
\leq \left( \sum |t_k|^2 \right)^{1/2} \left( \left( \sum_{k-1}^m 4\epsilon_k \right)^{1/2} + 2 \sup_m \left\| T_m \right\| \right) \\
\leq (2 + \epsilon) \sup_m \left\| T_m \right\| \left( \sum |t_k|^2 \right)^{1/2}.
\]

This completes the proof of Proposition 3.3 and therefore of Theorem 3.1.

REMARKS: (1) Let us sketch a proof of Paley’s result \((P)\) which uses the technique of the proof of Theorem 3.1.

Assume first that \((m_k)\) is a sequence of positive integers such that

\[
m_{k+1} \geq 2m_k \quad \text{for} \quad k = 1, 2, \ldots
\]

Let \(T_m = T_{\chi_{-m}}\) for \(m = 0, 1, \ldots\) be the compact operator on \(H^2\) which is the image of the coset \(\{\chi_{-m} + A_0\}\) by the isometry \(C_{2m}/A_0 \rightarrow K(H^2)\) defined in the proof of Proposition 3.2(ii). Then \(\langle T_m \chi_j, \chi_k \rangle = 0\) for \(j + k \neq m\) and \(\langle T_m \chi_j, \chi_k \rangle = 1\) for \(j + k = m\). Let \(P_m : H^2 \overset{\text{onto}}{\rightarrow} \text{span} (\chi_0, \chi_1, \ldots, \chi_{m-1})\) be the orthogonal projection. It follows from (3.7) that \(P_{n_{k-1}} T_{n_k} P_{n_{k-1}} = 0\) and \(T_{n_k} = P_{n_k} T_{n_k} P_{n_k}\) for \(k = 1, 2, \ldots\) (i.e. the sequences \((P_{n_k})\) and \((T_{n_k})\) satisfy (3.5) and (3.6) with \(n_k = m_k\) and \(\epsilon_k = 0\) for all \(k\)). Thus the argument used in the proof of Proposition
3.3 yields
\[ \left\| \sum t_k T_{mk} \right\| \leq 2 \left( \sum |t_k|^2 \right)^{1/2} \]
for every finite sequence of scalars \((t_k)\). Obviously \((\sum t_k T_{mk})(\sum \tilde{t}_k \chi_{mk}) = \sum_k |t_k|^2\). Hence
\[ \left\| \sum t_k T_{mk} \right\| \geq \left( \sum |t_k|^2 \right)^{1/2}. \]

Thus the subspace \([T_{mk}]\) is isomorphic to \(\ell^2\). Moreover \(Q\) defined by 
\[ Q(S) = \sum_k \langle S(x_0), \chi_{mk} \rangle T_{mk} \]
for \(S \in K(H^2)\) is a projection onto \([T_{mk}]\) with \(\|Q\| \leq 2\). Let us regard \(Q\) as an operator from \([T_{mk}]\) (the isometric image of \(C_{2m}/A_0\)) into itself and let \(P\) be the adjoint of \(Q\). Then, by Proposition 3.1(ii), \(P\) can be regarded as an operator from \(H^1\) into itself. Obviously \(\|P\| = \|Q\| \leq 2\). A direct computation shows that \(P\) is the orthogonal projection of \(H^1\) onto \([\chi_{mk}]\). To complete the proof of \((P)\) in the general case observe that every lacunary sequence admits a decomposition into a finite number of sequences satisfying (3.7).

(2) A similar argument gives also the following relative result.

Let \((f_n)\) be a sequence in \(H^1\). Assume that \(+ \infty > \sup_n \|f_n\|_\infty \approx \inf_n \|f_n\|_1 > 0\) and
\[ \lim_n \hat{f}_n(j) = 0 \quad \text{for every } j = 0, 1, \ldots \]
Then there exists an infinite subsequence \((n_k)\) and a \(1 \leq K < \infty\) such that the sequence \((f_{n_k})\) is \(K\)-equivalent to the unit vector basis of \(\ell^2\) and the orthogonal projection from \(H^1\) onto \([f_{n_k}]\) is a bounded operator.

Our next aim is to give a quantitative generalization of Theorem 3.1 to the case of \(H^p\) spaces \((1 < p \leq 2)\).

**Theorem 3.2:** Let \(1 < p \leq 2\) and let \(K \geq 1\). Then there exists an absolute constant \(c\) (independent of \(K\) and \(p\)) such that if \((f_n)\) is a sequence in \(H^p\) which is \(K\)-equivalent to the unit vector basis of \(\ell^2\), then there exists a subsequence \((n_k)\) such that there exists a projection \(P\) from \(H^p\) onto \([f_{n_k}]\) — the closed linear span of \((f_{n_k})\) with \(\|P\| \leq cK^2\).

