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INTERSECTION FORM FOR QUASI-HOMOGENEOUS SINGULARITIES

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Abstract

We consider quasi-homogeneous polynomials with an isolated singular point at the origin.

We calculate the mixed Hodge structure of the cohomology of the Milnor fiber and give a proof for a conjecture of V. I. Arnol'd concerning the intersection form on its homology.

AMS (MOS) subject classification scheme (1970) 14 B 05, 14 C 30, 14 F 10, 32 C 40.

Introduction

Let $f: \mathbb{C}^{n+1} \to \mathbb{C}$ be a quasi-homogeneous polynomial of type $(w_0, \ldots, w_n)$; this means that if $f = \Sigma\, a_\beta z^\beta$, $\beta = (\beta_0, \ldots, \beta_n)$, $z^\beta = z_0^{\beta_0} \ldots z_n^{\beta_n}$, $\beta_i \in \mathbb{Z}$, $\beta_i \geq 0$, $w_i \in \mathbb{Q}$, $w_i > 0$ and $a_\beta \neq 0$ then $\Sigma_{i=0}^n \beta_i w_i = 1$.

Equivalently:

$$f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda f(z_0, \ldots, z_n) \text{ for all } \lambda \in \mathbb{C}.$$

Denote V the affine variety in $\mathbb{C}^{n+1}$ with the equation $f(z) = 1$. The aim of this paper is, to compute the mixed Hodge structure on $H^n(V)$ in terms of the artinian ring

$$\mathbb{C}[[z_0, \ldots, z_n]]/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n).$$

The result can be described as follows.

Let $\{z^\alpha | \alpha \in I \subset \mathbb{N}^{n+1}\}$ be a set of monomials in $\mathbb{C}[z_0, \ldots, z_n]$ whose

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residue classes form a basis for the finite dimensional $\mathbb{C}$-vector-space $\mathbb{C}[[z_0, \ldots, z_n]]/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n)$. For $\alpha \in I$ let $l(\alpha) = \Sigma_{i=0}^{n}(\alpha_i + 1)w_i$. Define for every $\alpha \in I$ a rational $(n+1)$-form ω_α on $\mathbb{C}^{n+1}$ by

$$\omega_\alpha = z^\alpha (f(z) - 1)^{[-l(\alpha)]} dz_0 \wedge \ldots \wedge dz_n,$$

where [ ] denotes integral part. Using Griffiths's theory of rational integrals one associates with ω_α an element η_α of $H^n(V, \mathbb{C})$.

THEOREM 1: *Denote W and F the weight and Hodge filtrations on $H^n(V, \mathbb{C})$. Then*

$$Gr_k^W H^n(V) = 0 \quad \text{for} \quad k \neq n, n+1;$$

the forms η_α with $p < l(\alpha) < p+1$ form a basis for $Gr_F^p Gr_n^W H^n(V, \mathbb{C})$; the forms η_α with $l(\alpha) = p$ form a basis for $Gr_F^p Gr_{n+1}^W H^n(V, \mathbb{C})$.

For n even this theorem provides a proof for a conjecture of V. I. Arnol'd concerning the intersection form on $H_n(V, \mathbb{R})$:

THEOREM 2: *Let f, V, I be as above. Assume that n is even. Suppose that after diagonalization of the matrix of the intersection form on $H_n(V, \mathbb{R})$ one has μ_+ positive, μ_- negative entries and μ_0 zeroes on the diagonal. Then*

$$\mu_+ = \#\{\beta \in I \mid l(\beta) \notin \mathbb{Z} \text{ and } [l(\beta)] \text{ is even}\};$$
$$\mu_- = \#\{\beta \in I \mid l(\beta) \notin \mathbb{Z} \text{ and } [l(\beta)] \text{ is odd}\};$$
$$\mu_0 = \#\{\beta \in I \mid l(\beta) \in \mathbb{Z}\}.$$

The idea of the proof of theorem 1 is as follows. First one constructs a compactification $\bar{V}$ of V, which is the closure of V in a weighted projective space M. Because M, $\bar{V}$ and $V_\infty = \bar{V} - V$ have quotient singularities, in 2. we describe differential forms on spaces with quotient singularities. We extend Griffiths's theory of rational integrals to the weighted projective case in 4., using the proper generalization of Bott's vanishing theorem which plays a key role in it and which is proved in 3. Finally we show how to prove the conjecture of Arnol'd in 5.

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1. Weighted projective spaces

Let $w_0, \ldots, w_n$ be positive rational numbers; write $w_i = u_i/v_i$ with u_i, $v_i \in \mathbb{N}$ and $(u_i, v_i) = 1$ for $i = 0, \ldots, n$. Denote $d = \text{lcm}(v_0, \ldots, v_n)$ and

$b_i = dw_i$, $i = 0, \ldots, n$. Then each b_i is a natural number. Denote $e(\alpha) = \exp 2\pi i\alpha$, $\alpha \in \mathbb{C}$. Let G be the subgroup of $PGL(n+1, \mathbb{C})$ consisting of the elements

$$\begin{bmatrix} e(\beta_0/b_0) & & & 0 \\ & e(\beta_1/b_1) & & \\ & & \ddots & \\ 0 & & & e(\beta_n/b_n) \end{bmatrix}$$

with $\beta_i \in \mathbb{Z}$ and $0 \leq \beta_i < b_i$ for $i = 0, \ldots, n$. We define the weighted projective space of type $(w_0, \ldots, w_n)$ to be the quotient $\mathbb{P}^n(\mathbb{C})/G$ and denote it by M. If one considers $\mathbb{C}[z_0, \ldots, z_n]$ as a graded ring with degree of $z_i = b_i$, then $M \simeq$ set of prime ideals P of $\mathbb{C}[z_0, \ldots, z_n]$, generated by homogeneous elements, which are maximal in the set of prime ideals $\neq (z_0, \ldots, z_n)$, in other words, $M = \operatorname{Proj} \mathbb{C}[z_0, \ldots, z_n]$. M is covered by open affine pieces M_i, $i = 0, \ldots, n$; and $\Gamma(M_i, \mathcal{O}_M) =$ the ring of elements of $\mathbb{C}[z_0, \ldots, z_n, z_i^{-1}]$ which are homogeneous of degree 0 (with degree $z_i = b_i$ of course!)

