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THE HODGE CONJECTURE FOR CUBIC FOURFOLDS

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Introduction

In this paper, the Hodge Conjecture (the rational coefficient version) is proved for non-singular cubic hypersurfaces in $\mathbb{P}^5$. That is, it is shown that every rational cohomology class of type $(2, 2)$ on such a variety is the fundamental class of an algebraic cycle of dimension two. A proof using the method of *normal functions* is given, based on an outline presented by Phillip Griffiths. In the process, we describe the means by which one would attempt to use normal functions to prove the Hodge Conjecture, either in general or in certain cases. These steps contain several conditionals, and it is the main goal of this paper to show that all conditions are met in the case of cubic fourfolds. However, the limited present knowledge about intermediate Jacobians prevents extensive immediate applications.

In [5], it is proved that on a non-singular complex projective variety X of dimension $2m$, modulo torsion every primitive integral cohomology class of type (m, m) is the cohomology class of a normal function associated to a Lefschetz pencil of hyperplane sections. Let η be such a class, and suppose that one has an inversion theorem for the intermediate Jacobian of the general hyperplane section (e.g., when X is a cubic fourfold). It has been expected that the inversion process could be carried out in a sensible way to produce an algebraic cycle on X whose normal function is at worst an integer multiple of the given one, and whose fundamental class is, therefore, a multiple of η. The difficulty in realizing this seemed to come from the lack of information concerning the Abel-Jacobi mapping near singularities. We treat this matter in a general setting. It is shown herein that what one needs to know is that the Abel-Jacobi mapping is meromorphic

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across singularities. By appealing to known theorems about the extension of meromorphic mappings, we can determine conditions which imply the meromorphy of the Abel-Jacobi mapping. Moreover, these criteria do not seem to be terribly stringent and they in fact are satisfied in the case at hand.

We present the basic setting and notation in §1. In §2, we derive the criteria for meromorphy and demonstrate its relevance to the inversion process. We derive the Hodge Conjecture for cubic fourfolds in §3 by showing that the conditions in §2 are satisfied.

Two appendices have been included at the end of this article. The first gives an alternate proof of the Hodge Conjecture for cubic fourfolds, following ideas of H. Clemens. In an unpublished manuscript (Princeton Algebra Seminar, 1970), he gives a geometric argument to prove the same for *quartic* fourfolds. These ideas are adaptable to cubics, and I have attempted to simplify the presentation.

In the second appendix, it is shown that the Hodge Conjecture, as usually formulated, is false for general compact Kähler manifolds. The counterexamples are actually quite simple – they come from a class of complex tori – and they illustrate clearly the significance of the notion of *positivity*.

1. Preliminaries

(The reader is referred to [5] for a more thorough treatment of the material presented in this section.)

We begin with the basic geometric situation

$$\begin{array}{ccc} Y & \hookrightarrow & \bar{Y} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ S & \hookrightarrow & \bar{S} \end{array} \tag{1.1}$$

in which $\bar{S}$ is a smooth complete curve (over the complex numbers), $\bar{f}$ is a projective morphism which is smooth over the non-empty Zariski-open subset S of $\bar{S}$, and f is the restriction of $\bar{f}$ over S. The points of $\bar{S}-S$ are called *singular points.* Let $Y_s = \bar{f}^{-1}(s)$ for any $s \in \bar{S}$. Associated to Y are the families of *p-th intermediate Jacobians*

$$\pi_S^p : \mathbf{J}_S^p \to S$$

whose fibers $\{J_s\}$ are complex tori. (We will drop the p's from the notation.) Whenever the local monodromy transformations around the singular points are unipotent, there is a partial compactification of $\mathbb{J}_S$

$$\pi: \mathbb{J} \to \bar{S}$$

which is obtained by inserting *generalized intermediate Jacobians* over the singular points, yielding a family of complex Abelian Lie groups [5].

