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A HIRZEBRUCH-RIEMANN-ROCH FORMULA FOR ANALYTIC SPACES AND NON-PROJECTIVE ALGEBRAIC VARIETIES

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1. Introduction

The Hirzebruch-Riemann-Roch formula determines the Euler characteristic of an analytic or algebraic vector bundle on an analytic or algebraic space in terms of Chern classes of the bundle and certain invariants of the space. In this note we prove such a formula for arbitrary compact complex analytic spaces, and for arbitrary complete varieties over a field. This extends, and the proof depends on, the previous known results for complex manifolds [1] and projective varieties [2].

We show that such a space X has a class $\tau(X)$ in the homology of X (with rational coefficients), such that for any analytic (respectively algebraic) vector bundle E on X,

$$(*) \qquad \chi(X, E) = \epsilon(ch(E) \cap \tau(X)).$$

Here $\chi(X, E) = \Sigma(-1)^i \dim H^i(X, E)$ is the Euler characteristic of E, $ch(E)$ is the Chern character of E, $\cap$ is the cap product pairing of cohomology and homology to homology, and ϵ takes the degree of the zero-dimensional component of a homology class.

In particular, two bundles on X with the same Chern classes have the same Euler characteristic.

If X is nonsingular, with fundamental class $[X]$, then $\tau(X) = td(T_X) \cap [X]$ is dual to the Todd class of the tangent bundle, and $(*)$ reduces to Hirzebruch's formula [8, 4, 1]. If X is projective, $\tau(X)$ is the homology Todd class constructed in [2], where $(*)$ is a corollary of a theorem which constructs a functorial homomorphism from the Grothendieck group of coherent algebraic sheaves to homology. Such a Grothendieck-Riemann-Roch theorem is not even known for non-projective complex manifolds, however, and we have no progress to report on this.

The formula $(*)$ is proved for complex spaces in section 2, and for complete varieties in section 3. In section 4, we discuss the formula for curves and surfaces.

2. Analytic Spaces

Notation. We denote by K^0X (respectively K_0X) the Grothendieck group of analytic vector bundles (respectively coherent analytic sheaves) on a complex analytic space X. The functor K^0 is contravariant, while K_0 is covariant for proper mappings: if $f: X \to Y$ is proper, and $[\mathcal{F}]$ denotes the element of K_0X represented by a sheaf $\mathcal{F}$, then $f_*[\mathcal{F}] = \Sigma(-1)^i[R^if_*(\mathcal{F})]$ in K_0Y. The tensor product gives K^0X a ring structure and makes K_0X a module over K^0X, and there is a projection formula $f_*(f^*\beta \otimes \alpha) = \beta \otimes f_*\alpha$ for f as above, $\alpha \in K_0X$, $\beta \in K^0Y$. The structure sheaf of X is denoted $\mathcal{O}_X$.

Construction of the homology Todd class. For any compact analytic space X, apply the lemma below to find a (possibly disconnected) compact complex manifold X', a morphism $f: X' \to X$, and an element $\xi \in K^0X'$ such that $f_*(\xi \otimes [\mathcal{O}_{X'}]) = [\mathcal{O}_X]$ in K_0X. Let $\tau(X') = td(T_{X'}) \cap [X']$, and define the homology Todd class $\tau(X)$, in the singular homology of X with rational coefficients, by

$$\tau(X) = f_*(ch(\xi) \cap \tau(X')).$$

PROOF OF $(*)$: Let g be the map of X to a point, and identify $K_0(\text{point})$ with the integers. If $\beta \in K^0X$ is the class of a vector bundle E on X, then the left side of $(*)$ is $g_*(\beta \otimes [\mathcal{O}_X])$. And

$$\begin{aligned}
g_*(\beta \otimes [\mathcal{O}_X]) &= g_*(\beta \otimes f_*(\xi \otimes [\mathcal{O}_{X'}])) \\
&= g_*(f_*(f^*\beta \otimes \xi \otimes [\mathcal{O}_{X'}])) \text{ by the projection formula} \\
&= (gf)_*(f^*\beta \otimes \xi \otimes [\mathcal{O}_{X'}]) \\
&= \epsilon(ch(f^*\beta \otimes \xi) \cup td(T_{X'}) \cap [X']) \text{ by H-R-R for manifolds} \\
&= \epsilon(f^*ch(\beta) \cap (ch(\xi) \cap \tau(X'))) \\
&= \epsilon f_*(f^*ch(\beta) \cap (ch(\xi) \cap \tau(X'))) \text{ since } \epsilon \text{ is functorial} \\
&= \epsilon(ch(\beta) \cap f_*(ch(\xi) \cap \tau(X'))) \text{ by the projection formula} \\
&= \epsilon(ch(\beta) \cap \tau(X)) \text{ as desired.}
\end{aligned}$$

LEMMA: *Let X be an analytic space, $\eta \in K_0X$. Then there is a non-singular analytic space X', a proper morphism $f: X' \to X$, and an element $\xi \in K^0X'$ such that $\eta = f_*(\xi \otimes [\mathcal{O}_{X'}])$.*

PROOF: We use induction on the dimension of X. Standard arguments reduce it to the case where X is reduced and irreducible, and $\eta = [\mathcal{F}]$ for a torsion-free sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. Then there is an open set U of X where $\mathcal{F}$ is locally free and $X - U = Y$ is a proper analytic subspace of X; we may add the singular locus to Y to assure that U is a manifold. It suffices to find $f: X' \to X$ proper, X' non-singular, and $\zeta' \in K^0X'$ such that $f_*(\zeta' \otimes [\mathcal{O}_{X'}]) - \eta$ is in the image of the map from K_0Y to K_0X.