**Proof:** Let \(X = [f_n]\). By the assumption, there exists an isomorphism \(T: \ell^2 \xrightarrow{\text{onto}} X\) with \(\|T\| \|T^{-1}\| \leq K\). Hence, by Proposition 2.3, there exists a \(\varphi \in H^1\) which satisfies an outer (2.2), (2.3), (2.4). Let us set \(\|f\|_{q,a} = (1/(2\pi)) \int_0^{2\pi} |f(t)|^q |\varphi(t)| dt \) for \(f\) measurable and for \(1 \leq q < \infty\). It follows from (2.2) that there exists in the open unit disc a
holomorphic function, say $\tilde{g}$, such that $\tilde{g} = e^{p\tilde{g}}$. Let us set

$$\varphi^{-1/p} (t) = \lim_{r \to 1} e^{\tilde{g}(r \tilde{t})} \quad \text{for } t \in [0, 2\pi].$$

Since $0 \neq \varphi \in H^1$, the limit exists for almost all $t$ and $\varphi^{1/p} = \frac{1}{\varphi^{-1/p}} \in H^p$. Furthermore observe that (2.4) is equivalent to

$$(3.8) \quad \|f\varphi^{-1/p}\|_{\psi,2} \leq \gamma K \|f\varphi^{-1/p}\|_{\psi,p} \quad \text{for } f \in X,$$

where $\gamma$ is the absolute constant appearing in Proposition 2.2. On the other hand, by the logarithmic convexity of the function $q \to \|\varphi^{-1/p}\|_q$, we get

$$\|f\varphi^{-1/p}\|_{\psi,p} \leq \|f\varphi^{-1/p}\|_{\psi,1}^{(2/p)-1} \|f\varphi^{-1/p}\|_{\psi,2}^{2-(2/p)} \quad \text{for } f \in X.$$ 

Thus

$$(3.9) \quad \|f\varphi^{-1/p}\|_{\psi,p} \leq (\gamma K)^{(2p-2)/(2-p)} \|f\varphi^{-1/p}\|_{\psi,1} \quad \text{for } f \in X.$$ 

Now, let $H^1_{\varphi}$ denote the Banach space being the completion of the trigonometric polynomials $\sum_{n=0}^c \varphi_n \chi_n$ in the norm $\|\cdot\|_{1,\varphi}$. It easily follows from (2.2) and (2.3) that $H^1_{\varphi}$ is isometrically isomorphic to $H^1$. The desired isometry is defined by $f \to f\varphi$ for $f \in H^1_{\varphi}$. Next (3.9) and the obvious relation

$$\|f\|_p = \|f\varphi^{-1/p}\|_{\psi,p} \geq \|f\varphi^{-1/p}\|_{\psi,1} \quad \text{for } f \in H^p$$

imply that the sequence $(f_n \varphi^{-1/p})$ belongs to $H^1_{\varphi}$ and in $H^1_{\varphi}$ is $K^{(2p-2)/(2-p)+1} \gamma^{(2p-2)/(2-p)}$—equivalent to the unit vector basis of $\ell^2$. Hence, by Theorem 3.1 which we apply to $H^1_{\varphi}$—the isometric image of $H^1$, there exists a subsequence $(n_k)$ and a projection

$$Q : H^1_{\varphi} \to [f_n \varphi^{-1/p}] \quad \text{with } \|Q\| \leq 5 \gamma^{(2p-2)/(2-p)} K^{p/(2-p)}.$$

Let us set

$$P(f) = \varphi^{1/p} Q(f \varphi^{-1/p}) \quad \text{for } f \in H^p.$$ 

To see that $P$ is well defined observe first that if $f \in H^p$, then, by the Hölder inequality and by (2.3),

$$\|f\varphi^{-1/p}\|_{1,\varphi} = \|f\varphi\|_{(p-1)/p} \leq \|f\|_p \|\varphi\|_{(p-1)/p} = \|f\|_p.$$ 

Thus, by (3.9), for every $f \in H^p$, we have

$$\|P(f)\|_p = \|\varphi^{1/p} Q(f \varphi^{-1/p})\|_p = \|Q(f \varphi^{-1/p})\|_{\psi,p} \leq (\gamma K)^{(2p-2)/(2-p)} \|Q(f \varphi^{-1/p})\|_{\psi,1}$$

$$\leq 5 \gamma^{(2p-2)/(2-p)} K^{(3p-2)/(2-p)} \|f\varphi^{-1/p}\|_{\psi,1} \leq 5 \gamma^{(4p-4)/(2-p)} K^{(3p-2)/(2-p)} \|f\|_p.$$
Thus $P$ is bounded. Obviously $P(H^p) \subset X$ and $P(f) = f$ for $f \in [f_n]$. Hence $P$ is a projection. Now, for $p \leq \frac{6}{5}$ we get (remembering that $\gamma \geq 1$ and $K \geq 1$)

$$\|P\| \leq 5\gamma^{(4p-4)(2-p)}K^{(3p-3)(2-p)} \leq 5\gamma K^2.$$  

If $p > \frac{6}{5}$, then an inspection of the proof of Proposition 2.1 shows that there exists an isomorphism $T$ from $L^p$ onto a subspace of $H^1$ with $\|T\|\|T^{-1}\| \leq k = \gamma_{11/10,6/5} \cdot 2\beta_{11/10}$ (we put in (2.1) and further $p_2 = \frac{6}{5}$, $p_1 = \frac{11}{10}$). Thus, by Theorem 3.1, we infer that every sequence in $L^p$ (particularly in $H^p$) which is $K$-equivalent to the unit vector basis of $\ell^2$ contains an infinite subsequence whose closed linear span is the range of a projection from $L^p$ of norm $\leq 5k \cdot K$. This completes the proof.