Letting $\Theta(Y_s)$ denote the group of codimension p algebraic cycles on Y_s which are homologically equivalent to zero, we have the *Abel-Jacobi homomorphism*

$$\phi_s: \Theta(Y_s) \to J_s. \tag{1.2}$$

Let W be an algebraic variety parametrizing relative codimension p cycles $\{Z_w\}$ on Y. (More precisely stated, there is an S-variety $g: W \to S$ and a distinguished closed subscheme $T \subset W \times_S Y$, flat over S, of codimension p.) Let W' be another such variety, whose cycles are homologically equivalent to those of W. There is a holomorphic mapping (over S)

$$\Phi: W \times_S W' \to \mathbb{J}_S \tag{1.3}$$

defined by

$$\Phi(w, w') = \phi_s(Z_w - Z_{w'})$$

where $s = g(w) = g'(w')$. If g happens to have a section $\sigma: S \to W$, then we can define

$$\Phi: W \to \mathbb{J}_S \tag{1.4}$$

by $\Phi(w) = \phi_s(Z_w - Z_{\sigma(s)})$. Both Φ's will be called *Abel-Jacobi mappings*, and it is the meromorphy of such mappings on completions of W that we will examine.

2. On the meromorphy of the Abel-Jacobi mapping

Let M and N be complex analytic spaces, with N irreducible, and V a subvariety of N. Recall that an analytic mapping $\Phi: N - V \to M$ is said to be *meromorphic* on N if Γ, the closure of the graph of Φ, is an analytic subvariety of $N \times M$.

(2.1) PROPOSITION: *Let* $\bar{f}$: $\bar{Y} \to \bar{S}$ *be as in* (1.1); *let* $\bar{W}$ *be a normal projective variety parametrizing relative algebraic cycles of codimension p on* $\bar{Y}$, *with diagram*

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "g"'] & \bar{W} \arrow[d, "\bar{g}"] \\ S \arrow[r, hook] & \bar{S} \end{tikzcd}$$

Let Φ: $W \to \mathbf{J}_S \subset \mathbf{J}$ (*or* Φ: $W \times_S W' \to \mathbf{J}$; *for simplicity of notation, we stress the former case*) *be the Abel-Jacobi mapping. Then* Φ *is meromorphic on* $\bar{W}$ *if the following two statements hold*:

(A) Φ *extends analytically across a non-empty Zariski-open subset of the fiber* W_s *of* $\bar{g}$ *over each singular point s.*

(B) *Locally over* $\bar{S}$, $\mathbf{J}$ *embeds analytically in a complex Kähler manifold M which is proper over* $\bar{S}$.

PROOF: Condition (A) says that Φ is really defined outside a subset of codimension two in $\bar{W}$, yielding a diagram

$$\begin{tikzcd} \bar{W} - V \arrow[r, "\Phi"] & \mathbf{J} \subset M \arrow[d, "\pi"'] \arrow[d, "\bar{\pi}", shift left] \\ & \bar{S} \end{tikzcd}$$

Observe that the extension problem consists of local questions at the singular points, for V sits vertically above them. Thus, condition (B) is intended as a statement controlling the degeneration of $\mathbf{J}$ at the singular points. It allows one to apply the following simple variant of the theorem of Siu on extending meromorphic mappings [4].

(2.2) PROPOSITION: *Let*

$$\begin{tikzcd} N - V \arrow[r, "\Phi"] \arrow[dr, "p"'] & M \arrow[d, "\pi"] \\ & S \end{tikzcd}$$

be a commutative diagram of analytic mappings, in which V is of codimension two or more in the normal analytic space N, and π *is a proper mapping of complex manifolds, with M a Kähler manifold. Then* Φ *is meromorphic on N.*

REMARKS: 1. If M is $\mathbb{P}^n \times \bar{S}$ (locally on $\bar{S}$) in (B), the result follows from a classical fact about extending meromorphic functions [3, p. 133].