If $f \in \mathbb{C}[z_0, \ldots, z_n]$ is "homogeneous", then f defines a hypersurface in M, namely the set of homogeneous prime ideals containing f.

2. Differentials on V-manifolds

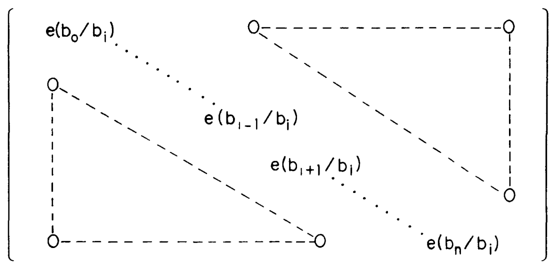
A V-manifold of dimension n is a complex analytic space which admits a covering $\{U_\alpha\}$ by open subsets, each of which is analytically isomorphic to Z_α/G_α where Z_α is an open ball in $\mathbb{C}^n$ and G_α is a finite subgroup of $GL(n, \mathbb{C})$. So a weighted projective space is a V-manifold. With notations as in 1., M_i is the quotient of $\mathbb{C}^n$ under the group generated in $GL(n, \mathbb{C})$ by the elements

$$\begin{pmatrix} e(\beta_0/b_0) & & & & & & 0 \\ & \ddots & & & & & \\ & & e(\beta_{i-1}/b_{i-1}) & & & & \\ & & & e(\beta_{i+1}/b_{i+1}) & & & \\ & & & & \ddots & & \\ 0 & & & & & & e(\beta_n/b_n) \end{pmatrix}$$

for $0 \leq \beta_j < b_j$, $j = 0, \ldots i-1, i+1, \ldots, n$ and $e(1/b_i)I$.

A finite subgroup G of $GL(n, \mathbb{C})$ is called *small* if no element of it is a nontrivial rotation around a hyperplane i.e. no element has 1 as an eigenvalue of multiplicity exactly $n-1$. If $G \subset GL(n, \mathbb{C})$ is finite, denote G_0 the subgroup generated by all rotations around hyperplanes. Then $\mathbb{C}[x_1, \ldots, x_n]^{G_0}$ is isomorphic to a polynomial ring and G/G_0 maps isomorphically to a small subgroup of $GL(n, \mathbb{C})$ acting on $\mathbb{C}^n/G_0 \cong \mathbb{C}^n$. Cf. [6] for a proof of these statements.

In the above example G/G_0 is isomorphic to the small quotient of the subgroup of $GL(n, \mathbb{C})$ generated by the matrix



If X is a V-manifold, one defines sheaves $\tilde{\Omega}^p_X$, $p \geq 0$, on X as follows. Denote $\Sigma = \mathrm{Sing}(X)$. Because X is normal, Σ has codimension at least two in X. Define

$$\tilde{\Omega}^p_X = i_* \Omega^p_{X-\Sigma}$$

with $i: X - \Sigma \to X$.

One can show that, if $\tilde{X} \xrightarrow{\pi} X$ is a resolution of singularities for X, then $\tilde{\Omega}^p_X = \pi_* \Omega^p_{\tilde{X}}$, hence $\tilde{\Omega}^p_X$ is coherent for all $p \geq 0$.

Moreover if $U \subset X$ and $U = Z/G$ with $Z \subset \mathbb{C}^n$ open and $G \subset GL(n, \mathbb{C})$ small, then $\tilde{\Omega}^p_X(U) = (\rho_* \Omega^p_Z)^G$ where $\rho: Z \to U$.

The sheaves $\tilde{\Omega}^p_X$, $p \geq 0$, form a complex which is a resolution of the constant sheaf $\mathbb{C}$.

THEOREM: *Let X be a projective V-manifold. Then the spectral sequence*

$$E_1^{pq} = H^q(X, \tilde{\Omega}^p_X) \Rightarrow \underline{\underline{H}}^{p+q}(X, \tilde{\Omega}_X) = H^{p+q}(X, \mathbb{C})$$

degenerates at E_1 and the resulting filtration on $H^*(X, \mathbb{C})$ coincides with the canonical Hodge filtration, constructed by Deligne.

Cf. [7] for proofs and for the definition of forms with logarithmic poles.