**Corollary 3.1:** There exists an absolute constant $c \geq 1$ such that, for $1 \leq p \leq 2$, every infinite-dimensional hilbertian subspace of $H^p$ contains an infinite dimensional subspace which is the range of a projection from $H^p$ of norm $\leq c$ and which is a range of an isomorphism from $\ell^2$, say $T$, with $\|T\|\|T^{-1}\| \leq c$.

**Proof:** Combine Theorems 3.1 and 3.2 with the recent result of Dacunha-Castelle and Krivine [5] from which, in particular, follows that every infinite-dimensional hilbertian subspace of $L^p$ contains, for every $\epsilon > 0$, a subspace which is $(1 + \epsilon)$—isomorphic to $\ell^2$.

Since the argument of Dacunha-Castelle and Krivine is quite involved, to make the paper self contained we include a proof of a slightly weaker Proposition 3.4 (which suffices for the proof of Corollary 3.1). This result and the argument below is due to H. P. Rosenthal\(^1\) and is published here with his permission.

**Proposition 3.4:** There exists an absolute constant $c$ such that every infinite-dimensional hilbertian subspace $X$ of $L^p$ (1 $\leq p \leq 2$) contains an infinite dimensional subspace $E$ such that there exists an isomorphism $T : \ell^2 \rightarrow E$ with $\|T\|\|T^{-1}\| \leq c$.

**Proof:** Since $L^p$ is isometrically isomorphic to a subspace of $L^1$ (1 $<$ $p \leq 2$), it is enough to consider the case $p = 1$. For $X \subset L^1$ and $X$

\(^1\)It was presented at the Functional Analysis Seminar in Warsaw in October 1973.
isomorphic to $\ell^2$ we put
\[ d(X, \ell^2) = \inf \{ \|S\| \|S^{-1}\| \colon S : \ell^2 \twoheadrightarrow X \text{ isomorphism} \} \]
\[ I_2(X) = \inf \left\{ \sup_{x \in X, \|x\|_1 = 1} \|T(x)\|_2 : T : L^1 \twoheadrightarrow L^1 \text{ positive isometry} \right\}. \]
\[ \tilde{I}_2(X) = \inf \{ I_2(Y) : Y \subset X, \dim X/Y < \infty \}. \]

Recall that, for the complex $L^1$, if $Z \subset L^1$ and $Z$ is isomorphic to $\ell^2$, then
\[ (3.10) \quad I_2(Z) \leq \frac{2}{\sqrt{\pi}} d(Z, \ell^2). \]

(This is a result of Grothendieck [12], cf. also Rosenthal [27]. It can be easily deduced from a result of Maurey [20], cf. the proof of our Proposition 2.3). Clearly
\[ I_2(Z) = \inf \left\{ \sup_{x \in Z, \|x\|_1 = 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |x(t)|^2 g^{-1}(t) dt \right)^{1/2} : g > 0, \|g\|_1 = 1 \right\}. \]

Now fix $X$ isomorphic to $\ell^2$ and pick $Y \subset X$ with $\dim X/Y < \infty$ so that $I_2(Y) < 2\tilde{I}_2(X)$. Replacing, if necessary $X$ by $T(X)$ for an appropriate positive isometry $T$ (depending only on $Y$ but not on subspaces of $Y$ of finite codimension), one may assume without loss of generality that
\[ (3.11) \quad I_2(Z) = \sup_{y \in Z, \|y\|_1 = 1} \|y\|_2 < 2\tilde{I}_2(X) \text{ for every } Z \subset Y \quad \text{with } \dim Y/Z < \infty. \]

We claim that (3.11) implies
\[ (3.12) \quad \text{for every } Z \subset Y \text{ with } \dim Y/Z < \infty \text{ there exists a } y \in Z \text{ such that} \]
\[ 1 = \|y\|_1 \leq \|y\|_2 < \frac{4}{\sqrt{\pi}}. \]

Indeed, let $m = \inf \{ \|y\|_1 : y \in Z \text{ and } \|y\|_1 = 1 \}$. Then, by (3.11), $m \|y\|_1 \leq \|y\|_2 < 2\tilde{I}_2(X) \|y\|_1$ for every $y \in Z$. Thus
\[ \frac{2\tilde{I}_2(X)}{m} > d(Z, \ell^2). \]

Hence, by (3.10),
\[ \frac{2\tilde{I}_2(X)}{m} > \frac{\sqrt{\pi}}{2} I_2(Z) \geq \frac{\sqrt{\pi}}{2} \tilde{I}_2(X). \]

Hence $m < 4/\sqrt{\pi}$ and this proves (3.12).

Let $(h_i)$ denote the Haar orthonormal basis. It follows from (3.12)
that one can define inductively a sequence \((y_n)\) in \(Y\) so that, for all \(n\),
\[
1 = \|y_n\|_2 \geq \|y_n\| > \frac{\sqrt{\pi}}{4},
\]
\(y_n\) is orthogonal to \(y_1, y_2, \ldots, y_{n-1}\) and \(h_1, h_2, \ldots, h_{n-1}\).