2. The properness assumption in (2.1) allows us to conclude that the projection $\Gamma \to N$ is proper (and surjective).

3. The requirement that $\bar{W}$ be normal is really no restriction, for we can always lift to the normalization.

Let $\nu: \bar{S} \to \mathbf{J}$ be a *normal function*, that is, a holomorphic cross-section of $\mathbf{J}$. We assume further that $\nu(s)$ is *invertible* for every $s \in S$. By this, we mean that $\nu(s)$ is in the image of the Abel-Jacobi homomorphism (1.2). It follows from the theory of Hilbert Schemes that there is an Abel-Jacobi mapping of the type (1.3) whose image contains $\nu(S)$.[1] If the conditions of (2.1) are met, let Γ be the closure of the graph of Φ. Since Γ is a Moishezon space, there is a projective variety X which dominates Γ [2], and there is a mapping $\tilde{\Phi}: X \to M$ extending Φ. Inside $\tilde{\Phi}^{-1}(\nu(\bar{S}))$, choose an algebraic curve C mapping finitely onto $\bar{S}$ via $h = \bar{\pi} \circ \tilde{\Phi}$. C also maps into $\bar{W}$, and therefore determines a cycle $Z \subset \bar{Y} \times_{\bar{S}} C$. Then $(1 \times_{\bar{S}} h)_* Z$ is a cycle on $\bar{Y}$ with normal function $(\deg h)\nu$. We have just proved

(2.3) THEOREM: *Under the conditions of Proposition* (2.1), *a normal function whose image consists of invertible points is a rational multiple of the normal function of an algebraic cycle.*

The cohomology class of a normal function ν is defined in [5, §3] as an integral element $[\nu] \in H^1(\bar{S}, R^{2p-1}\bar{f}_*\mathbb{C})$. If we assume that the situation (1.1) satisfies the condition

$$R^q\bar{f}_*\mathbb{C} \simeq j_*R^qf_*\mathbb{C}$$

we may write an exact sequence

$$(2.4) \quad H^{2p-2}(Y_s, \mathbb{C}) \xrightarrow{\gamma} H^{2p}(\bar{Y}, \mathbb{C})_0 \xrightarrow{\rho} H^1(\bar{S}, R^{2p-1}f_*\mathbb{C}) \to 0 \quad (s \in S),$$

where γ is the Gysin mapping and $H^{2p}(\bar{Y})_0 = \ker[H^{2p}(\bar{Y}) \to H^{2p}(Y_s)]$. When a cycle Z in $\bar{Y}$ of codimension p gives a normal function ν, its fundamental class $[Z] \in H^{2p}(\bar{Y})_0$ maps into $H^1(\bar{S}, R^{2p-1}\bar{f}_*\mathbb{C})$ and $\rho[Z] = [\nu]$ [5, (3.9)]. We conclude

(2.5) COROLLARY: *If η is the cohomology class of a normal function of the sort considered in* (2.3), *then η is the image of the fundamental class of an algebraic cycle on $\bar{Y}$ with rational coefficients.*

[1] For any Abel-Jacobi mapping Φ, either $(\mathrm{im}\,\Phi) \cap \nu(S)$ is discrete, or it equals all of $\nu(S)$.

3. The Hodge Conjecture for Cubic Fourfolds

As an application of the results in the preceding section, we will prove the Hodge Conjecture for cubic fourfolds.

Let X be a non-singular hypersurface of degree three in $\mathbb{P}^5$ (*cubic fourfold*). By taking a Lefschetz pencil of hyperplane sections (see [5, §4]), and then blowing up the base locus, one obtains a variety $\bar{Y}$ of the sort in (1.1), with $\bar{S} = \mathbb{P}^1$, whose general fiber is a non-singular cubic threefold.