3. Generalization of Bott's vanishing theorem

In this section we prove

THEOREM: *Let M be a weighted projective n-space and let $\pi: \mathbb{P}^n \to M$ be the quotient map. Let $\mathscr{L}$ be a coherent sheaf on M with $\pi^*\mathscr{L} \cong \mathcal{O}_{\mathbb{P}^n}(k)$ for some $k \in \mathbb{Z}$. Then $H^p(M, \tilde{\Omega}^q_M \otimes \mathscr{L}) = 0$ except possibly in the cases $p = q$ and $k = 0$, $p = 0$ and $k > q$ or $p = n$ and $k < q - n$.*

PROOF: If $\pi = \mathrm{id}$ one has Bott's theorem, cf. [2], p. 246. If Z is smooth of dimension n and H a group acting on Z, generated by a rotation around a hyperplane, then $\Omega^q_{Z/H} \cong (\rho_*\Omega^q_Z)^H$ if $\rho: Z \to Z/H$ is the quotient map. By induction the same is true if H is solvable and generated by such rotations. In particular $\tilde{\Omega}^q_M \cong (\pi_*\Omega^q_{\mathbb{P}^n})^G$ is a direct factor of $\pi_*\Omega^q_{\mathbb{P}^n}$ and $\tilde{\Omega}^q_M \otimes \mathscr{L}$ is a direct factor of $(\pi_*\Omega^q_{\mathbb{P}^n}) \otimes \mathscr{L} \cong \pi_*\Omega^q_{\mathbb{P}^n}(k)$ for all $q \geq 0$. Hence $H^p(M, \tilde{\Omega}^q_M \otimes \mathscr{L})$ is a direct factor of $H^p(M, \pi_*\Omega^q_{\mathbb{P}^n}(k)) \cong H^p(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k))$ so the theorem follows from its special case $\pi = \mathrm{id}$.

4. Quasi-homogeneous polynomials

Let $f: \mathbb{C}^{n+1} \to \mathbb{C}$ be a quasi-homogeneous polynomial of type $(w_0, \ldots, w_n)$. Denote $V \subset \mathbb{C}^{n+1}$ the affine variety with equation $f(z_0, \ldots, z_n) = 1$. Denote $w_i = u_i/v_i$ with $(u_i, v_i) = 1$, $d = \mathrm{lcm}\,(v_0, \ldots, v_n)$, $b_i = dw_i$ for $i = 0, \ldots, n$. Then the polynomial

$$f(z_0, \ldots, z_n) - z^d_{n+1}$$

is quasi-homogeneous of type $(w_0, \ldots, w_n, 1/d)$.

Denote M the weighted projective $(n+1)$-space of type $(w_0, \ldots, w_n, 1/d)$; $M = \mathrm{Proj}\,\mathbb{C}[Z_0, \ldots, Z_{n+1}]$ where $\mathbb{C}[Z_0, \ldots, Z_{n+1}]$ has the grading with degree $Z_i = b_i$, $i = 0, \ldots, n$, degree $Z_{n+1} = 1$. Then M is a compactification of $\mathbb{C}^{n+1}$ as one sees by putting $z_i = Z_i/Z^{b_i}_{n+1}$. Moreover the hypersurface in M with equation

$$f(Z_0, \ldots, Z_n) - Z^d_{n+1} = 0$$

is a compactification $\bar{V}$ of V.

Denote $M_\infty = M - \mathbb{C}^{n+1}$, $V_\infty = \bar{V} - V = \bar{V} \cap M_\infty$. Then M_∞ is isomorphic to the weighted projective space of type $(w_0, \ldots, w_n)$ and $V_\infty \subset M_\infty$ is given by the equation $f(Z_0, \ldots, Z_n) = 0$. From now on assume that f has an isolated singularity at 0.

LEMMA 1: *$\bar{V}$ and V_∞ are V-manifolds.*

PROOF: Let M_j be the subset of M given by $Z_j \neq 0$, $j = 0, \ldots, n+1$. Then M_j is a quotient of $\mathbb{C}^{n+1}$ as described in 2.: say $M_j \cong \mathbb{C}^{n+1}/G^{(j)}$. Let $x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}$ be coordinates on $\mathbb{C}^{n+1}$ and let $G_1^{(j)}$ be the subgroup of $G^{(j)}$ generated by the rotations around the hyperplanes $x_i = 0$ $(i \neq j)$ over angles $2\pi/b_i$. Then $\mathbb{C}[x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1}]^{G_1^{(j)}} = \mathbb{C}[\zeta_0, \ldots, \zeta_{j-1}, \zeta_{j+1}, \ldots, \zeta_{n+1}]$ where $\zeta_i = x_i^{b_i}$. Because $f(Z_0, \ldots, Z_n) - Z_{n+1}^d$ is $G_1^{(j)}$-invariant, there exists $g_j \in \mathbb{C}[\zeta_0, \ldots, \zeta_{j-1}, \zeta_{j+1}, \ldots, \zeta_{n+1}]$ such that $g_j = f(Z_0, \ldots, Z_{j-1}, 1, Z_{j+1}, \ldots, Z_n) - Z_{n+1}^d$.

Hence if $V_j \subset \mathbb{C}^{n+1}$ is the hypersurface with equation $g_j(\zeta) = 0$, then $\bar{V} \cap M_j$ is equal to a quotient of V_j by a finite group, isomorphic to $G^{(j)}/G_1^{(j)}$. Moreover V_j and $V_{j,\infty} = \{\zeta \in V_j \mid \zeta_{n+1} = 0\}$ are smooth: for a singular point one would have the relations

$$\begin{cases} \partial f/\partial \zeta_i = 0 & (i = 0, \ldots, j-1, j+1, \ldots, n) \\ \zeta_{n+1} = 0 \end{cases}$$

Consider these as equations in $(\zeta_0, \ldots, \zeta_{n+1})$. Using the relation $\zeta_i \partial f/\partial \zeta_i = f/w_i - \Sigma_{j \neq i}(w_j/w_i)\partial f/\partial \zeta_j)$ one concludes $f(\zeta) = 0$ and hence $\zeta_i = 0$ for all i, because f has an isolated singularity at 0. Because we only deal with points with $\zeta_j = 1$ we have finished the proof.