By a result of [2], passing again to a subsequence (if necessary) we may also assume that \((y_n)\) is equivalent to a block basic sequence with respect to the Haar basis regarded as a basis in \(L^{3/2}\). Now using the Orlicz inequality (cf. e.g. [25], p. 283), for arbitrary finite sequence of scalars \((t_n)\) we get
\[
\left\| \sum t_n y_n \right\|_2 \geq \left( \sum |t_n|^2 \|y_n\|^2 \right)^{1/2} \geq a \left( \sum |t_n|^2 \|y_n\|^2 \right)^{1/2} \geq \frac{\sqrt{\pi}}{4} \left( \sum |t_n|^2 \right)^{1/2} = \frac{a \sqrt{\pi}}{4} \left\| \sum t_n y_n \right\|_2.
\]
where \(a\) is an absolute constant depending only on the unconditional constant of the Haar basis in \(L^{3/2}\) and the constant in the Orlicz inequality for \(L^{3/2}\). Thus, for every \(f \in \text{span} (y_n)\),
\[
\|f\|_2 \geq \|f\|_{3/2} \geq \frac{a \sqrt{\pi}}{4} \|f\|_2.
\]
Hence by the logarithmic convexity of the function \(r \rightarrow \|f\|_r\)
\[
\|f\|_2 \geq \|f\|_{3/2} \geq \left( \frac{a \sqrt{\pi}}{4} \right)^3 \|f\|_2 \quad \text{for } f \in \text{span} (y_n).
\]
Thus the same inequality holds for \(f \in [y_n]\). Therefore \([y_n]\) is a subspace of \(X\) with \(d([y_n], \ell^2) \leq (4/(a \sqrt{\pi}))^3\). This completes the proof.

It is interesting to compare Corollary 2.1 with the following fact

**Proposition 3.5:** Let \(1 \leq p < 2\), let \(Y\) be a hilbertian subspace of \(H^p\). Then there exists a non complemented hilbertian subspace \(X\) of \(H^1\) which contains \(Y\).

**Proof:** Observe first that there exists a non complemented hilbertian subspace of \(H^p\) \((1 \leq p < 2)\). This follows from Proposition 1.2(iii) and from the corresponding fact for \(L^p\) \((1 < p < 2)\) (If \(1 < p \leq \frac{3}{2}\) then, by an observation of Rosenthal [26], p. 52, a result of Rudin [30] yields the existence of a non-complemented hilbertian subspace. If \(\frac{3}{2} < p < 2\), then the same fact for \(L^p\) was very recently observed by several mathematicians (cf. Bennet, Dor, Goodman, Johnson and
Newman [9]), finally $H^1$ contains an uncomplemented hilbertian subspace because, by Proposition 2.1, $H^1$ contains $H^p$ isomorphically for $2 > p > 1$.

Now Proposition 3.5 is an immediate consequence of the following general fact

**Proposition 3.6:** If a Banach space $Z$ contains a non complemented hilbertian subspace, say $E$, then every hilbertian subspace of $Z$ is contained in a non complemented hilbertian subspace.

**Proof:** Let $Y$ be a hilbertian subspace of $Z$. If $Y$ is finite dimensional, then the desired subspace is $Y + E$. If $Y$ is uncomplemented then there is nothing to prove. In the sequel suppose that $Y$ is infinite dimensional and that there exists a projection $P : Z \overset{onto}{\rightarrow} Y$. Let $E_1$ denote any subspace of $E$ with $\dim E/E_1 < \infty$. Let $P_{E_1}$ denote the restriction of $P$ to $E_1$. If $P_{E_1}$ were an isomorphic embedding, then the formula $SQP$ would define a projection from $Z$ onto $E_1$ where $Q$ is a projection from a hilbertian subspace $Y$ onto its closed subspace $P_{E_1}(E_1)$ and $S : P_{E_1}(E_1) \rightarrow E_1$, the inverse of $P_{E_1}$. Since $E$ is uncomplemented in $Z$, so is $E_1$. Hence the restriction of $P$ to no subspace of $E$ of finite codimension is an isomorphism. Combining this fact with the standard gliding hump procedure and the block homogeneity of the unit vector basis in $\ell^2$ (cf. [2]) we define a sequence $(e_n)$ in $E$ which is equivalent to the unit vector basis of $\ell^2$ and satisfies the condition $\|P(e_n)\| < 2^{-n}\|e_n\|$ for $n = 1, 2, \ldots$. This implies that, for some $n_0$, the perturbed sequence $(e_n - P(e_n))_{n > n_0}$ is equivalent to the unit vector basis of $\ell^2$; hence the space $F = \{e_n - P(e_n)\} \subset \ker P$ is hilbertian. If $F$ is not complemented in $Z$, then the desired subspace is $F + Y$. If $F$ is complemented in $Z$ and therefore in $\ker P$, then the standard decomposition method (cf. [22]) yields that $\ker P$ is isomorphic to $Z$. Thus $\ker P$ contains a non complemented hilbertian subspace, say $F_1$. The desired subspace can be defined now as $F_1 + Y$.

A modification of the above argument gives

**Proposition 3.7:** Let $Z$ be a separable Banach space such that (i) there exists a non complemented hilbertian subspace of $Z$, (ii) every infinite dimensional hilbertian subspace of $Z$ contains an infinite dimensional subspace which is complemented in $Z$. Then

(*) given infinite dimensional complemented hilbertian subspaces of $Z$, say $Y_1$ and $Y_2$, there exists an isomorphism of $Z$ onto itself which carries $Y_1$ onto $Y_2$. 

In particular $H^p$ satisfies (*) for $1 \leq p < 2$.

