COMPOSITIO MATHEMATICA

W. B. JOHNSON

On quotients of L_p which are quotients of ℓ_p

Compositio Mathematica, tome 34, nº 1 (1977), p. 69-89
<http://www.numdam.org/item?id=CM_1977__34_1_69_0>

COMPOSITIO MATHEMATICA, Vol. 34, Fasc. 1, 1977, pag. 69–89
Noordhoff International Publishing
Printed in the Netherlands

ON QUOTIENTS OF L_p WHICH ARE QUOTIENTS OF ℓ_p

W. B. Johnson*

Abstract

The main result is that a quotient of L_p $(2 < p < \infty)$ which is of type p-Banach–Saks is a quotient of ℓ_p. This is used to solve a problem left open in a paper of Odell and the author [4].

The techniques used also yield, for example, that every operator from L_p $(2 < p < \infty)$ into a subspace of a quotient of ℓ_p factors through ℓ_p, and that every quotient of a space which has a shrinking unconditional finite dimensional decomposition contains an unconditionally basic sequence.

I. Introduction

A natural problem which arises in an attempt to relate the structure of L_p $(= L_p[0, 1])$ to that of ℓ_p (we always assume $1 < p < \infty$, $p \neq 2$) is:

A. Give a Banach space condition so that if X is a subspace of L_p which satisfies the condition, then X embeds isomorphically into ℓ_p.

For $p > 2$, a good answer to A was given in [4]; the condition on X is that no subspace of X is isomorphic to ℓ_2. A reasonable solution to A for $p < 2$ was given when X has an unconditional finite dimensional decomposition (f.d.d.); the condition on X is that there is $\lambda < \infty$ so that every normalized basic sequence in X has a subsequence which is λ-equivalent to the unit vector basis for ℓ_p. (This really is a Banach space condition, since it can be reformulated as: there is $\lambda < \infty$ so that every normalized weakly null sequence has a subsequence whose closed linear span is λ-isomorphic to ℓ_p.) One consequence of our main

*Supported in part by NSF–MPS72–04634–A03.

result here is that the hypothesis that X have an unconditional f.d.d. is superfluous.

It happens that it is more convenient to consider the equivalent dual problem to A, which we now state:

B. Give a Banach space condition so that if X is a quotient of L_p which satisfies the condition, then X is isomorphic to a quotient of ℓ_p.

Of course, since we are interested in A for $p < 2$, B is of interest for $p > 2$.

Before stating our main result we need a definition. A Banach space X is said to be of *type p Banach–Saks* if there is a constant λ so that every normalized weakly null sequence in X has a subsequence (x_n) which satisfies $\|\sum_{i=1}^n x_i\| \leq \lambda n^{1/p}$ $(n = 1, 2, \ldots)$. It follows from, e.g., [5], that quotients of ℓ_p are of type p Banach–Saks. Our main result, Theorem III.2, yields that a quotient of L_p $(2 < p < \infty)$ which is of type p Banach–Saks is isomorphic to a quotient of ℓ_p. Actually, Theorem III.2 gives an analogous result for quotients of subspaces of L_p, as long as the subspace of L_p has the approximation property.

Another problem which arises when one is trying to relate L_p to ℓ_p is that of classifying operators from L_p which factor through ℓ_p. In Theorem III.1 we prove that every operator from L_p $(2 < p < \infty)$ into a subspace of a quotient of ℓ_p factors through ℓ_p. (We do not know whether this is true for $1 < p < 2$.)[1] Another way of stating this result is that every operator from a subspace of a quotient of ℓ_p $(1 < p < 2)$ into L_p can be factored through ℓ_p. We should mention that it is known (cf. [10] for $1 < p < 4/3$, [11] for $p > 2$, and [1] for $1 < p < 2$) that there is a subspace X of ℓ_p $(1 < p < \infty, p \neq 2)$ which is isomorphic to ℓ_p and an isomorphism from X into L_p which does not extend to an operator from ℓ_p into L_p. However, it might be true that if X embeds into ℓ_p then every isomorphism from X into L_p factors as $X \to \ell_p \to L_p$ with the map $\ell_p \to L_p$ an isomorphism.

The main technique used in the proofs of the above mentioned results is the blocking method developed in [5] and [4]. The simplest application of this technique given in this paper is presented in Theorem II.1, which says that every operator from a space with shrinking block p-Besselian f.d.d. $(1 < p < \infty)$ into a subspace of a space with block p-Hilbertian f.d.d. factors through a space of the form $(\sum E_n)_{\ell_p}$ with $\dim E_n < \infty$. (Recall that an f.d.d. (F_n) is called block p-Besselian (respectively, block p-Hilbertian) provided there is a positive constant λ so that for any blocking $E_i = F_{n(i)} + F_{n(i)+1} + \cdots +$

[1] Recently we have shown that this is also true for $1 < p < 2$; cf. *J. London Math. Soc.* (2) 14 (1976).

$F_{n(i+1)-1}$ $(n(1) < n(2) < \ldots)$ of (F_n) and any $e_i \in E_i$, $\|\Sigma\, e_i\| \geq \lambda (\Sigma\, \|e_i\|^p)^{1/p}$ (respectively, $\|\Sigma\, e_i\| \leq \lambda(\Sigma\, \|e_i\|^p)^{1/p}$)).

The blocking method in its simplest form can be described as follows: Suppose (E_n) is a shrinking f.d.d. for X and T is an operator from X into a space Y which has an f.d.d. (F_n). Then there is a blocking (E'_n) of (E_n) and a blocking (F'_n) of (F_n) so that TE'_n is essentially contained in $F'_n + F'_{n+1}$ $(n = 1, 2, \ldots)$. The "overlap" between TE'_n and TE'_{n+1} in F'_{n+1} causes difficulties which, however, one can get around in certain situations (cf. [5] and [4]).

The main new idea in the present paper involves the use of a technique for "killing the overlap" after employing the blocking method when T is a quotient map. This killing the overlap technique is most simply exposed in the proof of Theorem II.7, which yields, in particular, that every quotient of a space which has a shrinking unconditional f.d.d. contains an unconditionally basic sequence.

We use standard Banach space theory terminology, as may be found, for example, in the book of Lindenstrauss and Tzafriri [7].

II. General results

The first result we present uses a simple application of the blocking method of [5].

THEOREM II.1: *Suppose X has a shrinking block p-Besselian f.d.d. (E_n), Y embeds into a space Z which has a block p-Hilbertian f.d.d. (F_n), and T is an operator from X into Y. Then T factors through a space W of the form $(\Sigma\, W_n)_{\ell_p}$, where $W_k = [E_i]_{i=m(k)}^{m(k+1)-1}$ for some sequence $1 = m(1) < m(2) < \cdots$ of integers.*

PROOF: Since (E_n) is shrinking, the technique of Lemma 1 of [5] applies to yield a blocking $W_k = [E_i]_{i=m(k)}^{m(k+1)-1}$ of (E_n) and a blocking $G_k = [F_i]_{i=n(k)}^{n(k+1)-1}$ of (F_n) so that TW_k is essentially a subspace of $G_k + G_{k+1}$; i.e., for each $x \in W_k$, there exists $g \in G_k + G_{k+1}$ so that $\|Tx - g\| \leq 2^{-k}\|x\|$. (For a complete proof of an extension of this, see Lemma II.3 and the proof of Theorem II.4 below.)

Define $S: X \to (\Sigma\, W_k)_{\ell_p}$ by $S(\Sigma\, w_k) = (w_k)$ (where $w_k \in W_k$). If (E_n) is block p-Besselian with constant λ, then $\|S\| \leq \lambda^{-1}$.

Define $U: (\Sigma\, W_k)_{\ell_p} \to Y$ by $U((w_k)) = \Sigma\, Tw_k$. (More precisely we first define U on the linear span of the W_k's in $(\Sigma\, W_k)_{\ell_p}$. The argument below shows that U is bounded there and hence can be extended to all of $(\Sigma\, W_k)_{\ell_p}$.) Suppose $w_k \in W_k$. Choose $y_k \in G_k + G_{k+1}$ so that $\|Tw_k - y_k\| \leq 2^{-k}\|w_k\|$ and let β be the block p-Hilbertian constant of (F_n). Then

$\|U(w_k)\| = \|\Sigma\, Tw_k\| \le \|\Sigma\, Tw_{2k}\| + \|\Sigma\, Tw_{2k+1}\| \le \|\Sigma\, y_{2k}\| + \Sigma\, 2^{-2k}\|w_{2k}\| + \|\Sigma\, y_{2k+1}\| + \Sigma\, 2^{-2k-1}\|w_{2k+1}\| \le \beta(\Sigma\,\|y_{2k}\|^p)^{1/p} + \beta(\Sigma\,\|y_{2k+1}\|^p)^{1/p} + (\Sigma\,\|w_k\|^p)^{1/p} \le 2\beta(\Sigma\,\|y_k\|^p)^{1/p} + (\Sigma\,\|w_k\|^p)^{1/p} \le 2\beta[(\Sigma\,\|Tw_k\|^p)^{1/p} + (\Sigma\,\|w_k\|^p)^{1/p}] + (\Sigma\,\|w_k\|^p)^{1/p} \le (2\beta\|T\| + 2\beta + 1)(\Sigma\,\|w_k\|^p)^{1/p}$.

Thus $\|U\| \le 2\beta\|T\| + 2\beta + 1$. Since $US = T$, the proof is complete.

REMARK II.2: The hypothesis in Theorem II.1 that (E_n) is shrinking is necessary. Indeed, if $X = \ell_1$, $p = 2$, $Y = \ell_4$, and T is a quotient mapping from X onto Y, then T does not factor through any space of the form $(\Sigma\, W_n)_{\ell_2}$ with $\dim W_n < \infty$.

We would like to weaken the hypothesis on Y in Theorem II.1 to "Y is a subspace of a quotient of a space which has a block p-Hilbertian f.d.d." In preparation for this we need a version of Lemma 1 in [5]. We include a proof for the convenience of the reader.