Remark that in general $\pi^{-1}(\bar{V}) \subset \mathbb{P}^{n+1}$ is not smooth: let $f(z_0, z_1) = z_0^2 z_1 + z_1^4$. Here $w_0 = 3/8$, $w_1 = 1/4$, $d = 8$, $b_0 = 3$, $b_1 = 2$. Hence $\pi^{-1}(\bar{V}) \subset \mathbb{P}^2$ is given by the equation $z_0^6 z_1^2 + z_1^8 = z_2^8$, so $\pi^{-1}(\bar{V})$ is not smooth.

Lemma 1 and the theorem cited in 2. imply that $H^i(\bar{V})$ and $H^i(V_\infty)$, $i \geq 0$ carry Hodge structures which are purely of weight i. Therefore the canonical mixed Hodge structure on $H^i(V)$, $i \geq 0$, can be computed using the logarithmic complex $\tilde{\Omega}_{\bar{V}}^{\cdot}(\log V_\infty)$, which sits in the exact sequence

$$0 \to \tilde{\Omega}_{\bar{V}}^{\cdot} \to \tilde{\Omega}_{\bar{V}}^{\cdot}(\log V_\infty) \to \tilde{\Omega}_{V_\infty}^{\cdot-1} \to 0.$$

One obtains a long exact sequence

$$\cdots \to H^i(\bar{V}) \to H^i(V) \to H^{i-1}(V_\infty)(-1) \to H^{i+1}(\bar{V}) \to \cdots$$

as in [4]. Hence for $H^n(V)$ we get:

$$\begin{cases} Gr_n^W H^n(V) \cong H^n(\bar{V})_0; \\ Gr_{n+1}^W H^n(V) \cong H^{n-1}(V_\infty)(-1)_0, \end{cases}$$

where $H^n(\bar{V})_0 = \mathrm{Coker}\ (H^{n-2}(V_\infty)\ (-1) \to H^n(\bar{V}))$ denotes the primitive quotient and $H^{n-1}(V_\infty)(-1)_0 = \mathrm{Ker}\ (H^{n-1}(V_\infty)(-1) \to H^{n+1}(\bar{V}))$. Moreover $Gr_k^W H^n(V) = 0$ for $k \neq n, n+1$. Also note that $H^{n-1}(V_\infty)_0 \cong \mathrm{Coker}\ (H^{n-1}(M_\infty) \to H^{n-1}(V_\infty))$.

Let M be any weighted projective n-space and let $N \subset M$ be a hypersurface which is defined by a quasi-homogeneous polynomial g with an isolated singularity at 0. Then N is a V-manifold (lemma 1) and one can express the primitive cohomology $H^{n-1}(N)_0$ in terms of rational differential forms on $M - N$ as follows (cf. Griffiths [5], 10. for the smooth case).

The Hodge filtration on $H^i(N, \mathbb{C})$ satisfies $Gr_F^p H^i(N, \mathbb{C}) \cong H^{i-p}(N, \tilde{\Omega}_N^p)$ (cf. 2.). Denote $Z(\tilde{\Omega}_N^p) = \mathrm{Ker}\,(d\colon \tilde{\Omega}_N^p \to \tilde{\Omega}_N^{p+1})$. Then $F^p H^i(N, \mathbb{C}) \cong H^{i-p}(N, Z(\tilde{\Omega}_N^p))$. For $p \geq 0$, $k \geq 0$ denote $\tilde{\Omega}_M^p(kN) = \tilde{\Omega}_M^p \otimes_{\mathcal{O}_M} \mathcal{O}_M(kN)$ where $\mathcal{O}_M(kN)$ is the line bundle on M whose local sections are meromorphic functions ϕ such that $(\phi) + kN$ is a positive divisor. Then one has exact sequences

$$\text{(i)}\qquad 0 \to Z(\tilde{\Omega}_M^{q-1}(kN)) \to \tilde{\Omega}_M^{q-1}(kN) \xrightarrow{\ d\ } Z(\tilde{\Omega}_M^q((k+1)N)) \to 0 \qquad \text{for } k \geq 1;\ q \geq 1;$$

$$\text{(ii)}\qquad 0 \to Z(\tilde{\Omega}_M^q) \to Z(\tilde{\Omega}_M^q(N)) \xrightarrow{\ R\ } Z(\tilde{\Omega}_N^{q-1}) \to 0$$

where R is the Poincaré residue map. One shows this by taking the invariant parts of the sequences in [5], (10.9). From (ii) one obtains the exact sequences

$$\cdots \to F^{p+1}H^{i+1}(M, \mathbb{C}) \to H^{i-p}(M, Z(\tilde{\Omega}_M^{p+1}(N))) \xrightarrow{\ R\ } F^p H^i(N, \mathbb{C}) \to F^{p+1}H^{i+2}(M, \mathbb{C}) \to \cdots$$

showing that $F^p H^i(N, \mathbb{C})_0 \cong H^{i-p}(M, Z(\tilde{\Omega}_M^{p+1}(N)))$. Moreover from (ii) and repeated application of the vanishing theorem of 3. one obtains, because $\pi^* \mathcal{O}_M(kN) \cong \mathcal{O}(k')$ for some $k' > 0$:

$$\begin{aligned} H^{i-p}(M, Z(\tilde{\Omega}_M^{p+1}(N))) &\cong H^{i-p-1}(M, Z(\tilde{\Omega}_M^{p+2}(2N))) \cong \cdots \\ &\cong H^1(M, Z(\tilde{\Omega}_M^i((i-p)N))) \\ &\cong H^0(M, Z(\tilde{\Omega}_M^{i+1}((i-p+1)N)))/\, dH^0(M, \tilde{\Omega}_M^i((i-p)N))). \end{aligned}$$