**PROOF:** Let $P_j$ be a projection from $Z$ onto $Y$; ($j = 1, 2$). Using (i) we construct similarly as in the proof of Proposition 3.6 subspaces $F_j$ of $\ker P_j$ which are isomorphic to $\ell^2$. By (ii) we may assume without loss of generality that $F_j$ are complemented in $Z$ and therefore in $\ker P_j$ ($j = 1, 2$). Now the decomposition technique gives that $\ker P_j$ is isomorphic to $Z$ for $j = 1, 2$. This allows to construct an isomorphism of $Z$ onto itself which carries $\ker P_1$ onto $\ker P_2$ and $P_1(Z)$ onto $P_2(Z)$.

### 4. Remarks and open problems

We begin this section with a discussion of the behavior of the Banach Mazur distances $d(L^p, H^p)$, $d(L^p, L^p/H^p_0)$, $d(H^p, L^p/H^p_0)$ for $p \to \infty$ and for $p \to 1$.

Recall that if $X$ and $Y$ are isomorphic Banach spaces, then $d(X, Y) = \inf \{ \| T \| \| T^{-1} \| : T : X \to Y, T - \text{isomorphism} \}$; if $X$ and $Y$ are not isomorphic, then $d(X, Y) = \infty$. Let $p^* = p(p - 1)^{-1}$. Then

$$(H^p)^* = \left\{ f \in L^{p^*} : \int_0^{2\pi} f(t)g(t)dt = 0 \text{ for } g \in H^p \right\} = H_0^{p^*}$$

Hence the map $\{ f + H_0^{p^*} \} \to x^*$ where $x^*_f(g) = 1/(2\pi) \int_0^{2\pi} f(t)g(t)dt$ for $g \in H^p$ is a natural isometric isomorphism from $L^{p^*}/H_0^{p^*}$ onto the conjugate $(H^p)^*$. Thus, for $1 < p < \infty$,

\begin{align}
(4.1) \quad d(L^p, H^p) = d(L^{p^*}, L^{p^*}/H_0^{p^*}); \quad d(H^p, L^p/H^p_0) = d(H^{p^*}, L^{p^*}/H_0^{p^*}).
\end{align}

The formulae (4.1) allow us to restrict our attention to the case where $p \to 1$. In the sequel we assume that $1 \leq p \leq 2$.

The results enlisted in section 1 give upper estimates for the distances in question. We have

**Proposition 4.1:** There exists an absolute constant $K$ such that

$$\max (d(L^p, H^p), d(L^p, L^p/H^p_0), d(H^p, L^p/H^p_0)) \leq K \frac{p}{p - 1} \quad (1 < p \leq 2).$$

**Proof:** By Proposition 1.1(iii) and 1.2(iii), $d(L^p, H^p) \leq K(p/p - 1)$ and $d(L^{p^*}, H^{p^*}) \leq Kp^* = Kp/(p - 1)$. Hence, by (4.1), $d(L^p, L^{p^*}/H_0^{p^*}) \leq Kp/(p - 1)$. Let $\bar{H}^p = f \in L^p : \bar{f} \in H^p$ and let $V$ denote the restriction to $\bar{H}^p$ of the quotient map $L^p \to L^{p^*}/H_0^{p^*}$. Clearly
\[ \| V(f) \|_{L^p/\mathcal{H}^p} \leq \| f \|_p \text{ for } f \in \mathcal{H}^p. \] Since \( Q(\bar{g}) \) for \( g \in \mathcal{H}^p \) (cf. section 1 for the definition of \( Q \)), we have, for \( f \in \mathcal{H}^p \) and \( g \in \mathcal{H}^p \), \( \| f \|_p = \| \bar{f} \|_p = \| Q(\bar{f} - g) \|_p \leq \| Q \|_p \| f - g \|_p. \) Thus \( \| V(f) \|_{L^p/\mathcal{H}^p} = \inf_{g \in \mathcal{H}^p} \| f - g \|_p \geq \| Q \|_p \| f \|_p \) for \( f \in \mathcal{H}^p \). Therefore the range of \( V \) is closed in \( L^p/\mathcal{H}^p \) and since \( \mathcal{H}^p + \mathcal{H}^p \) is dense in \( L^p \), \( V(\mathcal{H}^p) \) maps \( \mathcal{H}^p \) onto \( L^p/\mathcal{H}^p \). Since \( \mathcal{H}^p \cap \mathcal{H}^0 = \{0\} \), we infer that \( V \) is one to one. Thus \( d(\mathcal{H}^p, L^p/\mathcal{H}^0) = \| V \| \| V^{-1} \| \leq \| Q \|_p \leq \| B \|_p \leq Kp/(p - 1). \) To complete the proof observe that \( \mathcal{H}^p \) is isometrically isomorphic to \( \mathcal{H}^p \) via the map \( f \to f^* \) where \( f^*(t) = f(-t) \).