LEMMA II.3: *Suppose that X has a shrinking f.d.d. (E_n), Y is a subspace of M, Q is a quotient mapping of Z onto M, (F_n) is an f.d.d. for Z, and $T: X \to Y$ is an operator. Then given any integer n and $\epsilon > 0$, there exists an integer $m = m(n, \epsilon)$ so that if $0 \ne x \in [E_k]_{k=m}^{\infty}$, then there is $z \in [F_k]_{k=n}^{\infty}$ so that $\|z\| < (2+\epsilon)\|T\|\|x\|$ and $\|Tx - Qz\| < \epsilon\|x\|$.*

PROOF: If the conclusion is false, then there are unit vectors $x_m \in [E_i]_{i=m}^{\infty}$ for $m = 1, 2, \dots$ so that $d(Tx_m, Q(2+\epsilon)\|T\| \operatorname{Ball} [F_i]_{i=n}^{\infty}) \ge \epsilon$. Thus by the separation theorem, there are norm one functionals f_m on M so that $f_m(Tx_m) \ge (2+\epsilon)\|T\| f_m Qz + \epsilon$ for each m and each $z \in \operatorname{Ball} [F_i]_{i=n}^{\infty}$.

Choose $z_m \in Z$ with $\|z_m\| \le (1 + 3^{-1}\epsilon)\|Tx_m\|$ and so that $Qz_m = Tx_m$. Now $x_m \overset{w}{\to} 0$, hence $Tx_m \overset{w}{\to} 0$, whence for each fixed k, $f_k(Qz_m) \to 0$ as $m \to \infty$. Thus by passing to a subsequence of (z_m) and the corresponding subsequence of (x_m), we can assume that $f_k(Qz_m) < \epsilon/2$ for $k < m$. Letting P_n be the natural projection from Z onto $[F_i]_{i=1}^{n}$, we can assume by passing to a further subsequence that $P_n z_m$ norm converges as $m \to \infty$, and thus, $\|P_n z_m - P_n z_{m+1}\| \to 0$ as $m \to \infty$. Thus $\|z_m - z_{m+1}\| < 2(1 + 3^{-1}\epsilon)\|T\|$ and $d(z_m - z_{m+1}, [F_i]_{i=n}^{\infty}) \to 0$ as $m \to \infty$. However, $f_m(Q(z_m - z_{m+1})) \ge f_m Qz_m - \epsilon/2 = f_m Tx_m - \epsilon/2$, hence when m is large enough, there is $z \in [F_i]_{i=n}^{\infty}$ with $\|z\| < (2+\epsilon)\|T\|$ so that $f_m(Qz) > f_m(Tx_m) - \epsilon$, which contradicts the choice of f_m. This completes the proof.

THEOREM II.4: *Suppose X has a shrinking block p-Besselian f.d.d. (E_n), Y is a subspace of M, M is a quotient of a space Z which has a block p-Hilbertian f.d.d. (F_n), and T is an operator from X into Y. Then*

T factors through a space W of the form $(\Sigma W_n)_{\ell_p}$, where $W_k = [E_i]_{i=m(k)}^{m(k+1)-1}$ for some sequence $1 = m(1) < m(2) < \cdots$ of integers.

PROOF: As in the proof of Theorem II.1, we want a blocking $W_k = [E_i]_{i=m(k)}^{m(k+1)-1}$ of (E_n) and a blocking $G_k = [F_i]_{i=n(k)}^{n(k+1)-1}$ of (F_n) so that TW_k is essentially contained in $Q(G_k + G_{k+1})$; i.e.,

(*) for each $0 \neq x \in W_k$, there exists $g \in G_k + G_{k+1}$ so that $\|g\| < 3\|T\|\|x\|$ and $\|Tx - Qg\| < 2^{-k}\|x\|$.

Once we have (*), we finish the proof by mimicing the proof of Theorem II.1. Indeed, define $S : X \to (\Sigma W_k)_{\ell_p}$ by $S(\Sigma w_k) = (w_k)$ (where $w_k \in W_k$). If (E_n) is block p-Besselian with constant λ, then $\|S\| \leq \lambda^{-1}$.

Define $U : (\Sigma W_k)_{\ell_p} \to M$ by $U(w_k) = \Sigma Tw_k$. Suppose $w_k \in W_k$. Using (*), we choose $g_k \in G_k + G_{k+1}$ so that $\|g_k\| \leq 3\|T\|\|w_k\|$ and $\|Tw_k - Qg_k\| \leq 2^{-k}\|w_k\|$. Letting β be the block p-Hilbertian constant of (F_n), we have $\|U(w_k)\| = \|\Sigma Tw_k\| \leq \|\Sigma Qg_k\| + \Sigma 2^{-k}\|w_k\| \leq \|\Sigma g_k\| + (\Sigma \|w_k\|^p)^{1/p} \leq \|\Sigma g_{2k}\| + \|\Sigma g_{2k+1}\| + (\Sigma \|w_k\|^p)^{1/p} \leq 2\beta(\Sigma \|g_k\|^p)^{1/p} + (\Sigma \|w_k\|^p)^{1/p} \leq (6\beta\|T\| + 1)(\Sigma \|w_k\|^p)^{1/p}$. Thus $\|U\| \leq 6\beta\|T\| + 1$. Since $US = T$, this will complete the proof.

It remains to produce the blockings (W_k) and (G_k). For this we apply Lemma II.3 and use the argument of Proposition 1 of [5]. By induction we choose increasing sequences $1 = n(1) < n(2) < \cdots$ and $1 = m(1) < m(1) < \cdots$ of integers to satisfy for all $1 \leq k < \infty$:

(A) For each $x \in [E_j]_{j=m(k)}^{m(k+1)-1}$ with $\|x\| = 1$, there is z in $[F_j]_{n(k)}^{n(k+2)-1}$ so that $\|z\| < 3\|T\|$ and $\|Tx - Qz\| < 2^{-k}$.

(B) For each x in $[E_j]_{j=m(k)}^{\infty}$ with $\|x\| = 1$, there is z in $[F_j]_{j=n(k)}^{\infty}$ so that $\|z\| < 3\|T\|$ and $\|Tx - Qz\| < 2^{-k}$.

Set $n(1) = m(1) = 1$, let $n(2) > 1$, and, using Lemma II.3, choose $m(2) > m(1)$ so that (B) holds for $k = 2$. (B) obviously is true for $k = 1$, so choose $n(3) > n(2)$ to make (A) valid for $k = 1$. Now select $m(3) > m(2)$ so that (B) holds for $k = 3$. Using (B) for $k = 2$ choose $n(4) > n(3)$ to make (A) true for $k = 3$. Continue in this way to define $(m(k))$ and $(n(k))$ by induction.

Of course, (A) is equivalent to (*) and thus the proof of the theorem is complete. By duality we obtain:

THEOREM II.5: *Suppose X is isomorphic to a subspace of a quotient of a space which has a shrinking block p-Besselian f.d.d., T is an operator from X into Y, and Y is isomorphic to a subspace of a reflexive* *space Z which has a block p-Hilbertian f.d.d. Then T factors through a subspace of* $(\Sigma\ W_k)_{\ell_p}$ *for some sequence* (W_k) *of finite dimensional spaces.*

PROOF: Use Theorem II.4 to factor $T^*Q : Z^* \to X^*$ through $(\Sigma\ W_k^*)_{\ell_q}$ $(p^{-1}+q^{-1}=1)$ with $\dim W_k < \infty$, where Q is the natural quotient mapping from Z^* onto Y^*. Then obviously T^{**}, consider as an operator from X^{**} into $Z^{**}=Z$, factors through $(\Sigma\ W_k)_{\ell_p}$, hence T, considered as an operator from X into Y, factors through a subspace of $(\Sigma\ W_k)_{\ell_p}$.

REMARK II.6: The reflexivity of Z is used to insure that the natural block p-Besselian f.d.d. for a subspace of Z^* is actually a shrinking block p-Besselian f.d.d. for Z^*, so that T^*Q factors through $(\Sigma\ W_k^*)_{\ell_q}$. After this paper was submitted, J. Arazy proved that the reflexivity condition on Z can be dropped. It might be possible to improve further Arazy's version of Theorem II.5 by weakening the hypotheses on Y to "Y is a subspace of a quotient of a space which has a block p-Hilbertian f.d.d."

THEOREM II.7: *Suppose that X has a shrinking unconditional f.d.d.* (E_n) *with unconditional constant* λ, *Q is a quotient mapping from X onto Y and the identity operator on* c_o *does not factor through Q (i.e., if Z is any subspace of X which is isomorphic to* c_o, *then* $Q|Z$ *is not an isomorphism). Then for each* $\epsilon > 0$, *every normalized weakly null sequence in Y has a subsequence which is an unconditionally basic sequence with unconditional constant at most* $\lambda + \epsilon$.

REMARK II.8: Maurey and Rosenthal [8] have recently given examples of normalized weakly null sequences which have no unconditionally basic subsequences.

PROOF OF THEOREM II.7: Suppose $y_n \in Y$, $\|y_n\|=1$, $y_n \overset{w}{\to} 0$. Let us regard Y as embedded in a space Z which has an f.d.d. (F_n) (e.g., $Z = C[0,1]$). By passing to a small perturbation of a subsequence of (y_n), we can assume that (y_n) is a block basic sequence of (F_n). Thus by replacing (F_n) with a blocking of (F_n), we can assume that $y_n \in F_n$ for $n = 1, 2, \ldots$.

We apply the blocking method to get blockings (E'_n) of (E_n) and (F'_n) of (F_n) so that QE'_k is essentially contained in $F'_k + F'_{k+1}$; i.e., $d(Qx, F'_k + F'_{k+1}) \leq 2^{-k}\|x\|$ for each $x \in E'_k$ and $k = 1, 2, \ldots$. Let us assume, by passing to a further subsequence of (y_n), that $y_n \in F'_n$ for

$n = 1, 2, \ldots$. Further, in order to avoid complicated notation, let us assume that $QE'_k \subseteq F'_k + F'_{k+1}$ for each $k = 1, 2 \ldots$.