In particular for $i = n-1$ one obtains

$$F^p H^{n-1}(N, \mathbb{C})_0 \cong H^0(M, \tilde{\Omega}_M^n((n-p)N)))/dH^0(M, \tilde{\Omega}_M^{n-1}((n-p-1)N)))$$

EXAMPLE: Let $X \subset M$ be a curve in a weighted projective plane. To compute $H^1(X, \mathbb{C})$ one uses the exactness of

$$0 \to Z(\tilde{\Omega}^1_M) \to Z(\tilde{\Omega}^1_M(X)) \to \mathbb{C}_X \to 0$$

to get $H^1(X, \mathbb{C}) \xleftarrow{\sim} H^1(M, Z(\tilde{\Omega}^1_M(X))$ because $H^0(X, \mathbb{C}) \xrightarrow{\sim} H^1(M, Z(\tilde{\Omega}^1_M))$ $= \mathbb{C}$, and one uses

$$0 \to Z(\tilde{\Omega}^1_M(X)) \to \tilde{\Omega}^1_M(X) \to \tilde{\Omega}^2_M(2X) \to 0$$

to get $H^1(M, Z(\tilde{\Omega}^1_M(X))) \cong H^0(M, \tilde{\Omega}^2_M(2X))/dH^0(M, \tilde{\Omega}^1_M(X))$ because $H^1(M, \tilde{\Omega}^1_M(X)) = 0$.

We now apply this in the cases $(M, N) = (M, \bar{V})$ or (M_∞, V_∞).

LEMMA 2: *If* $\omega = g(z_0, \ldots, z_n)\ (f(z) - 1)^{-k}\, dz_0 \wedge \ldots \wedge dz_n,\ \ k \geq 0,$ *describes a rational* $(n+1)$*-form on* M, *then*

$$\omega \in H^0(M, \tilde{\Omega}^{n+1}_M(k\bar{V})) \Leftrightarrow g = \sum_\beta g_\beta z^\beta \text{ with } g_\beta = 0$$

for all β *with* $l(\beta) \geq k$.

PROOF: Rewrite ω in coordinates $(x_1, \ldots, x_{n+1})$ on $\tilde{M}_0$ by putting $z_j = x_j x_{n+1}^{-b_j}$ $(j = 1, \ldots, n)$ and $z_0 = x_{n+1}^{-b_0}$. Then

$$\omega = \sum_\beta (-1)^{n+1} g_\beta x_1^{\beta_1} \ldots x_n^{\beta_n} x_{n+1}^{kd-1-l(\beta)d}$$
$$\times (f(1, x_1, \ldots, x_n) - x_{n+1}^d)^{-k}\, dx_1 \wedge \ldots \wedge dx_{n+1}.$$

So ω is regular at $M_\infty \Leftrightarrow g$ is linear combination of monomials z^β with $l(\beta) < k$ (or equivalently: $l(\beta) \leq k - 1/d$). Moreover ω has at most a logarithmic pole at $M_\infty \Leftrightarrow g$ is linear combination of z^β for which $l(\beta) \leq k$.

Denote $d\hat{z}_i$ the form $dz_0 \wedge \ldots \wedge dz_{i-1} \wedge dz_{i+1} \wedge \ldots \wedge dz_n$ for $i = 0, \ldots, n$. Every rational n-form on M can be written as $\Sigma_{i=0}^n g_i d\hat{z}_i$ with g_i rational functions on M for $i = 0, \ldots, n$. Analogous to lemma 2 we determine a basis for $H^0(M, \tilde{\Omega}^n_M(k\bar{V}))$. A rational n-form on M with a pole of order $\leq k$ along $\bar{V}$ which is regular on $M - (\bar{V} \cup M_\infty)$ can be written as

$$\omega = \sum_{i=0}^n \sum_\beta h_{i\beta}\, (f(z) - 1)^{-k}\, z^\beta d\hat{z}_i\ (h_{i\alpha} \in \mathbb{C}).$$

In coordinates $x_1, \ldots, x_{n+1}$ on $\tilde{M}_0$ one obtains

$\cdot z^{\beta} = x_1^{\beta_1} \dots x_n^{\beta_n} x_{n+1}^{-b_0\beta_0 - \dots - b_n\beta_n}$;

$\cdot (f(z)-1)^{-k} = x_{n+1}^{kd}(f(1, x_1, \dots, x_n) - x_{n+1}^d)^{-k}$;

$\cdot d\hat{z}_i = (-1)^n b_0 x_{n+1}^{b_i - \Sigma_j b_j - 1} d\hat{x}_i$ for $i \neq 0$;

$\cdot d\hat{z}_0 = \sum_{i=1}^{n} x_i b_i (-1)^{n-i+1} x_{n+1}^{-b_1 - \dots - b_n - 1} d\hat{x}_i + x_{n+1}^{-b_1 - \dots - b_n} d\hat{x}_{n+1}$.