Associated to a cubic threefold is the *Fano surface* of lines in $\mathbb{P}^4$ which are contained in it. The family of Fano surfaces for $\bar{Y}$ gives a diagram

$$\begin{array}{ccc} F & \hookrightarrow & \bar{F} \\ \downarrow & & \downarrow \\ S & \hookrightarrow & \bar{S} = \mathbb{P}^1 \end{array}$$

If $s \in S$, F_s is non-singular; otherwise, there is on Γ_s a double curve corresponding to lines passing through the double point of Y_s. One has a Abel-Jacobi mapping of type (1.4), for $\bar{F}$ has a section corresponding to each of the 27 lines lying in the base locus of the Lefschetz pencil (cubic surface). According to [1], the mapping $(F_s)^3 \to J_s^2$ is surjective if $s \in S$, and therefore $\Phi: F \times_S F \times_S F \to \mathbf{J}_S$ is surjective. Let D be the codimension two subvariety of $\bar{W} = \bar{F} \times_{\bar{S}} \bar{F} \times_{\bar{S}} \bar{F}$ consisting of all points whose projection on at least one factor lies on the double curve of a singular Fano surface. If Z_w is a cycle parametrized by $w \in \bar{W} - D$, Proposition (4.58) of [5] and its proof assert that the Abel-Jacobi mapping is defined for this cycle by evaluating certain integrals over Z_w. Thus, the Abel-Jacobi map Φ extends as a holomorphic mapping Φ: $\bar{W} - D \to \mathbf{J}$.

The situation is ripe for Proposition (2.1). As condition (A) has already been verified, it remains to check (B). Here, $\mathbf{J}_S$ is a family of principally-polarized Abelian varieties, in fact the family of Albanese varieties associated to F [1].[1] The desired conclusion is a consequence of the following proposition of a local analytic nature:

(3.1) Proposition: *Locally on $\bar{S}$, $\mathbf{J}$ admits an embedding in $\mathbb{P}^N \times \bar{S}$.*

The above is a special case of a general theorem on the degeneration of Abelian varieties. The proof, which can be achieved by a direct calculation using theta functions [6], will be omitted here.

[1]Deligne has pointed out that this fact permits the use of a global algebraic argument.

By the theorem on normal functions [5, (4.17) or (5.15)], every primitive integral cohomology class ξ of type $(2,2)$ on X is the class of a normal function, and in the present situation all normal functions are invertible. Thus, we may use (2.5) to produce a rational algebraic cycle Z on $\bar{Y}$ with $[Z]=p^*\xi$ modulo the Gysin image (2.4), where $p\colon \bar{Y}\to X$ is the natural projection. Adjusting Z by a divisor on some fiber Y_s, we can arrange $[Z]=p^*\xi$. Then $[p_*Z]=\xi$; and we can state finally:

(3.2) Theorem (Hodge Conjecture for cubic fourfolds): *Let X be a non-singular cubic fourfold. Then every rational cohomology class in $H^{2,2}(X)$ is the fundamental class of an algebraic cycle with rational coefficients.*

Appendix A: An alternate approach

We provide a proof of Theorem (3.2) that avoids the use of normal functions, following ideas of Clemens. While the proof is considerably more direct than the one presented earlier, it suffers from the disadvantage of being an argument that is of limited generality.

Let $\bar{F}$ be the relative Fano surface introduced in §3, representing lines contained in the hyperplane sections of a Lefschetz pencil on X, a cubic fourfold. Desingularizing $\bar{F}$ if necessary, we may assume $\bar{F}$ is non-singular. Let $\pi\colon E\to\bar{F}$ be the tautological projective line bundle over $\bar{F}$; as a pullback of the universal bundle over a Grassmannian, E is the projectivization of a rank two vector bundle over $\bar{F}$. Let $\mu\colon E\to X$ denote the obvious map. From [1], it follows that μ is a mapping of degree 6.