We would now like to outline the rest of the proof. We will pass to a subsequence $(y_{n(k)})$ of (y_k) so that $(n(k))$ is very lacunary and show that $(y_{n(k)})$ is unconditionally basic. For that, we need show that if $\|\Sigma_{k=1}^m \alpha_k y_{n(k)}\| < 1$, then $\|\Sigma_{k=1}^m \pm \alpha_k y_{n(k)}\| < \lambda + \epsilon$ for all choices of $\pm$ signs. Naturally we write $\Sigma_{k=1}^m \alpha_k y_{n(k)} = Qx$, where $x \in X$ and $\|x\| < 1$, and expand x as $\Sigma_{k=1}^\infty x_k$ with $x_k \in E'_k$. Since $QE'_k \subseteq F'_k + F'_{k+1}$, we expect that $Q(x_{n(k)-1} + x_{n(k)}) = \alpha_k y_{n(k)}$. This is not quite right, since the overlap between QE'_k and QE'_{k+1} in F'_{k+1} might produce cancellation. However, we will show that, if $(n(k))$ is sufficiently lacunary, there is $i(k)$ with $n(k) + 1 \leq i(k) \leq n(k+1) - 1$ (where $i(0) \equiv 0$) so that $Qx_{i(k)} \approx 0$. Suppose that we can guarantee that $Qx_{i(k)} = 0$. Then $Q\ \Sigma_{j=i(k-1)+1}^{i(k)-1} x_j = \alpha_k y_{n(k)}$ and $x = \Sigma_{k=1}^m \Sigma_{j=i(k-1)+1}^{i(k)-1} x_j + \Sigma_{j=i(m)+1}^\infty x_j + \Sigma_{k=1}^m x_{i(k)}$. But if $y = \Sigma_{k=1}^m \pm \Sigma_{j=i(k-1)+1}^{i(k)-1} x_j$, then $\|y\| < \lambda$ and $Qy = \Sigma_{j=1}^m \pm \alpha_j y_{n(j)}$, hence $\|\Sigma_{j=1}^m \pm \alpha_j y_{n(j)}\| < \lambda$.

Actually we cannot guarantee that $Qx_{i(k)} = 0$, but we will be able to get that $\|Qx_{i(k)}\| < \epsilon_k$, where $\epsilon_k \downarrow 0$ as fast as we please. It thus follows that $\|Q\ \Sigma_{j=i(k-1)+1}^{i(k)-1} x_j - \Sigma\ \alpha_k y_{n(k)}\| < 2\beta(\epsilon_k + \epsilon_{k+1})$, where β is the basis constant of (F_n). Thus if we choose $\epsilon_k \downarrow 0$ fast enough and define y as above, we get that $\|Qy - \Sigma_{j=1}^m \pm \alpha_j y_{n(j)}\| < \epsilon$, so that $\|\Sigma_{j=1}^m \pm \alpha_j y_{n(j)}\| < \lambda + \epsilon$.

To complete the proof, we need to produce the $n(k)$'s and $i(k)$'s. For this it is obviously sufficient to prove the following lemma.

LEMMA II.9: *Suppose (E_n) is an unconditional f.d.d. for X, $T: X \to Y$ is an operator, and $T|Z$ is not an isomorphism for any subspace Z of X with Z isomorphic to c_o. Then for each integer m and $\epsilon > 0$, there is $n > m$ so that if $x \in X$, $\|x\| \leq 1$, $x = \Sigma_{k=1}^\infty x_k$ with $x_k \in E_k$, then there is i, $m < i < n$, so that $\|Tx_i\| < \epsilon$.*

PROOF: If the conclusion is false for a given m and ϵ, then there is a sequence (x^k) of unit vectors in X, $x^k = \Sigma_{j=1}^\infty x_j^k$ with $x_j^k \in E_j$, and $\|Tx_j^k\| \geq \epsilon$ for $m \leq j \leq m + k$. Since $\dim E_j < \infty$, we can assume by passing to a subsequence of (x^k) that $\lim_{k\to\infty} x_j^k = x_j$ for each $j = 1, 2, \ldots$, and hence $\|Tx_j\| \geq \epsilon$ for $j = m, m+1, \ldots$. Clearly $\sup_n \|\Sigma_{j=m}^n x_j\| < \infty$, and $(x_j)_{j=m}^\infty$ is unconditional, so (x_j) is equivalent to the unit vector basis of c_o. Since $(Tx_j)_{j=m}^\infty \xrightarrow{w} 0$ and $\inf_j \|Tx_j\| \geq \epsilon$, (Tx_j) has a basic subsequence, which is, necessarily, equivalent to the unit vector basis for c_o. This contradicts the hypothesis on T and completes the proof of the lemma and hence also the proof of Theorem II.7.

REMARK II.10: Of course it is immediate from Theorem II.7 that every quotient space of a space which has a shrinking unconditional f.d.d. contains an unconditionally basic sequence.

REMARK II.11: The (probably superfluous) hypothesis in Theorem II.7 that the identity on c_o does not factor through Q is, of course, satisfied whenever Y is reflexive. Actually, this hypothesis probably implies that Y is reflexive, but we did not check this out.

III. Operators on L_p $(2 < p < \infty)$

We begin with applications of the results in section II to L_p.

THEOREM III.1: *Let $2 < p < \infty$ and let Y be a subspace of a quotient of a space which has a block p-Hilbertian f.d.d. (e.g., Y can be a subspace of a quotient of ℓ_p).*

(a) *Every operator from L_p into Y factors through ℓ_p.*

(b) *Every operator from a subspace X of L_p which has the approximation property into Y factors through a space of the form $(\Sigma\, W_n)_{\ell_p}$ with each W_n a finite dimensional subspace of ℓ_p.*

(c) *If Y is a subspace of a reflexive space which has a block p-Hilbertian f.d.d. (e.g., by* [6], *Y can be a subspace of a quotient of ℓ_p), then every operator from a subspace of L_p into Y factors through a subspace of ℓ_p.*

PROOF:

(a) It is known that the Haar functions (h_n) form an unconditional basis for L_p and that every unconditional basis for L_p is block p-Besselian (cf., e.g., [12]). Thus by Theorem II.4, every operator from L_p into Y factors through a space W of the form $(\Sigma\, W_k)_{\ell_p}$, where (W_k) is a blocking of $([h_k])$. Since the W_k's are uniformly complemented in L_p, $(\Sigma\, W_k)_{\ell_p}$ is isomorphic to a complemented subspace of ℓ_p and hence by [9] to ℓ_p.

(b) Since X has the approximation property, the technique of the appendix to [3] yields that $W = X \oplus (\Sigma\, E_n)_{\ell_p}$ has an f.d.d. for some sequence (E_n) of finite dimensional subspaces of ℓ_p. Since L_p has a block p-Besselian f.d.d., the blocking method of Proposition 1 of [5] yields that W also has a block p-Besselian f.d.d. Thus if P is a projection from W onto X, we have from Theorem II.4 that TP factors through a space of the desired form, hence T does also.

(c) Since L_p has a block p-Besselian f.d.d., (c) is a special case of

Theorem II.5. (By the result of Arazy's mentioned in Remark II.6, "reflexive" can be omitted in the statement of (c).)

We now state the main result of the paper.

THEOREM III.2: *Suppose X is a subspace of* L_p $(2 < p < \infty)$ *which has the approximation property and Y is a quotient of X which is of type p Banach–Saks. Then Y is isomorphic to a quotient of a subspace of* ℓ_p.

REMARK III.3: Of course, by Theorem III.1, if $X = L_p$ then Y is isomorphic to a quotient of ℓ_p and every operator from L_p into Y factors through ℓ_p. Actually, the proof below shows directly that the quotient mapping from L_p onto Y factors through ℓ_p.

PROOF OF THEOREM III.2: As mentioned in Theorem III.1, X is isomorphic to a complemented subspace of a subspace of L_p which has a block p-Besselian f.d.d. so without loss of generality we may assume that X has a block p-Besselian f.d.d. (E_n). Let $P_n : X \to [E_i]_{i=1}^n$ be the natural projections. We can assume, by perturbing (E_n) slightly (cf., Lemma III.5 below), that the P_n's are continuous on X in the L_2 norm. Let Q be the quotient mapping from X onto Y.

As in the proof of Theorem II.1, it is sufficient to produce a blocking (W_n) of (E_n) so that if $x \in X$, $x = \Sigma\, w_n$ $(w_n \in W_n)$, then $\|Qx\| \leq$ constant $(\Sigma \|w_n\|^p)^{1/p}$. What we shall do is produce a blocking (W_n) of (E_n) so that if $w_n \in W_{4n}$ and $\|Qw_n\|$ is not essentially zero, then there is a vector $v_n \in W_{4n-1} + W_{4n} + W_{4n+1}$ which has about the same norm as w_n, $Qv_n \approx Qw_n$, and $\|v_n\|_2 \equiv (\int |v_n|^2)^{1/2}$ is small relative to $\|w_n\|$. This means that v_n is essentially supported on a set of small measure. The construction will be made so that in fact the supports of (v_n) are essentially pairwise disjoint, so that $\|\Sigma\, v_n\| \approx (\Sigma \|v_n\|^p)^{1/p} \approx (\Sigma \|w_n\|^p)^{1/p}$. Since $Q\, \Sigma\, v_n \approx Q\, \Sigma\, w_n$, we will have that $\|Q\, \Sigma\, w_n\| \leq$ constant $(\Sigma \|w_n\|^p)^{1/p}$. (For $w_n \in W_{4n+1}$ or $w_n \in W_{4n+2}$ or $w_n \in W_{4n+3}$ the blocking will work in a similar fashion.)

In the construction of the blocking of (E_n), we make use of the following lemma:

LEMMA III.4: *Suppose X is a subspace of* L_p, *Y is of type p-Banach–Saks with constant* α, *and Q is a quotient mapping from X onto Y. Then there is a constant* λ *so that for every* $\epsilon > 0$ *and every weakly null sequence* (y_k) *of unit vectors in Y, there is an integer* $n = n(\epsilon)$ *so that for all* $k \geq n$, *there is* $x_k \in X$ *so that* $\|x_k\| < \lambda$, $Qx_k = y_k$, *and* $\|x_k\|_2 < \epsilon \|x_k\|_p$.

We shall postpone the proof of the lemma, and return to the proof of Theorem III.2. Observe that Lemma III.4 yields

(*) For each $\epsilon > 0$ and $\delta > 0$, there is an integer n so that if $z \in [E_i]_{i=n}^{\infty}$, $\|z\| = 1$, and $\|Qz\| \geq \delta$, then there is $x \in X$, so that $\|x\| < \lambda \|Qz\|$, $Qx = Qz$, and $\|x\|_2 < \epsilon \|x\|_p$.