Hence for ω we get $\omega = \Sigma_{i=1}^{n+1} g_i d\hat{x}_i/(f - x_{n+1}^d)^k$ where

$$g_i = \sum_{\beta} (-1)^n b_0 h_{i\beta} x_1^{\beta_1} \dots x_n^{\beta_n} x_{n+1}^{b_i + kd - dl(\beta) - 1}$$

$$+ \sum_{\beta} (-1)^{n-i+1} b_i h_{0\beta} x_1^{\beta_1} \dots x_n^{\beta_n} x_i x_{n+1}^{kd + b_0 - dl(\beta) - 1}$$

if $i = 1, \dots, n$ and $g_{n+1} = \Sigma_{\beta}\, h_{0\beta} x_1^{\beta_1} \dots x_n^{\beta_n} x_{n+1}^{kd - dl(\beta) + b_0}$. So:

LEMMA 3: *A basis for $H^0(M, \tilde{\Omega}_M^n(k\bar{V}))$ is given by the forms*

$$\begin{cases} z^{\beta}(f-1)^{-k} d\hat{z}_i \quad (i = 0, \dots, n;\, l(\beta) \leq k + w_i - 1/d); \\ z^{\gamma} \Sigma_{i=0}^{n} (-1)^i b_i z_i (f-1)^{-k} d\hat{z}_i \quad (l(\gamma) = k). \end{cases}$$

PROOF: With notations as above, g_{n+1} regular implies that $l(\beta) \leq k + w_0$ if $h_{0\beta} \neq 0$. By symmetry one gets $h_{i\beta} = 0$ if $l(\beta) > k + w_i$. If for all i with $0 \leq i \leq n$ and for all β with $h_{i\beta} \neq 0$ one has $l(\beta) \leq k + w_i - 1/d$, then clearly g_i is regular for all i. This gives the first set of generators. If one considers forms ω with $h_{i\beta} = 0$ if $l(\beta) \neq k + w_i$, the regularity condition leads to the second set of generators.

Let I be as in the introduction. Denote $I_1 = \{\alpha \in I \mid l(\alpha) \notin \mathbb{Z}\}$, $I_2 = I \backslash I_1$.

LEMMA 4: *The forms ω_α ($\alpha \in I_1$, $k < l(\alpha) < k+1$) given by $\omega_\alpha = z^{\alpha}(f-1)^{-k-1} dz_0 \wedge \dots \wedge dz_n$ map to a $\mathbb{C}$-basis for*

$$H^0(M, \tilde{\Omega}_M^{n+1}((k+1)\bar{V}))/[H^0(M, \tilde{\Omega}_M^{n+1}(k\bar{V})) + dH^0(M, \tilde{\Omega}_M^n(k\bar{V}))].$$

PROOF: Let E be the linear subspace of $\mathbb{C}[z_0, \dots, z_n]$ spanned by all monomials z^{β} for which $l(\beta) < k+1$. The map $c: E \to H^0(M, \tilde{\Omega}_M^{n+1}((k+1)\bar{V}))$ defined by $c(z^{\beta}) = z^{\beta}(f-1)^{-k-1}\, dz_0 \wedge \dots \wedge dz_n$ is an isomorphism by lemma 2. Let $E_1 = c^{-1}H^0(M, \tilde{\Omega}_M^{n+1}(k\bar{V}))$ and $E_2 = c^{-1}dH^0(M, \tilde{\Omega}_M^n(k\bar{V}))$. Write $E = E_3 \oplus E_4$ where $E_3(E_4)$ is spanned by monomials z^{β} with $l(\beta) \leq k$ (resp. $k < l(\beta) < k+1$). Denote $\Delta(f) =$

$(\partial f/\partial z_0, \ldots, \partial f/\partial z_n)$. The statement of the lemma is equivalent to

$$E_4/(E_4 \cap \Delta(f)) \cong E/(E_1 + E_2).$$

Let $p: E_4 \to E/(E_1 + E_2)$ be the natural map. Consider $\mathbb{C}[z_0, \ldots, z_n]$ as a graded ring with $\deg z_i = b_i$, $i = 0, \ldots, n$. One has $\deg z^\beta = dl(\beta) - \Sigma_{i=0}^n b_i$. With this notation

$$E = \left\{h \in \mathbb{C}[z_0, \ldots, z_n] | \deg(h) < d(k+1) - \sum_{i=0}^n b_i\right\};$$
$$E_3 = \left\{h \in E | \deg(h) \leq dk - \sum_{i=0}^n b_i\right\}.$$

Remark that $E_1 = E \cap (f-1)E$ and that the map $h \to h(f-1)$ gives an isomorphism between $\{h \in E | \deg(h) < dk - \Sigma_{i=0}^n b_i\}$ and E_1. Hence if $g \in E_1$, $g = h(f-1) = hf - h \in (E \cap (f)) + E_3 \subset (E \cap \Delta(f)) + E_3$. Hence $E_1 \subset E_3 + (E \cap \Delta(f))$. Computation of the differentials of the generators of $H^0(M, \tilde{\Omega}_M^n(k\bar{V}))$ as given in lemma 3 shows that $E_2 \subset E_1 + E \cap \Delta(f)$.

If $l(\beta) < k$ then $z^\beta \in E_1 + E_2$, for in that case $\deg fz^\beta = \deg(f-1)z^\beta \leq d(k+1) - \Sigma_{i=0}^n b_i$ and $z^\beta = (1-f)z^\beta + fz^\beta$. If $l(\beta) = k$, then $z^\beta \Sigma_{i=0}^n (-1)^i b_i z_i (f-1)^{-k} d\hat{z}_i \in H^0(M, \tilde{\Omega}_M^n(k\bar{V}))$ and its differential equals $-z^\beta(f-1)^{-k-1} dz_0 \wedge \ldots \wedge dz_n$ (use $\Sigma_{i=0}^n b_i(\beta_i + 1) = dk$ and $fd = \Sigma_{i=0}^n b_i z_i \, \partial f/\partial z_i$). Hence $z^\beta \in E_2$. This shows $E_3 \subset E_1 + E_2$.