**Problem 4.1:** Does there exist an absolute constant \( k > 0 \) such that, for \( 1 < p < 2 \),

\[ \min \{ d(L^p, \mathcal{H}^p), d(L^p, L^p/\mathcal{H}^0), d(\mathcal{H}^p, L^p/\mathcal{H}^0) \} > k \frac{p}{p - 1}. \]

We are able to prove only

**Proposition 4.2:** There exists an absolute constant \( k > 0 \) such that

(a) \( d(L^p, \mathcal{H}^p) \geq k \sqrt{\frac{p}{p - 1}} \) \((1 < p \leq 2),\)

(b) \( d(\mathcal{H}^p, L^p/\mathcal{H}^0) \geq k \sqrt{\frac{p}{p - 1}} \) \((1 < p \leq 2),\)

(c) \( \lim_{p \to 1} d(L^p, L^p/\mathcal{H}^0) = \infty. \)

**Proof:** (a) is an immediate consequence of the following stronger result.

(a') There exists an absolute constant \( k > 0 \) such that if \( X \) is a subspace of \( \mathcal{H}^p \) \((1 < p \leq 2),\) if \( X \) contains a subspace isomorphic to \( \ell^2, \) and if \( X \xrightarrow{S} L^p \xrightarrow{T} X \) is a factorization of identity (i.e. \( TS = \) the identity on \( X \)), then \( \| T \| \| S \| \geq \sqrt{p/(p - 1)} \).

**Proof** Applying Corollary 3.1: we can choose a subspace \( E \subset X \) an isomorphism \( U : E \xrightarrow{\text{onto}} \ell^2 \) and a projection \( P : X \xrightarrow{\text{onto}} E \) so that \( \| U \| \| U^{-1} \| \leq c \) and \( \| P \| \leq c \) where \( c \) is an absolute constant. Let \( S_1 = SU^{-1} \) and \( T_1 = UPT. \) Then \( \ell^2 \xrightarrow{S_1} L^p \xrightarrow{T_1} \ell^2 \) is a factorization of identity with \( \| S \| \| T \| \| T_1 \| \geq \| S \| \| T \| \cdot c^2. \) Now the desired conclusion follows from a result of Gordon, Lewis and Retherford [11], Remark (1) to Corollary 5.7 which asserts that there exists an absolute constant \( k_1 \) such that if \( \ell^2 \xrightarrow{S_1} L^p \xrightarrow{T_1} \ell^2 \) is any factorization of identity, then \( \| T \| \| S \| \geq k_1 \sqrt{p/(p - 1)} \) \((1 < p \leq 2). \) This completes the proof of (a').
(b') There exists an absolute constant \( k > 0 \) such that if \( U \) is an isomorphism from \( L^p / H_0^p \) onto a subspace \( X \) of \( H^p \) \((1 < p \leq 2)\) then \( \|U\| \|U^{-1}\| \geq k \sqrt{p/(p-1)} \).

**Proof:** Let \( X_p \) denote the closed linear subspace of \( L^p \) \((1 < p \leq 2)\) generated by the sequence \( \langle x_{-2k} \rangle \). Let \( I_p : L^p \rightarrow L^1 \) and \( j_p : L^p / H_0^p \rightarrow L^1 / H_0^p \) denote natural embeddings (i.e. \( j_p(\{f + H_0^p\}) = \{f + H_0^p\} \)) and let \( q_p : L^p \rightarrow L^p / H_0^p \) denote the quotient map. Clearly \( \|q_p\| \leq 1 \) and, we have \( q_p I_p = q_p I_p \). A direct computation shows that \( \|f\|_p \leq 2^{1/2} \|f\|_1 \) for \( f \in X_2 \). Thus the logarithmic convexity of the function \( p \rightarrow \|f\|_p \) yields
\[
\|f\|_2 \geq \|f\|_p \geq \|f\|_1 \geq 2^{-1/2} \|f\|_2 \quad \text{for } f \in X_p.
\]
It follows from the above inequality and from the proof of Proposition 2.4 that the operator \( V_p \) — the restriction of \( q_p \) to \( X_p \) is invertible and \( \|V_p^{-1}\| \leq c \) where \( c \) is an absolute constant independent of \( p \). Since \( X_p \) is isometric to \( \ell^2 \), so is \( UV_p(X_p) \). Hence, by Corollary 3.1, there exist a subspace \( E \) of \( UV_p(X_p) \) an isomorphism \( T : E \overset{\text{onto}}{\rightarrow} \ell^2 \) and a projection \( P : X \overset{\text{onto}}{\rightarrow} E \) with \( \|T\| \|T^{-1}\| \leq c_1 \) and \( \|P\| \leq c_1 \) where \( c_1 \) is an absolute constant. Now we consider the factorization of identity.
\[
\ell^2 \overset{T^{-1}}{\rightarrow} E \overset{U^{-1}}{\rightarrow} V_p(X_p) \overset{V_p^{-1}}{\rightarrow} L^p \overset{q_p}{\rightarrow} L^p / H_0^p \overset{U}{\rightarrow} X \overset{P}{\rightarrow} E \overset{T}{\rightarrow} \ell^2
\]
By a result of [11], Remark (1) to Corollary 5.7, there exists an absolute constant \( k_1 > 0 \) such that
\[
k_1 \sqrt{\frac{P}{p-1}} \leq \|V_p^{-1}T^{-1}\| \|TPUq_p\|
\leq \|T\| \|T^{-1}\| \|V_p^{-1}\| \|q_p\| \|P\| \|U\| \|U^{-1}\|
\leq c_1^2 \|U\| \|U^{-1}\|.
\]
Thus \( \|U\| \|U^{-1}\| \geq k \sqrt{p/(p-1)} \) for \( k = k_1 c_1^2 c^{-1} \). This completes the proof of (b').