Indeed, if (*) is false there are $\epsilon > 0$, $\delta > 0$, and unit vectors $z_n \in [E_i]_{i=n}^{\infty}$ for $n = 1, 2, \dots$ so that $\|Qz_n\| \geq \delta$ and $\|x_n\|_2 \geq \epsilon \|x_n\|$ whenever $x_n \in X$, $\|x_n\| < \lambda \|Qz_n\|$, and $Qx_n = Qz_n$. Now $z_n \overset{w}{\to} 0$, hence $\|Qz_n\|^{-1} Qz_n \overset{w}{\to} 0$, so by Lemma III.4 there are an integer n and vectors $u_k \in X$ for $k \geq n$ so that $\|u_k\| < \lambda$, $Qu_k = \|Qz_k\|^{-1} Qz_k$, and $\|u_k\|_2 < \epsilon \|u_k\|$. Letting $x_k = \|Qz_k\| u_k$, we have that $\|x_k\| < \lambda \|Qz_k\|$, $Qx_k = Qz_k$, and $\|x_k\|_2 < \epsilon \|x_k\|$.

Using (*) and standard properties of L_p, we choose a sequence $\epsilon_k \downarrow 0$ with $\epsilon_1 = 4^p$ and a sequence $1 = m(1) < m(2) < \cdots$ of integers to satisfy for each k,

(1) If $x \in [E_i]_{i=1}^{m(k)-1}$ then $2^{-k} \epsilon_k \|x\| \leq \|x\|_2$,

(2) If $0 \neq y \in [E_i]_{i=m(k+1)}^{m(k+2)-1}$ and $\|Qy\| \geq 2^{-k} \|y\|$, then there is $x \in [E_i]_{i=m(k)}^{m(k+3)-1}$ so that $\|x\| < 2\lambda \|Qy\|$, $\|Qx - Qy\| < 2^{-k} \|y\|$, and $\|x\|_2 < 2^{-kp} \epsilon_k \|x\|$,

(3) If $0 \neq y \in [E_i]_{i=m(k+1)}^{\infty}$ and $\|Qy\| \geq 2^{-k-1} \|y\|$, then there is $x \in [E_i]_{i=m(k)}^{\infty}$ so that $\|x\| < 2\lambda \|Qy\|$, $\|Qx - Qy\| < 2^{-k} \|y\|$, and $\|x\|_2 \leq 2^{-(k+1)p} \epsilon_k \|x\|$,

(4) If $x \in [E_i]_{i=1}^{m(k)-1}$ and meas $A \leq \epsilon_k^{p/p-1}$, then $(\int_A |x|^p)^{1/p} \leq 2^{-k} \beta^{-1} \|x\|$ (where β is the basis constant for the f.d.d. (E_n)).

To see that this is possible, let $m(1) = 1$ and $m(2) = 2$, so that (1), (3), and (4) are satisfied for $k = 1$. (Observe that $2^{-2p} \epsilon_1 = 1$, so (3) is trivially satisfied for $k = 1$, while (1) and (4) are vacuous for $k = 1$.) Now choose ϵ_2 so that (1) and (4) are satisfied for $k = 2$. We would like to choose $m(3)$ large enough so that (3) is true for $k = 2$. Given $\tau > 0$, we can by (*) choose $m(3)$ so that given $0 \neq y \in [E_i]_{i=m(3)}^{\infty}$ with $\|Qy\| \geq 2^{-3} \|y\|$, there is $w \in X$ so that $\|w\| < \lambda \|Qy\|$, $Qw = Qy$, and $\|w\|_2 < \tau \|w\|$. Write $w = v + x$ with $v = P_{m(2)-1} w \in [W_i]_{i=1}^{m(2)-1}$ and $x \in [W_i]_{i=m(2)}^{\infty}$. We claim that if τ is sufficiently small, then $\|v\|$ is small enough so that $\|Qx - Qy\| < 2^{-3} \|y\|$, $\|x\| < 2\lambda \|Qy\|$, and $\|x\|_2 < 2^{-3p} \epsilon_2 \|x\|$. Indeed from (1) we have that $\|v\| \leq 2^2 \epsilon_2^{-1} \|v\|_2 = 4\epsilon_2^{-1} \|P_{m(2)-1} w\|_2 \leq 4\epsilon_2^{-1} \|P_{m(2)-1}\|_2 \|w\|_2 \leq 4\epsilon_2^{-1} \|P_{m(2)-1}\|_2 \tau \|w\|$. Thus if τ is chosen small enough, we have that $\|v\| \leq 2^{-4p} \epsilon_2 \|w\|$, hence $\|w - x\| \leq 2^{-4p} \epsilon_2 \|w\|$, whence $\|x\| < 2\lambda \|Qy\|$ (if ϵ_2 is specified to be less than $2^{4p}(\lambda - 1)$) and $\|Qx - Qw\| = \|Qx - Qy\| \leq 2^{-4} \epsilon_2 \lambda \|Qy\| < 2^{-3} \|y\|$ (as long as ϵ_2 is specified smaller than $2\lambda^{-1}$). Finally, we can guarantee that $\|x\|_2 < 2^{-3p} \epsilon_2 \|x\|$ if τ is sufficiently small, since $\|x-w\|_2 \le \|x-w\| \le 2^{-4p}\epsilon_2\|w\|$ and $\|w\|_2 \le \tau\|w\|$.

We have thus verified that (3) holds for $k=2$. Now we choose ϵ_3 small enough so that (1) and (4) hold for $k=3$. Observe that, since (3) holds for $k=1$, (2) will automatically be true for $k=1$ if $m(4)$ is chosen sufficiently large. Now we can repeat the argument of the preceeding paragraph to choose $m(4)$ so that (3) holds for $k=3$. Clearly we can continue in this way and select $m(k)$ and ϵ_k to satisfy (1)–(4).

Let $W_k = [E_i]_{i=m(k)}^{m(k+1)-1}$ and set $W = (\Sigma\, W_k)_{\ell_p}$. We would like to show that Q factors through W, so that Y is isomorphic to a quotient of W. Define $S: X \to W$ by $S(\Sigma\, w_k) = (w_k)_{k=1}^{\infty}$ $(w_k \in W_k)$. S is well defined and continuous, since (W_k) is p-Besselian. Define $U: W \to Y$ by $U(w_k)_{k=1}^{\infty} = \Sigma\, Qw_k$ $(w_k \in W_k)$. We need to verify that U is bounded. To do that, it is sufficient to show that if $w_k \in W_{4k}$ (resp., $w_k \in W_{4k-1}$ resp., $w_k \in W_{4k-2}$ resp., $w_k \in W_{4k-3}$), then $\|\Sigma\, Qw_k\| \le$ constant $(\Sigma\,\|w_k\|^p)^{1/p}$.

Assume, for definiteness, that $w_k \in W_{4k+1}$ $(k=1,\ 2,\dots)$ and $\Sigma\,\|w_k\|^p = 1$. Let

$$A = \{k : \|Qw_k\| < 2^{-4k}\|w_k\|\} \quad B = \{k : \|Qw_k\| \ge 2^{-4k}\|w_k\|\}.$$

Then $\|\Sigma_{k\in A}\, Qw_k\| < 2^{-3}$, so we need show only that $\|\Sigma_{k\in B}\, Qw_k\|$ is less than some absolute constant. But by (2), if $k \in B$ and $w_k \ne 0$ we can choose $x_k \in [W_i]_{i=4k}^{4k+3}$ so that $\|x_k\| < 2\lambda\|Qw_k\|$, $\|Qx_k - Qw_k\| < 2^{-4k}\|w_k\|$, and $\|x_k\|_2 < 2^{-4kp}\epsilon_{4k}\|x_k\|$. (For other values of k we let $x_k = 0$.) It is a standard fact that this last inequality and (4) mean qualitatively that the x_k's are essentially pairwise disjointly supported, so that $\|\Sigma\, x_k\| \approx (\Sigma\,\|x_k\|^p)^{1/p}$. To make this precise, let $A_1 = [0,1]$, set $A_k = \{t : |x_k(t)| \ge \epsilon_{4k}^{1/(1-p)}\|x_k\|\}$ for $k = 2, 3, \dots$, and let $B_k = A_k \sim \bigcup_{i=k+1}^{\infty} A_i$. Observe that by the definition of A_k, meas $B_k \le$ meas $A_k \le \epsilon_{4k}^{p/(p-1)}$, so that if $\Sigma_k\, \epsilon_k^{p/(p-1)} < \infty$, then meas $([0,1] \sim \bigcup_k B_k) = 0$. Thus $\|\Sigma_i\, x_i\| = (\Sigma_k \int_{B_k} |\Sigma_i\, x_i|^p)^{1/p} \le 3^{1/q}[\Sigma_k \int_{B_k} (|\Sigma_{i=1}^{k-1} x_i|^p + |x_k| + \Sigma_{i=k+1}^{\infty} |x_i|^p)]^{1/p} \le 3[\Sigma_k (\int_{B_k} |\Sigma_{i=1}^{k-1} x_i|^p)^{1/p} + \Sigma_k (\int_{B_k} |x_k|^p)^{1/p} + \Sigma_i (\int_{\cup_{k=1}^{i-1} B_k} |x_i|^p)^{1/p}] \equiv 3[a+b+c]$.
Observe that by (2) and Schwartz's inequality,

$$\begin{aligned}(\textstyle\int_{\sim A_k} |x_k|^p)^{1/p} &= (\textstyle\int_{\sim A_k} |x_k|\,|x_k|^{p-1})^{1/p} \le (\textstyle\int_{\sim A_k} |x_k|^2)^{1/2p}(\textstyle\int_{\sim A_k} |x_k|^{2p-2})^{1/2p}\\ &\le \|x_k\|_2^{1/p}(\epsilon_{4k}^{1/(1-p)}\|x_k\|)^{(2p-2)/(2p)} \le 2^{-4k}\epsilon_{4k}^{1/p}\|x_k\|^{1/p}\epsilon_{4k}^{-1/p}\|x_k\|^{(p-1)/p} = 2^{-4k}\|x\|_p.\end{aligned}$$

Thus letting $D_k = \bigcup_{i=1}^{k-1} B_i$, we have $c \le \Sigma_k (\int_{D_k} |x_k|^p)^{1/p} \le \Sigma_k (\int_{\sim A_k} |x_k|^p)^{1/p} \le \Sigma\, 2^{-4k}\|x_k\| \le (\Sigma\,\|x_k\|^p)^{1/p}$.