(A.1) Lemma: *Let $\pi\colon G\to V$ be a flag bundle associated to a vector bundle over the non-singular projective variety V. Then if the Hodge Conjecture is true for V, it is also true for G.*

Proof: $H^*(G)$ is a free module over $H^*(V)$, generated by monomials in the universal Chern classes of G. In particular, these generators are integral cohomology classes of type (p,p) (various p's). It follows that if $\xi\in H^{p,p}(G,\mathbb{Q})$, then each coefficient in $H^*(V)$ is also a rational class of type (p',p') (some p'). As the Chern classes and their coefficients are representable by algebraic cycles, the same is true for ξ.

(A.2) LEMMA: *Let $\pi: X' \to X$ be a mapping of degree d, $d > 0$ where X and X' are non-singular projective varieties. Suppose that the Hodge Conjecture is true for X'. Then it is also true for X.*

PROOF: If $\xi \in H^{p,p}(X, \mathbb{Q})$, $\mu^*(\xi) \in H^{p,p}(X', \mathbb{Q})$, hence $\mu^*(\xi)$ is the fundamental class $[Z]$ of a rational cycle Z. But then $\xi = d^{-1}\mu_*\mu^*(\xi) = d^{-1}[\mu_* Z]$.

Applying (A.1) to $\pi: E \to \bar{F}$, and (A.2) to $\mu: E \to X$, we see that in order to prove the Hodge Conjecture on X, it suffices to know it for $\bar{F}$. But $\bar{F}$ is a threefold, and the Hodge Conjecture is true through dimension three.

Appendix B:

Complex Tori With Non-Analytic Rational Cohomology of Type (p, p)

In this appendix, we show that it is possible to construct compact Kähler manifolds (which are not, however, projective algebraic varieties) possessing rational cohomology classes of type (p, p) that cannot be fundamental classes of analytic cycles. That is to say, there are counterexamples to the Hodge Conjecture if it is formulated in the category of Kähler manifolds. While some people are aware of this phenomenon, it does not seem to be widely known, hence it is included here.

Let $V = \mathbb{C}^n \oplus \mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_n)$, to be abbreviated (z, w), and let $J: V \to V$ be the complex-linear mapping $J(z, w) = (iz, -iw)$. Let L be a lattice in V with $JL = L$. Form the torus $T = V/L$. By construction, J induces an automorphism of T, which we also denote by J. T will be called a *J-torus.*

REMARK: If one adds the assumption that T be an Abelian variety, one obtains the candidate presented by David Mumford as a test case for the Hodge Conjecture for algebraic varieties. Here, we will insist that T be non-algebraic.

Choosing a basis for L of the form $\{v_1, \ldots, v_{2n}, Jv_1, \ldots, Jv_{2n}\}$, the period matrix of T takes the form

$$M = \begin{pmatrix} A & iA \\ B & -iB \end{pmatrix}$$

where A and B are $n \times (2n)$ matrices.

LEMMA (Mumford): *There is a complex number α such that if $\omega = \alpha dz \wedge d\bar{w}$, $\xi = Re\,\omega$ and $\eta = Im\,\omega$ represent integral cohomology classes of type (n, n).*

PROOF: As $\xi = \frac{1}{2}(\omega + \bar{\omega})$ and $\eta = -\frac{1}{2}i(\omega - \bar{\omega})$, the classes in question are certainly of type (n, n). Let

$$\alpha = \det^{-1}\begin{pmatrix} A \\ \bar{B} \end{pmatrix}$$

To check that this choice of α works, we must compute the periods of ω over the real sub-tori $\{e_I\}$ generated by any $2n$ elements of the lattice basis, for these tori provide a basis for the homology of T. This involves multiplying by α the $(2n)\times(2n)$ minors of the matrix

$$\begin{pmatrix} A & iA \\ \bar{B} & i\bar{B} \end{pmatrix}.$$

By inspection, the numbers obtained are all of the form i^k or 0. Q.E.D.