To prove (c), in view of the fact that, for \( 1 < p \leq 2 \) \( H^p \subset L^p \) is isometrically isomorphic to a subspace of \( L^1 \) (cf. e.g. [27], p. 354), it is enough to show
\[
(c') \text{Let } d_p = \inf \{d(L^p / H_0^p, X) : X \subset L^1 \} \text{ for } (1 < p \leq 2). \text{Then } \lim_{p \rightarrow 1} d_p = \infty.
\]

**Proof of (c'):** Fix \( \epsilon > 0 \) and a finite-dimensional subspace \( B \) of \( L^1 / H_0^1 \). Since the continuous \( 2\pi \)-periodic functions are dense in \( L^1 \), the standard perturbation argument (cf. e.g. [2]) yields the existence of a
(dim $B$)-dimensional subspace $G$ of $C_{2\pi}$ with $G \cap H_0 = \{0\}$ such that
\[d(B, (G + H_0)/H_0) < (1 + \epsilon)^{1/2}\]
($G + H_0$ is regarded as a subspace of $L$). Let us put
\[\|g\|_p = \inf \{\|g + h\|_p : h \in H_0\} \quad (g \in G, p \geq 1)\]
and let $G_p$ stand for $G$ equipped with the norm $\|\cdot\|_p$. We claim that
\[(4.2) \quad \text{If } g \downarrow p, \text{ then } \|g\|_q \downarrow \|g\|_p \quad (g \in G, p \geq 1).\]
To see (4.2) observe first that
\[\text{because } A_0 \text{ is dense in each } H_0^c. \text{ Next note that, for every } g \in G \text{ and } h \in A_0, \text{ the function } p \rightarrow \|g + h\|_p \text{ is (finite) continuous and non decreasing. Thus}
\]
\[\lim_{q \downarrow p} \|g\|_q \leq \|g\|_p \text{ and } \|g\|_q \geq \|g\|_p \quad (g \in G, 1 \leq p < q)\]
which yield (4.2).

Let $S^1_G = \{g \in G : \|g\|_1 = 1\}$. Since $G$ is finite-dimensional, $S^1_G$ is compact. Hence Dini’s Theorem combined with (4.2) implies that
\[\|g\|_p \rightarrow \|g\|_1 = 1 \text{ uniformly on } S^1_G \text{ as } p \rightarrow 1.\]
Therefore there exists a $p_0 = p_0(B, \epsilon) > 1$ such that
\[(1 + \epsilon)^{1/2} \geq \|g\|_p \geq 1 \text{ for } g \in S^1_G \text{ and for } 1 < p < p_0.\]
Equivalently the formal identity map $j_p : G_p \rightarrow G$ is an isomorphism with
\[\|j_p\| \leq (1 + \epsilon)^{1/2}.\]
Clearly $G_p$ is isometrically isomorphic to the subspace $(G + H_p)/H_p$ of $L^q/H_0$. Using this fact for $p = 1$ we get
\[(4.3) \quad d(B, G_p) \leq 1 + \epsilon \quad (1 < p < p_0).\]

Now suppose to the contrary that there exist a sequence $(p(n))$ with $\lim n p(n) = 1$, a constant $\lambda > 0$ and a sequence $(\mathcal{X}_n)$ of subspaces of $L^1$ such that
\[d(L^{p(n)}/H_0^{p(n)}, \mathcal{X}_n) < \lambda \quad \text{for all } n.\]
Then (4.3) would imply that for every finite-dimensional subspace $B$ of $L^1/H_0^c$ there exists a subspace $B_1$ in $L^1$ with $d(B, B_1) < \lambda$. Hence, by [16], Proposition 7.1, $L^1/H_0^c$ would be isomorphic to a subspace of some $L^1(\mu)$-space which contradicts [24]. This completes the proof of (c’) and therefore of Proposition 4.2.

There are several problems related to Proposition 2.1.
PROBLEM 4.2: Does there exist an absolute constant $\lambda \geq 1$ such that, for every $p$ and $q$ with $1 \leq q < p < 2$, there exists a subspace $X_{p,q}$ of $H^q$ such that $d(H^p, X_{p,q}) \leq \lambda$? In particular is $H^p$ isometrically isomorphic to a subspace of $H^q$?

The recent result of Dacunha-Castelle and Krivine [5] yields that, for every $p$ with $1 \leq p < \infty$ and for every $\lambda > 1$, there exists a subspace $X$ of $H^p$ such that $d(X, \ell^2) \leq \lambda$. In fact a subspace $X$ with the above property can be defined as the closed linear span of a sequence $(\sum_{k=0}^{m+1} \chi_{n_k})_{m=1,2,...}$ where $k$ and the “lacunary” sequence $(n_k)$ depend on $p$ and $q$. We do not know, however, whether $\ell^2$ is isometrically isomorphic to a subspace of $H^p$ for any $p \neq 2$? On the other hand there is no subspace of $H^p$ which is isometrically isomorphic to the 2-dimensional space $\ell^2_2$ ($p \neq 2$). Otherwise there would exist in $H^p$ functions $f_1$ and $f_2$ of norm one such that $\|f_1 + f_2\|_p + \|f_1 - f_2\|_p = 2(\|f_1\|_p + \|f_2\|_p)$. Then (cf. e.g. [22]) $f_1 \cdot f_2 = 0$. Thus the analyticity of the $f_i$'s would imply that either $f_1$ or $f_2$ is zero, a contradiction. This remark answers negatively a question of Boas [4] who asked whether $H^p$ is isometrically isomorphic to $L^p$ for some $p \neq 2$.