Of course we have $b \le (\Sigma\,\|x_k\|^p)^{1/p}$, so it remains to estimate (a). By (4), $(\int_{B_k} |\Sigma_{i=1}^{k-1} x_i|^p)^{1/p} \le 2^{-4k}\beta^{-1}\|\Sigma_{i=1}^{k-1} x_i\| \le 2^{-4k}\|\Sigma_{i=1}^{\infty} x_i\|$. Thus $a \le 2^{-3}\|\Sigma\, x_k\|$, hence $\|\Sigma\, x_k\| \le 3\cdot 2^{-3}\|\Sigma\, x_k\| + 6(\Sigma\,\|x_k\|^p)^{1/p}$, whence $\|\Sigma\, x_k\| \le 12(\Sigma\,\|x_k\|^p)^{1/p}$.

Returning now to the w_k's we have that $\|\Sigma_{k\in B}\, Qw_k\| \le \|\Sigma_{k\in B}\, Qx_k\| + \Sigma\, 2^{-4k}\|w_k\| \le \|\Sigma_{k\in B}\, x_k\| + (\Sigma\,\|w_k\|^p)^{1/p} \le 12(\Sigma\,\|x_k\|^p)^{1/p} + (\Sigma\,\|w_k\|^p)^{1/p} \le$

$(12 \cdot 2\lambda + 1)(\Sigma \|w_k\|^p)^{1/p}$. This completes the proof that $U: W \to Y$ is bounded. Since clearly $Q = US$, this completes the proof that Q factors through a subspace of ℓ_p.

We turn now to the proof of Lemma III.4. Since we used Lemma III.4 only when X has an f.d.d., we assume X has a block p-Besselian f.d.d. (E_n) with p-Besselian constant β. (In section IV we will indicate how this restriction can be removed.) We can use the blocking method and assume, by replacing (E_n) with a blocking of (E_n), that for each n, $[E_i]_{i=1}^{n-1}$ and $[E_i]_{i=n+1}^{\infty}$ are disjointly supported relative to the Haar basis (h_i) for L_p; more precisely, we can assume that for each n, there is $m = m(n)$ so that if $x \in [E_i]_{i=1}^{n-1}$, $y \in [E_i]_{i=n+1}^{\infty}$, then $\|x - H_m x\| \leq 2^{-n}\|x\|$ and $\|y - H^m y\| \leq 2^{-n}\|y\|$, where H_m is the natural projection from L_p onto $[h_i]_{i=1}^m$ and $H^m = I - H_m$. For simplicity of exposition, we assume that $[E_i]_{i=1}^{n-1} \subseteq [h_i]_{i=1}^m$ and $[E_i]_{i=n+1}^{\infty} \subseteq [h_i]_{i=m+1}^{\infty}$. The reason for doing this is that if $x_k \in [E_i]_{i=n(k)+1}^{n(k+1)-1}$ for some sequence $n(1) < n(2) < \cdots$, then (x_k) is disjointly supported relative to the Haar basis for L_p and thus is orthogonal.

Let us assume that Y is embedded into $C[0,1]$ isometrically. We can get a blocking (B_n) of a monotone basis for $C[0,1]$ and a blocking (E'_n) of (E_n) so that QE'_n is essentially contained in $B_n + B_{n+1}$, i.e., $d(Qx, B_n + B_{n+1}) \leq \epsilon_n\|x\|$ for $x \in E'_n$, where $\epsilon_n \downarrow 0$ as fast as we like. In order to avoid keeping track of approximations, we assume that $QE'_n \subseteq B_n + B_{n+1}$ for $n = 1, 2, \ldots$.

Now suppose that the conclusion of Lemma III.4 is false for λ, where $\lambda > 2\alpha\beta^{-1}$. Then there are $\epsilon > 0$ and a weakly null sequence (y_n) of unit vectors in Y so that if $x_n \in X$ with $\|x_n\| < \lambda$ and $Qx_n = y_n$, then $\|x_n\|_2 \geq \epsilon\|x_n\|$. Since $y_n \xrightarrow{w} 0$, some subsequence of (y_n) is a small perturbation of a block basis of (B_n). Thus by replacing (B_n) with a blocking of (B_n), we can assume that some subsequence $(y_{n(k)})$ of (y_n) satisfies $\|y_{n(k)} - z_k\| < 8^{-k}\epsilon$, where $z_k \in B_{n(k)}$, $\|z_k\| = 1$, and $n(k+1) - n(k)$ is as big as we want.

Since Y is type p Banach–Saks, we can also assume that $\|\Sigma_{i=1}^k z_i\| \leq \alpha k^{1/p}$ for $k = 1, 2, \ldots$.

Fix k and choose $x \in X$ so that $\|x\| \leq \alpha k^{1/p}$ and $Qx = \Sigma_{i=1}^k z_i$. Write $x = \Sigma x_i$ $(x_i \in E'_i)$. We claim that for each $1 \leq i \leq k$, there is $j(i)$, $n(i) + 1 \leq j(i) \leq n(i+1) - 1$, so that $\|x_{j(i)}\| < 8^{-3i}\epsilon$. Indeed, since (E'_i) is block p-Besselian with constant β, then $\alpha k^{1/p} \geq \|x\| \geq \beta(\Sigma_{j=n(i)+1}^{n(i+1)-1}\|x_j\|^p)^{1/p}$, so if $n(i+1) - n(i)$ is large enough, such a $j(i)$ must exist.

Let $u_i = \Sigma_{j=j(i-1)+1}^{j(i)-1} x_j$ (where $j(o) = 0$). Then $Qu_i \approx y_{n(i)}$ for $1 \leq i \leq k$. More precisely, let P_i be the natural projection from $C[0,1]$ onto $[B_j]_{j=j(i-1)+1}^{j(i)}$. Then since $QE'_n \subseteq B_n + B_{n+1}$ for each n, for $j \neq i$, $P_iQu_j = 0$

and for $\ell \neq i-1$, i $P_i x_{j(\ell)} = 0$. Of course, $\|P_i x_{j(i-1)}\| \leq \|P_i\| \|x_{j(i-1)}\| \leq 2 \cdot 8^{-3(i-1)}\epsilon$ and $\|P_i x_{j(i)}\| \leq \|P_i\| \|x_{j(i)}\| \leq 2 \cdot 8^{-3i}\epsilon$. Since $P_i Qx = P_i \sum_{j=1}^k z_j = z_i$, we thus have that $\|P_i Qu_i - z_i\| < 8^{-2}\epsilon$ for $1 \leq i \leq k$. Since $P_i Qu_i = Qu_i$, we have that $\|Qu_i - z_i\| < 8^{-2}\epsilon$.

Finally, we need to count the set $A = \{i : \|u_i\| \geq \lambda/2\}$. However, $\alpha k^{1/p} \geq \|x\| \geq \beta(\sum_{i=1}^k \|u_i\|^p)^{1/p} \geq \beta\lambda\frac{1}{2}(\text{card } A)^{1/p}$, so that $([\text{card } A]/k)^{1/p} \leq 2\alpha(\beta\lambda)^{-1} < 1$.

Let $B = \{1, \ldots, k\} \sim A$. We claim that if k is large enough then $\|u_i\|_2 \leq 2^{-1}\epsilon\|u_i\|$ for some $i \in B$. Suppose not. Now the u_i's are orthogonal by the remarks at the beginning of the proof of this lemma, hence $\alpha k^{1/p} \geq \|x\| \geq \|\sum_{i=1}^k u_i\| - \sum_{i=1}^k \|x_{j(i)}\| \geq \|\sum_{i=1}^k u_i\|_2 - \sum_{i=1}^k 8^{-3i}\epsilon \geq (\sum_{i\in B} \|u_i\|_2^2)^{1/2} - 1 > 2^{-2}\epsilon(\text{card } B)^{1/2} - 1$ (since $\|u_i\| \geq 2^{-1}$ for each i) $\geq 2^{-2}\epsilon(\gamma k)^{1/2} - 1$, where $\gamma = 1 - (2\alpha)^p(\beta\lambda)^{-p} > 0$. Since $p > 2$, this inequality is impossible for k sufficiently large.

Thus $\|u_i\|_2 < 2^{-1}\epsilon\|u_i\|$ for some $i \in B$. Recalling that $\|Qu_i - y_{n(i)}\| \leq \|Qu_i - z_i\| + \|z_i - y_{n(i)}\| < 8^{-1}\epsilon$, we can find $w_i \in X$ so that $\|u_i - w_i\| < 8^{-1}\epsilon$ and $Qw_i = y_{n(i)}$. Since $1/2 \leq \|u_i\| < \lambda/2$ and $\|u_i\|_2 < \epsilon/2\|u_i\|$, we have that $\|w_i\| < \lambda$ and $\|w_i\|_2 < \epsilon\|w_i\|$. This contradicts the choice of $y_{n(i)}$ and completes the proof of Lemma III.4.

REMARK: The constant λ in Lemma III.4 appears to depend on α and also on the p-Besselian constant of an f.d.d. for X. Actually, the proof in Section IV shows that λ depends only on α and p.

In the proof of Theorem III.2 we made use of the following lemma, which is probably known.

LEMMA III.5: *Let (E_n) be an f.d.d. for a subspace X of L_p $(p > 2)$. Then there is an f.d.d. (F_n) for X with (F_n) equivalent to (E_n) and $[F_i]_{i=1}^n = [E_i]_{i=1}^n$ for $n = 1, 2, \ldots$ so that for each n, the natural projection from X onto F_n is continuous when X is given the L_2 norm.*

PROOF: Let Q_n be the natural projection from X onto $[E_i]_{i=1}^n$. It is sufficient to show that for any sequence $\epsilon_n \downarrow 0$, there is a sequence (T_m) of operators on X, each continuous in the L_2 norm, so that $T_n T_m = T_{\min(n,m)}$, and $\|T_m - Q_{n(m)}\| < \epsilon_m$ with $T_m X = Q_{n(m)} X$ for some sequence $n(1) < n(2) < \cdots$. Indeed, a standard perturbation argument shows that if $\epsilon_m \downarrow 0$ fast enough and $G_m = (T_m - T_{m-1})X$ (where $T_o = 0$), then (G_m) is equivalent to the f.d.d. $([E_i]_{i=n(m-1)+1}^{n(m)})$. One can fill out the f.d.d. (G_m) to produce the desired f.d.d. (F_n).