PROPOSITION: *For general M, the two classes ξ and η generate $H^{n,n}(T, \mathbb{Z})$.*

PROOF: For an integral $2n$-homology class $e_{\underline{m}} = \Sigma\, m_I e_I$ to be dual to a cohomology class of type (n, n), it must annihilate every form, with constant coefficients, of types $(2n, 0) \oplus \cdots \oplus (n+1, n-1)$, and in particular, $dz \wedge dw$. For each collection of integers $\{m_I\}$, the relation

$$(*) \qquad \langle e_{\underline{m}}, dz \wedge dw \rangle = 0$$

is a polynomial in the entries of A and B, for $\langle e_I, dz \wedge dw \rangle$ is a $(2n)\times(2n)$ minor of M. Thus, $\{M : \langle e_{\underline{m}}, dz \wedge dw \rangle = 0\}$ is an algebraic subvariety of the space of all $(2n)\times(2n)$ matrices.

It remains to show that these are proper subvarieties unless $e_{\underline{m}}$ is dual to $r\xi + s\eta$ $(r, s \in \mathbb{Q})$. For simplicity, and since it suffices anyway for our purposes, let's assume $n = 1$, so

$$M = \begin{pmatrix} a & b & ia & ib \\ c & d & -ic & -id \end{pmatrix}$$

One calculates that

$$\xi = (dx_1 \wedge dx_2 - dx_3 \wedge dx_4) \sim (e_{34} - e_{12}) = Jv_1 \wedge Jv_2 - v_1 \wedge v_2$$

$$\eta = (dx_1 \wedge dx_4 - dx_2 \wedge dx_3) \sim (e_{23} - e_{14}) = v_2 \wedge Jv_1 - v_1 \wedge Jv_2$$

where (x_1, x_2, x_3, x_4) are coordinates with respect to the lattice basis. We may eliminate e_{23} and e_{34} from (*) to obtain

$$\langle k_1e_{12} + k_2e_{13} + k_3e_{14} + k_4e_{24}, dz \wedge dw\rangle = 0.$$

$$(**) \quad k_1(ad - bc) + k_2(-2iac) + k_3(-i)(ad - bc) + k_4(-2ibd) = 0.$$

If ab^{-1} is transcendental over $\mathbb{Q}$, and $c = b^{-1}$, $d = 2a^{-1}$, (**) reads

$$k_1 - 2ik_2ab^{-1} + ik_3 - 4ik_4ba^{-1} = 0,$$

which clearly is satisfied by no non-trivial 4-tuple of integers (k_1, k_2, k_3, k_4). Q.E.D.

Now, let T be a J-torus which is generic in the sense of the Proposition. We can see immediately that $J^*\omega = i^{2n}\omega = (-1)^n\omega$. Thus, if n is *odd*, $J^*\omega = -\omega$, and therefore $J^*\xi = -\xi$ and $J^*\eta = -\eta$, so $J^*\beta = -\beta$ for all $\beta \in H^{n,n}(T, \mathbb{Z})$. No element $\beta \in H^{n,n}(T, \mathbb{Z})$ can be the fundamental class $[Z]$ of an effective analytic cycle Z with rational coefficients, for if $\beta = [Z]$, $[Z + J^{-1}Z] = \beta + J^*\beta = 0$; but no effective analytic cycle can be homologous to zero on a compact Kähler manifold. We have shown:

THEOREM: *For the general J-torus T, $H^{n,n}(T, \mathbb{Z})$ is of rank two, yet T has no analytic subvarieties of dimension n.*

Thus, even the Lefschetz theorem for divisors is false on general Kähler manifolds. However, we can screen out these examples by imposing a positivity assumption, thereby sharpening the conjecture: when X is a compact Kähler manifold, is every *sufficiently positive* element of $H^{p,p}(X, \mathbb{Z})$ the fundamental class of a *positive* analytic cycle of codimension p? If X is projective algebraic, the positive classes span $H^{p,p}(X, \mathbb{Z})$, but we have seen that this is not the case in general.

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