Finally we would like to mention the well known open problems concerning the existence of unconditional structures in $H^1$.

PROBLEM 4.3: (a) Does $H^1$ have an unconditional basis?

(b) Is $H^1$ isomorphic to a subspace of a Banach space with an unconditional basis? (c) Does $H^1$ have a local unconditional structure either in the sense of [6] or of [10]?

Let us mention that the basis for $H^1$ which has been constructed by Billard [3] is conditional.

Let us recall briefly Billard’s construction. Let $H^1_k$ denote the real Banach space of functions $f \in L^1_k$ such that $\mathcal{H}(f) \in L^1_k$ equipped with the norm $\|f\|_1 = \sqrt{\|f\|_1^1 + \|\mathcal{H}(f)\|_1^1}$. It is easy to see that the complexification of $H^1_k$ is isomorphic to $H^1$. Therefore every basis for $H^1_k$ induces a basis for $H^1$. Billard [3] has proved that the classical Haar system $(h_k)_{0 \leq k < \infty}$ is a basis for $H^1_k$. (In our convention the $h_k$'s are defined on the whole real line, are $2\pi$-periodic, and restricted to $[0,2\pi)$ consist the Haar orthonormal system i.e. $h_0 = 1$ and for $j = 0, 1, ..., r = 0, 1, ..., 2^{j-1} - 1$, $h_{2^{j-1}r}(t) = 2^{j/2}(I_{\Delta(j+1,2r)} - I_{\Delta(j+1,2r+1)})(t)$ for $0 \leq t < 2\pi$ where $\Delta(j+1,k) = \{t \in \mathbb{R} : 2\pi k 2^{-j-1} < t < 2\pi (k+1) 2^{-j-1}\}$ and $I_A$ denotes the characteristic function of a set $A \subseteq \mathbb{R}$.)

PROPOSITION: The sequence $(h_k)_{0 \leq k < \infty}$ is a conditional basis for $H^1_k$. 

"
PROOF: Let us set \( g_0 = 2h_1, \ g_0^* = 2h_1, \)

\[
    g_n = 2h_1 + \sum_{j=1}^{n} 2^{\mu_j} (h_{2^j} + h_{2^{j+1} - 1}),
\]

\[
    g_n^* = 2h_1 + \sum_{k=1}^{n/2} 2^{(2k+1)\mu_k} (h_{2^{2k+1}} + h_{2^{2k+2} - 1}).
\]

Since \( \|g_n^*\|_1 \geq \|g_n^*\|_1 \geq n/4 \) for all \( n \) (an easy computation), to complete the proof it suffices to show that \( \sup_n \|g_n\|_1 < \infty \). Observe that, for all \( n \),

\[
    g_n(t) = 2^{n+1}(I_{\Delta(n+1,0)} - I_{\Delta(n+1,2^{n+1} - 1)})(t) \quad \text{for} \ 0 \leq t < 2\pi.
\]

Thus \( \|g_n\|_1 = 2 \) for all \( n \). Therefore our task is to show that \( \sup_n \|H(g_n)\|_1 < \infty \).

We have almost everywhere (cf. [33], [7])

\[
    H(g_n)(t) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{t-\varepsilon}^{t+\varepsilon} \frac{\text{ctg} \left( \frac{s}{2} \right)}{s} \left[ g_n(t-s) - g_n(t+s) \right] ds
\]

\[
    = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{t-\varepsilon}^{t+\varepsilon} \left[ \frac{\text{ctg} \left( \frac{s}{2} \right)}{s} - \frac{2}{s} \right] \left[ g_n(t-s) - g_n(t+s) \right] ds
\]

\[
    + \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{t-\varepsilon}^{t+\varepsilon} \frac{2}{s} \left[ g_n(t-s) - g_n(t+s) \right] ds.
\]

Since

\[
    \left| \frac{\text{ctg} \left( \frac{s}{2} \right)}{s} - \frac{2}{s} \right| < \frac{2}{\pi} \quad \text{for} \ 0 < s < \pi \ \text{and} \ \|g_n\|_1 = 2,
\]

we infer that

\[
    \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{t-\varepsilon}^{t+\varepsilon} \left[ \frac{\text{ctg} \left( \frac{s}{2} \right)}{s} - \frac{2}{s} \right] \left[ g_n(t-s) - g_n(t+s) \right] ds \|_1 \leq c_1
\]

for some constant \( c_1 \) independent of \( n \). On the other hand, evaluating the second integral, we get

\[
    \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{t-\varepsilon}^{t+\varepsilon} \frac{2}{s} \left[ g_n(t-s) - g_n(t+s) \right] ds = \frac{2^n}{\pi} \ln \left( \frac{(t - 2^{-n}\pi)(t + 2^{-n}\pi)}{t^2} \right)
\]

\[
    = \frac{2^n}{\pi} \ln \left( 1 - \frac{\pi^2}{(2^n t)^2} \right).
\]

Since

\[
    2^n \int_0^{2^n} \ln \left| 1 - \frac{\pi^2}{(2^n t)^2} \right| dt = c_2 < +\infty,
\]

we infer that \( \|H(g_n)\|_1 \leq c_1 + c_2 \) for all \( n \). This completes the proof.
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