Given $\delta_m \downarrow 0$, we can find projections P_n from X onto $[E_i]_{i=1}^n$ which are continuous in the L_2 norm and which satisfy $\|Q_n - P_n\| < \delta_n$. Indeed, since L_2 is dense in $L_q = L_p^*$, this is a special case of

Sublemma: *If P is a bounded projection from X onto a finite-dimensional subspace E, Y is a norm dense subspace of X^*, and $\epsilon > 0$, then there is a projection Q from X onto E so that $\|P - Q\| < \epsilon$ and Q is $\sigma(x, Y)$ continuous (i.e., $Q^*X^* \subseteq Y$).*

Proof of Sublemma: Write $Px = \Sigma_{i=1}^n x_i^*(x)e_i$, where $(e_i)_{i=1}^n$ is a normalized basis for E and $x_i^* \in X^*$. Since Y is separating over X, there are $(f_i)_{i=1}^n$ in Y so that $(e_i, f_i)_{i=1}^n$ is biorthogonal. Let $\lambda = \max \|f_i\|$ and let $0 < \delta = \delta(\epsilon, n, \lambda)$ be small. Since Y is dense in X^*, we can select $(y_i^*)_{i=1}^n$ in Y so that $\max \|x_i^* - y_i^*\| < \delta$. The operator $\tilde{Q} : X \to E$ defined by $\tilde{Q}x = \Sigma_{i=1}^n y_i^*(x)e_i$ satisfies $\|\tilde{Q} - P\| \le \delta n$, and $\tilde{Q}$ is $\sigma(X, Y)$ continuous, but $\tilde{Q}$ is not quite a projection. However, the operator $S : X \to E$ defined by $Sx = \Sigma_{i=1}^n f_i(x)e_i$ is a projection which is $\sigma(X, Y)$ continuous and S satisfies $\|S\| \le \lambda n$, hence if we set $Q = \tilde{Q} + (P - \tilde{Q})S$, then also $Q = \tilde{Q}(1 - S) + S$, so Q is a $\sigma(X, Y)$ continuous projection from X onto E and $\|Q - P\| \le \delta n + \lambda\delta n^2$.

Let $T_1 = P_1$. We will show that $n(1) < n(2) < \cdots$ can be defined so that if $T_m = P_{n(m)} + T_{m-1}(I - P_{n(m)})$, then (T_m) satisfies the desired conditions.

Using the fact that $P_nP_m = P_m$ for $n \ge m$, one can show that $T_nT_m = T_{\min(n,m)}$ and Range T_m = Range $Q_{n(m)}$ no matter how $(n(m))_{m=1}^\infty$ is defined. Indeed, one has that $T_mP_{n(k)} = P_{n(k)}$ for $m \ge k$. Also, by induction we have Range $T_m \subseteq$ Range $P_{n(m)}$ so that $P_{n(k)}T_m = T_m$ for $k \ge m$. Thus if $n \ge m$, $T_nT_m = T_n(P_{n(m)}T_m) = (T_nP_{n(m)})T_m = P_{n(m)}T_m = T_m$. On the other hand, $T_{m-1}T_m = T_{m-1}[P_{n(m)} + T_{m-1}(I - P_{n(m)})] = T_{m-1}P_{n(m)} + T_{m-1}(I - P_{n(m)}) = T_{m-1}$, and hence by induction we have that $T_nT_m = T_n$ if $n < m$. Thus we have $T_nT_m = T_{\min(n,m)}$ and, of course, Range T_m = Range $P_{n(m)}$ = Range $Q_{n(m)}$.

Finally, we need to show that $\|T_m - Q_{n(m)}\| < \epsilon_m$ if $(n(m))$ grows fast enough. Assume that $n(1) < n(2) < \cdots < n(k-1)$ have been defined so that this inequality is satisfied for $m = 1, \ldots, k-1$. For $n > n(k-1)$ we have that $\|P_n + T_{k-1}(I - P_n) - Q_n\| \le \|P_n - Q_n\| + \|(I - P_n^*)T_{k-1}^*\| < \delta_n + \|(I - P_n^*)T_{k-1}^*\|$.

Since $\|P_n - Q_n\| \to 0$ and X is reflexive, we have that $P_n^* \to I$ strongly, and thus $\|(I - P_n^*)T_{k-1}^*\| \to 0$ as $n \to \infty$ because T_{k-1}^* has finite rank. Therefore, if $n(k)$ is chosen large enough, we will have that $\|T_k - Q_{n(k)}\| < \epsilon_k$. This completes the proof.

Let us say that a Banach space X satisfies the q basic sequence condition with constant λ if every normalized basic sequence in X has a subsequence which is λ-equivalent to the unit vector basis for ℓ_q. Let X be a subspace of L_q $(1 < q < 2)$ which satisfies the q-basic sequence condition. In [4] it was shown that X embeds isomorphically into ℓ_q provided X has an unconditional f.d.d. In his dissertation, E. Odell proved that if X satisfies the q basic sequence condition with constant λ for every $\lambda > 1$, then X embeds into ℓ_q. Here we show that X^* is of type p Banach–Saks ($p^{-1}+q^{-1}=1$), so that by Theorems III.1 and III.2, X embeds into ℓ_q.

We begin with a lemma.

LEMMA III.7: *Suppose X is reflexive, Y is a subspace of X, (E_n) is an f.d.d. for X, and $P_n : X \to [E_i]_{i=1}^n$ are the natural projections. Given $\epsilon > 0$ and an integer m, there is an integer $n = n(m, \epsilon)$ so that if $y \in Y$ then $d(P_m y, Y) \le \max(2\|(P_n - P_m)y\|, \epsilon)\|y\|$.*

PROOF: Suppose not. Then there are unit vectors (y_n) in Y for $n = m+1,\ m+2, \dots$ so that $d(P_m y_n, Y) > \max(2\|(P_n - P_m)y_n\|, \epsilon)$. Since X is reflexive we can assume that $y_n \xrightarrow{w} y$. Then $P_n y_n \xrightarrow{w} y$ so that $d(P_m y, Y) \le \|y - P_m y\| \le \lim_n \inf \|P_n y_n - P_m y_n\|$. However, $\|P_m y_n - P_m y\| \to 0$ since P_m has finite rank and $y_n \xrightarrow{w} y$, so that $d(P_m y, Y) \ge \limsup_n d(P_m y_n, Y) \ge \limsup_n \max(2\|P_n y_n - P_m y_n\|, \epsilon)$. This contradiction completes the proof.

In order to simplify the computations in Theorem III.9, we want to show that the type p Banach–Saks property follows from a formally weaker condition. Actually, the observant reader will notice that only the weaker condition was needed for the proof of Theorem III.2.

LEMMA III.8: *Suppose that λ is a constant so that for any normalized weakly null sequence (x_i) in X and any integer k, there is a subsequence (y_i) of (x_i) ((y_i) depends on k) so that $\|\Sigma_{i=1}^k y_i\| \le \lambda k^{1/p}$. Then X is of type p Banach–Saks with constant $\lambda + 1$.*

PROOF: Let (x_i) be a normalized weakly null sequence in X. We say that a length k subsequence $(y_i)_{i=1}^k$ of (x_i) is *k-good* provided $\|\Sigma_{i=1}^k y_i\| \le \lambda k^{1/p}$. Our hypothesis on X yields that every subsequence of (x_i) contains a further subsequence whose length k initial segment is k-good. Thus by Ramsey's theorem, there is for each fixed k a subsequence of (x_i) all of whose k element subsets are k-good. By a diagonal process, we extract a subsequence (y_i) of (x_i) so that for every k, every k element subset of $(y_i)_{i=n(k)}^\infty$ is k-good, where $n(k) = [2^{-1}k^{1/p}]$. Then for each k,

$$\Big\|\sum_{i=1}^k y_i\Big\| \le \Big\|\sum_{i=1}^{n(k)} y_i\Big\| + \Big\|\sum_{i=n(k)+1}^{n(k)+k} y_i\Big\| + \Big\|\sum_{i=k+1}^{k+n(k)} y_i\Big\| \le 2^{-1}k^{1/p} + \lambda k^{1/p} + 2^{-1}k^{1/p} = (\lambda+1)k^{1/p}.$$

This completes the proof.

THEOREM III.9: *If X is a subspace of L_q $(1 < q < 2)$ which satisfies the q-basic sequence condition, then X^* is of type p Banach–Saks* $(p^{-1} + q^{-1} = 1)$.

PROOF: In the first step of the proof we find a blocking (E_n) of the Haar basis for L_q, $\epsilon_1 > \epsilon_2 > \cdots > 0$, and $\alpha > 0$ so that if $y_i \in [E_j]_{j=n(i)+1}^{n(i+1)-1}$, $d(y_i, X) \le \epsilon_{n(i)}$, and $2^{-n(i)-5} \le \|y_i\| \le 2$, then $\|\Sigma\, y_i\| \ge \alpha(\Sigma\, \|y_i\|^q)^{1/q}$. The proof that we can do this is essentially contained in the proof of Theorem 2 of [4]. As in that proof, we have from the truncation lemma of [2] that for each $k < \infty$ and $\beta < \lambda^{-1}$, there is an integer m so that if $x \in [h_i]_{i=m}^{\infty}$ (where (h_i) is the Haar basis for L_q), $\|x\| = 1$, and $x \in X$, then $\|{}^k x - x\| > \beta$. Here

$$ {}^k x(t) = \begin{cases} x(t), & \text{if } |x(t)| \le k \\ 0, & \text{otherwise} \end{cases} $$

and X satisfies the q-basic sequence condition with constant λ. The same kind of argument yields that there is $\epsilon > 0$ and m so that if $x \in [h_i]_{i=m}^{\infty}$, $\|x\| = 1$, and $d(x, Y) < \epsilon$, then $\|{}^k x - x\| > \beta$. The further argument from Theorem 2 of [4] then yields the desired blocking (E_n) of (h_n). For the convenience of the reader, we sketch this argument. Choose $\beta < \lambda^{-1}$, $k(1) < k(2) < \cdots$, integers $1 = m(1) < m(2) < \cdots$, $1 > \epsilon_1 > \epsilon_2 > \cdots$, and $\beta/2 > \delta_1 > \delta_2 > \cdots$ to satisfy

(1) if meas $A < \delta_n$ and $0 \ne x \in [h_i]_{i=1}^{m(n)-1}$, then $(\int_A |x|^q)^{1/q} < \delta_{n-1}\|x\|$

(2) $\|{}^{k(n)}x - x\| > \beta$ for $x \in [h_i]_{i=m(n+1)}^{\infty}$ with $\|x\| = 1$ and $d(x, X) < 2^{n+5}\epsilon_n$

(3) $k(n)^{-q} < 2^{-n}\delta_n$.