Finally if $g = \Sigma_{j=0}^n g_j \, \partial f/\partial z_j \in \Delta(f) \cap E$, one may write $g = \Sigma_{j=0}^n h_j \, \partial f/\partial z_j$ with $\deg(h_i) \leq dk - \Sigma_{j \neq i} b_j$ (use the fact that $\partial f/\partial z_j$ is homogeneous of degree $d - b_j$). This implies that $\eta = \Sigma_{i=0}^n h_i(f-1)^{-k} d\hat{z}_i \in H^0(M, \tilde{\Omega}_M^n(k\bar{V}))$ and $d\eta = (g+h)(f-1)^{-k-1} \; dz_0 \wedge \ldots \wedge dz_n$ for some $h \in E_1$. Hence $g = g + h - h \in E_1 + E_2$. So $\Delta(f) \cap E \subset E_1 + E_2$. Because $E_3 \subset E_1 + E_2$ one has $E = E_3 + E_4 = E_1 + E_2 + E_4$ hence p is surjective. Moreover $E_1 + E_2 \subset E_1 + (E \cap \Delta(f)) \subset E_3 + (E \cap \Delta(f)) \subset E_1 + E_2$, hence $E_1 + E_2 = E_3 + (E \cap \Delta(f)) = E_3 \oplus (E_4 \cap \Delta(f))$. So $(E_1 + E_2) \cap E_4 = E_4 \cap \Delta(f) = \operatorname{Ker}(p)$.

LEMMA 5: *For β with $l(\beta) \in \mathbb{Z}$ denote $\omega_\beta = z^\beta(f-1)^{-l(\beta)} dz_0 \wedge \ldots \wedge dz_n$ and $\eta_\beta = res_{M_\infty}\omega_\beta$. Then the forms $\eta_\beta(\beta \in I_2, l(\beta) = k)$ map to a basis for*

$$H^0(M_\infty, \tilde{\Omega}_{M_\infty}^n(kV_\infty))/[H^0(M_\infty, \tilde{\Omega}_{M_\infty}^n((k-1)V_\infty)) + dH^0(M_\infty, \tilde{\Omega}_{M_\infty}^{n-1}((k-1)V_\infty))].$$

PROOF: For all $i \geq 0$ one has the exact sequence

$$0 \to \tilde{\Omega}_M^{i+1}(k\bar{V}) \to \tilde{\Omega}_M^{i+1}(\log M_\infty)(k\bar{V}) \to \tilde{\Omega}_{M_\infty}^i(kV_\infty) \to 0.$$

By the generalized vanishing theorem this gives

$$H^0(M_\infty, \tilde{\Omega}_{M_\infty}^i(kV_\infty)) \approx H^0(M, \tilde{\Omega}_M^{i+1}(\log M_\infty)(k\bar{V}))/H^0(M, \tilde{\Omega}_M^{i+1}(k\bar{V})).$$

This implies (cf. the proof of lemma 2) that a basis for $H^0(M_\infty, \tilde{\Omega}_{M_\infty}^n(kV_\infty))$ is given by the forms η_β with $l(\beta) = k$. Let $E \subset \mathbb{C}[z_0, \ldots, z_n]$ be spanned by all monomials z^β with $l(\beta) = k$ let c: $E \xrightarrow{\sim} H^0(M_\infty, \tilde{\Omega}_{M_\infty}^n(kV_\infty))$ be given by $c(z^\beta) = \eta_\beta$. Denote $E_1 = c^{-1}H^0(M_\infty, \tilde{\Omega}_{M_\infty}^n((k-1)V_\infty)$ and $E_2 = c^{-1}dH^0(M_\infty, \tilde{\Omega}_{V_\infty}^{n-1}((k-1)V_\infty))$. We have to show that $E_1 + E_2 = E \cap \Delta(f)$. One checks easily that $E_1 = (f) \cap E$. To determine E_2, remark that $H^0(M, \tilde{\Omega}_M^n(\log M_\infty)((k-1)\bar{V}))$ is generated by the forms

$$\{z^\beta(f-1)^{-k+1}d\hat{z}_i | l(\beta) \leq k - 1 + w_i, \quad i = 0, \ldots, n\}.$$

Denote $\omega_{\beta,i}$ a typical generator and $\eta_{\beta,i}$ its residue at M_∞. Then

$$c^{-1}d\eta_{\beta,i} = (-1)^i\{\beta_i z_i^{-1}z^\beta f + (1-k)z^\beta\, \partial f/\partial z_i\}$$

if $l(\beta) = k + w_i - 1$ and $c^{-1}d\eta_{\beta,i} = 0$ elsewhere. This implies that $E_2 + ((f) \cap E) = \Delta(f) \cap E$ as required.

This lemma gives a method to calculate explicitly the cohomology of every smooth projective hypersurface.

Lemma 4 and lemma 5 together prove theorem 1.

5. The intersection form

We preserve the notations of the preceding sections. Denote $H_c^*(V)$ the cohomology with compact support. Then $H_c^n(V)$ is isomorphic to the dual of $H^n(V)$; the mixed Hodge structure on $H_c^n(V)$ satisfies $Gr_k^W H_c^n(V) = 0$ for $k \neq n$, $n-1$ and $W_{n-1}H_c^n(V) = \{\omega \in H_c^n(V) | \langle \omega, \eta \rangle = 0$ for all $\eta \in W_n H^n(V)\}$.