Set $E_n = [h_i]_{i=m(n)}^{m(n+1)-1}$ and suppose $y_i \in [E_j]_{j=n(i)+1}^{n(i+1)-1}$ with $2^{-n(i)-5} \le \|y_i\| \le 2$ and $d(y_i, X) < \epsilon_{n(i)}$.

Let $A_i = \{t : k_{n(i)}\|y_i\| \le |y_i(t)|\}$ and set $B_i = A_i \sim \bigcup_{j=i+1}^{\infty} A_j$. From (3) we have that meas $\bigcup_{j=i+1}^{\infty} A_j < \delta_{n(i+1)}$, so it follows from (2) and (1) that

$$ \left(\int_{B_i} |y_i|^q\right)^{1/q} > (\beta - \delta_{n(i)})\|y_i\| > 2^{-1}\beta\|y_i\|. $$

Since (B_i) is pairwise disjoint and (y_i) is a block of the Haar basis, we get from Lemma 2 of [4] that there is a constant $\alpha > 0$, depending only on β and the unconditional constant of the Haar system, so that $\|\Sigma\, y_i\| \ge \alpha(\Sigma\, \|y_i\|^q)^{1/q}$. This completes the first step of the proof.

Using Lemma III.7 and the first step of the proof, we can, for any fixed $\epsilon > 0$, define a blocking (F_n) of the Haar basis with natural projections $P_n : L_q \to [F_i]_{i=1}^{n}$, a constant $\alpha > 0$, and a sequence $\epsilon > \tau_0 > \tau_1 > \cdots > 0$ with $\tau_i < 2^{-i-5}$ so that

(4) If $y_i \in [F_j]_{j=n(i)+1}^{n(i+1)-1}$, $d(y_i, X) \le \tau_{n(i)}$, and $2^{-n(i)-5} \le \|y_i\| \le 2$, then $\|\Sigma\, y_i\| \ge \alpha(\Sigma\, \|y_i\|^q)^{1/q}$

(5) If $y \in X$ then $d(P_n y, X) \le \max(2\|P_{n+1}y - P_n y\|, 3^{-1}\tau_n)\|y\|$.

Suppose that (f_i) is a weakly null sequence of unit vectors in X^* and k is a fixed positive integer. We claim that, given any $\epsilon > 0$,

(6) there is a subsequence (g_i) of (f_i) and integers $\ell(1) < r(1) < \ell(2) < r(2) < \cdots$ with $(\ell(n+1) - r(n))$ growing as fast as we like so that if $y \in X$ with $\|y\| \le 3$ and $\|P_{r(n)}y - P_{\ell(n)}y\| < 6k\epsilon$, then $|g_n(y)| < 15k\epsilon$.

To see this, observe first that we can find $h_i \in L_q^*$ with $\|h_i\| \le 2$, $h_i \xrightarrow{w} 0$, and $(h_i|_X)$ a subsequence of (f_i). Indeed, let $h_i' \in L_q^*$, $\|h_i'\| = 1$, and $h_i|_X = f_i$. Pick a subsequence (h_i'') of (h_i') so that $h_i'' \xrightarrow{w} h$ and let $h_i = h_i'' - h$. Of course, $h|_X = 0$ since $f_i \xrightarrow{w} 0$, so (h_i) satisfies the desired condition. Next observe that, since $h_i \xrightarrow{w} 0$, some subsequence of (h_i) is essentially a block basis of the f.d.d. (F_n^*) for L_q^*. That is, there are integers $\ell(1) < r(1) < \ell(2) < r(2) < \cdots$ and a subsequence (h_i^*) of (h_i) so that $\|h_i^* - (P_{r(i)}^* h_i^* - P_{\ell(i)}^* h_i^*)\| < \epsilon$ for all $i = 1, 2, \ldots$. (Of course, by thinning the subsequence (h_i^*), we can make $(\ell(n+1) - r(n))$ grow as fast as we want.) This last inequality means that

$$|h_i^*(y) - h_i^*(P_{r(i)} - P_{\ell(i)})(y)| < \epsilon\|y\|$$

for $y \in X$ and $i = 1, 2, \ldots$, hence $|h_i^*(y)| \le \|h_i^*\|\|(P_{r(i)} - P_{\ell(i)})y\| + \epsilon\|y\|$. Therefore $(g_i) = (h_i^*|_X)$ is the desired subsequence of (f_i) which satisfies (6).

Finally, we wish to show that, if $\epsilon > 0$ is small enough relative to k, then (4), (5), and (6) imply that $\|\Sigma_{i=1}^k g_i\| \le 5\alpha^{-1}k^{1/p}$. In view of Lemma III.8 this will complete the proof.

Let $g = \Sigma_{i=1}^k g_i$ and pick $y \in X$ with $\|y\| = 1$ and $g(y) = \|g\|$. Write $y = \Sigma_{i=1}^\infty x_i$ with $x_i \in F_i$. Observe that for each $1 \le i \le k$, there is $n(i)$, $r(i) < n(i) < \ell(i+1)$, for $1 \le i < k$ and $r(k) < n(k)$, so that $\|x_{n(i)}\| < 6^{-1}\tau_{r(i)}$. This is true if $(\ell(n+1) - r(n))$ grows fast enough, because $\|y\| \ge K_q(\Sigma\, \|x_i\|^2)^{1/2}$, where K_q depends only on the unconditional constant of the Haar system (c.f., e.g., [12]). Thus, from (5) we have that $d(\Sigma_{j=1}^{n(i)-1} x_j, X) < \max(2\|x_{n(i)}\|, 3^{-1}\tau_{n(i)-1})\|\Sigma_{j=1}^\infty x_j\| \le 3^{-1}\tau_{r(i)}$. Setting $y_i = \Sigma_{j=n(i-1)+1}^{n(i)-1} x_j$ (where $n(o) = 0$), we have that $d(y_i, X) < 3^{-1}(\tau_{r(i)} + \tau_{r(i-1)}) + 6^{-1}\tau_{r(i-1)}) < \tau_{r(i-1)}$ for $i \le i \le k$. Let

$$A = \{i : \|y_i\| \ge 2^{-n(i-1)-5}\} \quad \text{and}$$
$$B = \{1, \ldots, k\} \sim A.$$

Since $\|y_i\| \le 2\|y\| = 2$ for each i, we have from (4) that $\|\Sigma_{i\in A}\, y_i\| \ge$

$\alpha(\Sigma_{i\in A}\|y_i\|^q)^{1/q}$. On the other hand, $(\Sigma_{i\in B}\|y_i\|^q)^{1/q} \leq \Sigma_{i\in B} 2^{-n(i-1)-5} < 2^{-4}$ so that $(\Sigma_{i=1}^k \|y_i\|^q)^{1/q} < 2^{-4} + \alpha^{-1}\|\Sigma_{i\in A} y_i\|$. However, $\|\Sigma_{i\in A} y_i\| \leq \|y\| + \Sigma_{i\in B}\|y_i\| + \Sigma_{i=1}^k \|x_{n(i)}\| + \|\Sigma_{j>n(k)} x_j\|$, which is $\leq 3/2$ if $n(k)$ is sufficiently large, and hence $(\Sigma_{i=1}^k \|y_i\|^q)^{1/q} \leq 2\alpha^{-1}$.

Since $d(y_i, X) < \tau_{n(i-1)} < \epsilon$ for $1 \leq i \leq k$, we can choose $u_i \in X$ so that $\|y_i - u_i\| < \epsilon$, and thus $(\Sigma_{i=1}^k \|u_i\|^q)^{1/q} \leq 3\alpha^{-1}$, $\|y - \Sigma_{i=1}^k u_i\| < 5^{-1}$, and $\|\Sigma_{i=1}^k u_i\| \leq 2$ if ϵ is small enough relative to k.

Also, since $(P_{n(i)-1} - P_{n(i-1)})y_i = y_i$, we have $\|u_i - (P_{n(i)-1} - P_{n(i-1)})u_i\| \leq \|u_i - y_i\| + \|P_{n(i)-1} - P_{n(i-1)}\| \ \|u_i - y_i\| \leq 3\epsilon$, hence for each $j \neq i$, $\|P_{r(j)}u_i - P_{\ell(j)}u_i\| = \|P_{r(j)} - P_{\ell(j)})(u_i - [P_{n(i)-1} - P_{n(i-1)}]u_i\| \leq 6\epsilon$, whence for each $j \neq i$, $\|P_{r(j)} - P_{\ell(j)}) \Sigma_{\substack{i=1\\i\neq j}}^k u_i\| < 6k\epsilon$. Of course, $\|\Sigma_{\substack{i=1\\i\neq j}}^k u_i\| < 3$ if ϵ is sufficiently small, so by (6) we have $|g_j(\Sigma_{\substack{i=1\\i\neq j}}^k u_i)| < 15k\epsilon$ and therefore $g(\Sigma_{i=1}^k u_i)| \leq \Sigma_{i=1}^k |g_i(u_i)| + \Sigma_{j=1}^k |g_j(\Sigma_{\substack{i=1\\i\neq j}}^k u_i) \leq (\Sigma_{i=1}^k \|g_i\|^p)^{1/p}(\Sigma_{i=1}^k \|u_i\|^q)^{1/q} + 15k^2\epsilon \leq k^{1/p}3\alpha^{-1} + 15k^2\epsilon \leq 4\alpha^{-1}k^{1/p}$, if ϵ is small enough relative to k.

Hence $\|g\| = |g(y)| \leq |g(\Sigma_{i=1}^k u_i)| + \|g\|\|\Sigma_{i=1}^k u_i - y\| \leq 4\alpha^{-1}k^{1/p} + 5^{-1}\|g\|$, whence $\|g\| \leq 5\alpha^{-1}k^{1/p}$. This completes the proof.

IV. Concluding remarks and open problems

PROBLEM IV.1: If X is a quotient of a space which has a shrinking unconditional f.d.d., then does every normalized weakly null sequence in X have an unconditionally basic subsequence?

PROBLEM IV.2: Give an intrinsic characterization of reflexive Banach spaces which embed into a space with block p-Hilbertian (respectively, block p-Besselian) f.d.d. In particular, if there is a constant λ so that every normalized weakly null sequence in the separable reflexive space X has a block p-Hilbertian subsequence (respectively, block p-Besselian subsequence), with constant λ, then must X embed into a (reflexive) space which has a block p-Hilbertian (respectively, block p-Besselian) f.d.d.?

The results of [6] suggest that the following restricted version of Problem IV.2 may have a positive solution.

PROBLEM IV.3: For a reflexive space X, are the following equivalent? 1. X is a subspace of a reflexive space which has a block p-Hilbertian f.d.d. 2. X is a quotient of a space with block p-Hilbertian f.d.d. 3. X is a subspace of a quotient of a space with block p-Hilbertian f.d.d.

PROBLEM IV.4: Prove Theorem III.1b without the assumption that X has the approximation property.

PROBLEM IV.5: Prove Theorem III.2 without the assumption that X has the approximation property.

We think that Problem IV.5 is the most significant problem mentioned so far and would like to make some further comments on it. One natural approach on IV.5 is to embed Y into $C[0,1]$ in such a way that the quotient mapping $Q: X \to Y$ extends to an operator $T: L_p \to C[0,1]$. We can get a blocking (E_n) of the Haar basis for L_p and a blocking (F_n) of a basis for $C[0,1]$ so that TE_n is essentially contained in $F_n + F_{n+1}$. Lemma III.4 is proved below in the case when X fails the approximation property, so that a version of the (*) condition of Theorem III.2 will hold; namely, there will be n so that if $z \in [E_i]_{i=n}^{\infty}$ and $d(z, X)$ is small enough, then there is $x \in X$ with $\|x\|_2$ small relative to $\|x\|_p$ and $Qx \approx Tz$. Using this and, probably, a version of Lemma III.7, it should be possible to define a blocking (E'_n) of (E_n) so that if $x \in X$, $x = \Sigma\, e_n$ $(e_n \in E'_n)$, then $\|Qx\| \leq \text{constant}\, (\Sigma\, \|e_n\|^p)^{1/p}$. Of course, $\|x\| \geq \text{constant}\, (\Sigma\, \|e_n\|^p)^{1/p}$, since the Haar system is unconditional and thus block p-Besselian, so this would show that Q factors through a subspace of $(\Sigma\, E'_n)_{\ell_p}$.

Actually, the approach just suggested is used in the

PROOF OF LEMMA III.4: First observe that we can embed Y into $C[0,1]$ in such a way that Q extends to a norm one operator from L_p into $C[0,1]$. Indeed, regard Y as a subspace of ℓ_∞ and let $T: L_p \to \ell_\infty$ be a norm one extension. Then since TL_p is separable, it can be embedded isometrically into $C[0,1]$. Now define a blocking (E_n) of the Haar system and a block (F_n) of a monotone basis for $C[0,1]$ so that TE_n is essentially contained in $F_n + F_{n+1}$. For simplicity of exposition, we assume that $TE_n \subseteq F_n + F_{n+1}$.

By Lemma III.7, given any $\epsilon_1 > \epsilon_2 > \cdots > 0$, we can find a blocking (E'_n) of (E_n) with natural projections $P_n: L_p \to [E'_i]_{i=1}^{n}$ so that if $x \in X$, then $d(P_n x, X) \leq \max(2\|P_{n+1}x - P_n x\|, \epsilon_n)\|x\|$. Let (B_n) be the corresponding blocking of (F_n), so that $TE'_n \subseteq B_n + B_{n+1}$.

Suppose that the conclusion of Lemma III.4 is false for λ, where $\lambda > 2\alpha\beta^{-1}$, and β is the block p-Besselian constant for the Haar basis of L_p. Then there are $\epsilon > 0$ and a normalized weakly null sequence (y_n)

in Y so that if $x_n \in X$ with $\|x_n\| < \lambda$ and $Qx_n = y_n$, then $\|x_n\| \geq \epsilon \|x_n\|$. Since $y_n \xrightarrow{w} 0$, some subsequence of (y_n) is a small perturbation of a block f.d.d. of (B_n), so by replacing (B_n) with a blocking of (B_n), we can assume that some subsequence $(y_{n(k)})$ of (y_k) satisfies $d(y_{n(k)}, B_{n(k)}) \to 0$. For simplicity, assume that $y_{n(k)} \in B_{n(k)}$. By the type p-Banach–Saks condition, we can also assume that $\|\sum_{i=1}^k y_{n(i)}\| \leq \alpha k^{1/p}$ for each $k = 1, 2, \ldots$. (Actually, as in the proof of the special case of III.4 given in Section III, we need this only for one value of k which is sufficiently large relative to ϵ.)

So far we have just repeated the proof of Lemma III.4 given in Section III, except we assumed for simplicity that, in the notation of the proof in Section III, $y_{n(i)} = z_i$. As in that proof, we let $x \in X$ so that $\|x\| \leq \alpha k^{1/p}$ and $Qx = \sum_{i=1}^k y_{n(i)}$. As in that proof, we have (if $n(i+1) - n(i)$ grows fast enough) that for each $1 \leq i \leq k$, there is $j(i)$, $n(i)+1 < j(i) < n(i+1) - 1$, so that $\|x_{j(i)}\| < \min(8^{-3i}\epsilon, 6^{-1}\epsilon_{n(i)})$. As in the proof of Theorem III.9, we have by the use of Lemma III.7 that $u_i = \sum_{j=j(i)-1}^{j(i+1)-1} x_j$ is almost in X; i.e., $d(u_i, X) < \epsilon_{j(i)}$, if $\epsilon_n \downarrow O$ fast enough. Just as in the proof of Lemma III.4 in Section III, it follows that $\|u_i\|_2 < 2^{-1}\epsilon\|u_i\|$ for some i such that $\|u_i\| < 2^{-1}\lambda$. Since $Tu_i \approx y_{n(i)}$ and $d(u_i, X) < \epsilon_{j(i)}$, we have (if the ϵ_i's are chosen small enough relative to ϵ) as in the argument in Section III that there are $w_i \in X$ with $\|w_i\| < \lambda$, $\|w_i\|_2 < \epsilon\|w_i\|$, and such that $Qw_i = y_{n(i)}$. This contradiction completes the sketch of the proof of Lemma III.4.

In the present paper we have been concerned with quotients of L_p which are quotients of ℓ_p. It is also natural to ask what quotients of L_p are quotients of $\ell_p \oplus \ell_2$.

PROBLEM IV.6: Give a condition on X so that if X is a quotient of L_p $(2 < p < \infty)$ and X satisfies the condition, then X is a quotient of $\ell_p \oplus \ell_2$.

PROBLEM IV.7: Does every operator from L_p $(2 < p < \infty)$ into a quotient of $\ell_p \oplus \ell_2$ factor through $\ell_p \oplus \ell_2$?

If Problem IV.7 has an affirmative answer, it would follow that the only $\mathcal{L}_p$ subspaces of $\ell_p \oplus \ell_2$ for $1 < p < 2$ are ℓ_p, ℓ_2, and $\ell_p \oplus \ell_2$. For $p > 2$, this is not the case (cf. [11]).

In this paper and in [4], we studied L_p for $p > 1$. It would also be nice to have a condition on a subspace of L_1 which would guarantee that the subspace embeds into ℓ_1.

PROBLEM IV.8: Suppose X is a subspace of L_1 such that there is a λ so that every basic sequence in X has a subsequence whose closed linear span is λ-isomorphic to ℓ_1. Then must X embed into ℓ_1? (One should note that subspaces of ℓ_1 do satisfy the given condition for X.)

REMARK IV.9: The proof of Theorem 2 in [4] shows that Problem IV.8 has an affirmative answer in case X has an unconditional f.d.d.

REFERENCES

[1] G. BENNETT, L. E. DOR, V. GOODMAN, W. B. JOHNSON and C. M. NEWMAN: On uncomplemented subspaces of L_p, $1 < p < 2$. *Israel J. Math.*

[2] P. ENFLO and H. P. ROSENTHAL: Some results concerning $L^p(\mu)$ spaces, *J. Functional Analysis, 14* (1973) 325–348.

[3] W. B. JOHNSON: Factoring compact operators. *Israel J. Math., 9* (1971) 337–345.

[4] W. B. JOHNSON and E. ODELL: Subspaces of L_p which embed into ℓ_p. *Comp. Math., 28* (1974) 37–49.

[5] W. B. JOHNSON and M. ZIPPIN: On subspaces of quotients of $(\Sigma G_n)_{\ell_p}$ and $(\Sigma G_n)_{c_0}$. *Israel J. Math., 13* (1972) 311–316.

[6] W. B. JOHNSON and M. ZIPPIN: Subspaces and quotient spaces of $(\Sigma G_n)_{\ell_p}$ and $(\Sigma G_n)_{c_0}$. *Israel J. Math., 17* (1974) 50–55.

[7] J. LINDENSTRAUSS and L. TZAFRIRI: Classical Banach spaces. Springer-Verlag Lecture Notes 338 (1973).

[8] B. MAUREY and H. P. ROSENTHAL: Normalized weakly null sequences with no unconditional subsequence. *Studio Math.*

[9] A. PELCZYNSKI: Projections in certain Banach spaces. *Studia Math., 19* (1960) 209–228.

[10] H. P. ROSENTHAL: Projections onto translation-invariant subspaces of $L_p(G)$. *Memoirs AMS, 63* (1966).

[11] H. P. ROSENTHAL: On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables. *Israel J. Math., 8* (1970) 273–303.

[12] H. P. ROSENTHAL: On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L_p(\mu)$ to $L_r(\nu)$. *J. Functional Analysis, 2* (1969) 176–214.

(Oblatum 15–XI–1975)

The Ohio State University
Columbus, Ohio 43210
and
The University of Texas at Austin
Austin, Texas 78712