Consider the commutative diagram:

$$\begin{array}{ccc} H_c^n(V) & \xrightarrow{i_*} & H_c^n(\bar{V}) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \bar{j}}\,{\scriptstyle \cong} \\ H^n(V) & \xleftarrow{i^*} & H^n(\bar{V}) \end{array}$$

All arrows in this diagram are morphisms of Hodge structures, j and $\bar{j}$ are the natural maps and $i\colon V \to \bar{V}$ is the inclusion. Denote S the bilinear form (intersection form) on $H^n_c(V)$ given by $S(x, y) = \langle x, j(y)\rangle$. Because j is a morphism of Hodge structures, $S(x, y) = 0$ if x or $y \in W_{n-1}H^n_c(V)$, because in that case $j(x) = 0$ or $j(y) = 0$ and $S(y, x) = (-1)^{n(n-1)/2}S(x, y)$. Moreover i_* identifies $Gr^W_n H^n_c(V)$ with the primitive part of $H^n_c(\bar{V})$ and hence S is described as follows on $Gr^W_n H^n_c(V)$: Denote $Gr^W_n H^n_c(V, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(V)$ the Hodge decomposition. Then

(i) $S(x, y) = 0$ if $x \in H^{p,q}$, $y \in H^{r,s}$, $(p, q) \neq (s, r)$;
(ii) If $x \in H^{p,q}$, $x \neq 0$ then $(-1)^{n(n-1)/2} i^{p-q}\, S(x, \bar{x}) > 0$.

COROLLARY: *Suppose that n is even and that the matrix S has the diagonal form on some basis for $H^n_c(V, \mathbb{Q})$. Suppose there are on the diagonal of this matrix μ_0 zeros, μ_+ positive and μ_- negative rational numbers. Then*:

$$\mu_0 = \dim Gr^W_{n+1} H^n(V);$$

$$\mu_+ = \sum_{\substack{q \text{ even} \\ p+q=n}} \dim H^{p,q};$$

$$\mu_- = \sum_{\substack{q \text{ odd} \\ p+q=n}} \dim H^{p,q}.$$

V.I. Arnol'd has conjectured that one may calculate μ_0, μ_+ and μ_- as follows. Let λ_i, $i = 1, \ldots, \mu$ be the eigenvalues of the residue of the Gauss-Manin connection [3] considered as an endomorphism of the ring $\mathbb{C}[[z_0, \ldots, z_n]]/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n)$. Then if n is even:

$$\mu_0 = \#\{j \mid \exp \pi i \lambda_j \in \mathbb{R}\}$$
$$\mu_+ = \#\{j \mid \operatorname{Im} \exp \pi i \lambda_j > 0\}$$
$$\mu_- = \#\{j \mid \operatorname{Im} \exp \pi i \lambda_j < 0\}$$

This has been communicated to me by A. Varchenko. We now show how one deduces this from the theorem of the introduction. If $\{z^\alpha \mid \alpha \in I\}$ is a basis of monomials for $\mathbb{C}[[z_0, \ldots, z_n]]/(\partial f/\partial z_0, \ldots, \partial f/\partial z_n)$, then they are eigenvectors for the Gauss-Manin connection: $\nabla z^\alpha = l(\alpha) z^\alpha$, so the eigenvalues for ∇ are precisely $\{l(a) \mid \alpha \in I\}$, and

$$\mu_0 = \dim Gr^W_{n+1} H^n(V) = \# I_2 = \#\{\alpha \mid l(\alpha) \in \mathbb{Z}\} = \#\{\alpha \mid \exp \pi i l(\alpha) \in \mathbb{R}\}.$$

Moreover for $\alpha \in I_1$ one has $\omega_\alpha \in Gr_F^p H^n(V) \Leftrightarrow p < l(\alpha) < p+1$, so $\#\{\alpha | \operatorname{Im} \exp \pi i l(\alpha) > 0\} = \#\{\alpha \in I_1 | [l(\alpha)] \text{ is even}\} = \Sigma_{q \text{ even}} \dim H^{p,q} = \mu_+$ and similarly for μ_-.

REMARK: With this method one also obtains the intersection form for semi-quasi-homogeneous polynomials (see Arnol'd [1]).

We end with some examples. Let $h^{p,q} = \dim H^{p,q}$

$$f(x, y, z) = x^3 + y^3 + z^3 + 3\lambda xyz \ (\tilde{E}_6) \ (\lambda^3 \neq +1).$$

Monomials:	1	x	y	z	xy	xz	yz	xyz
$l(\alpha)$:	1	4/3	4/3	4/3	5/3	5/3	5/3	2

Get: $h^{2,0} = h^{0,2} = 0$, $h^{1,1} = 6$, $h^{1,2} = h^{2,1} = 1$.
So $\mu_+ = 0$, $\mu_- = 6$, $\mu_0 = 2$.

$$f(x, y, z) = x^2 z + y^3 + z^4 \quad (Q_{10})$$

Monomials:	1	x	x^2	z	z^2	y	xy	x^2y	yz	yz^2
$24l(\alpha)$:	23	32	41	29	35	31	40	49	37	43

$$h^{2,0} = h^{0,2} = 1,\ h^{1,1} = 8,\ h^{1,2} = h^{2,1} = 0$$

So $\mu_+ = 2$, $\mu_- = 8$, $\mu_0 = 0$.

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