Let $P = \mathrm{Proj}(S_{\mathcal{O}_X}\mathcal{F})$ be the projective fibre space defined by $\mathcal{F}$ [7, 9], $p: P \to X$ the structural morphism, $p^*\mathcal{F} \to \mathcal{O}(1)$ the universal line bundle quotient, or fundamental sheaf, on P. This corresponds to a morphism $\phi: \mathcal{F} \to p_*\mathcal{O}(1)$ on X. When we restrict to U, we have a projectivized vector bundle. It follows that $\mathrm{Ker}(\phi)$, $\mathrm{Coker}(\phi)$, and all $R^ip_*\mathcal{O}(1)$, $i > 0$, are supported on Y. So if ζ is the class of $\mathcal{O}(1)$ in K^0P, then

$$p_*(\zeta \otimes [\mathcal{O}_P]) - \eta \in \mathrm{Im}(K_0Y \to K_0X). \tag{1}$$

Now by Hironaka's resolution of singularities, we may construct a non-singular X' and a proper morphism $\pi: X' \to P$ which maps $\pi^{-1}(p^{-1}(U))$ isomorphically onto $p^{-1}(U)$. Set $f = p \circ \pi$, and let ζ' be the class of $\pi^*\mathcal{O}(1)$ in K^0X'. Then the natural map $\mathcal{O}(1) \to \pi_*\pi^*\mathcal{O}(1)$ is an isomorphism on $p^{-1}(U)$, and $R^i\pi_*(\pi^*\mathcal{O}(1))$ has support on $p^{-1}(Y)$ for $i > 0$, so

$$\pi_*(\zeta' \otimes [\mathcal{O}_{X'}]) - \zeta \otimes [\mathcal{O}_P] \in \mathrm{Im}(K_0(p^{-1}Y) \to K_0P).$$

Therefore, applying p_* to this,

$$f_*(\zeta' \otimes [\mathcal{O}_{X'}]) - p_*(\zeta \otimes [\mathcal{O}_P]) \in \mathrm{Im}(K_0Y \to K_0X) \tag{2}$$

The desired result follows by adding (1) and (2).

3. Complete Varieties

The construction of the Todd class and the proof of the formula $(*)$ for complete algebraic schemes over a field is quite similar. We indicate how the discussion in section 2 needs to be changed.

The coherent sheaves and vector bundles are algebraic, of course. Instead of singular homology and cohomology, one may use the étale theory [10], or the Chow theory [5, 6], or in fact any homology-cohomology theory with coefficients in a field of characteristic zero which has cap products, a projection formula, Chern classes, and fundamental classes for subvarieties.

In the statement of the lemma, X' is projective (not necessarily non-singular), and it is constructed by using Chow's lemma (cf. [9]) instead of resolution of singularities. The construction of the homology Todd class is then the same, and the proof of $(*)$ is the same except that one appeals to the known theorem for projective varieties [2] instead of the non-singular version.

4. The Todd Class

We see from the construction that the homology Todd class of an irreducible space X has the fundamental class $[X]$ as its top-dimensional term. Formula $(*)$ shows that the degree of its zero-dimensional term is $\chi(X, \mathcal{O}_X)$. So for a vector bundle E of rank r on a curve X we recover the formula

$$\chi(X, E) = c_1(E) \cdot [X] + r\chi(X, \mathcal{O}_X),$$

where we have written $\beta \cdot \alpha$ instead of $\epsilon(\beta \cap \alpha)$ for a cohomology class β and a homology class α.

If X is a normal surface, it has a canonical Weil divisor K, which may be defined as the divisor of a meromorphic one-form on X; it is well-defined up to rational equivalence. If $\pi: \tilde{X} \to X$ is a resolution of singularities of X, then the homology class $[K]$ of K is equal to $\pi_*(-c_1(T_{\tilde{X}}) \cap [\tilde{X}])$. From the construction, since $\tilde{X} \to X$ is an isomorphism except over a zero-dimensional set in X, we see that $\tau(X)$ has $-\frac{1}{2}[K]$ as its middle-dimensional component, and so $(*)$ gives

$$\chi(X, E) = \tfrac{1}{2}(c_1(E)^2 - 2c_2(E)) \cdot [X] - \tfrac{1}{2}c_1(E) \cdot [K] + r\chi(X, \mathcal{O}_X)$$

for a vector bundle E of rank r on X.

If we use homology K-theory instead of singular homology, the reasoning of section 2 constructs an orientation class in the homology K-theory of any compact complex analytic space (cf. [3]).

Although the Todd class is not characterized by the fact that $(*)$

holds for all bundles, the construction in section 2 will give a well-defined class, provided, following Hironaka, one has a definite choice for each resolution of singularities that occurs. If the main theorem of [2] could be extended to analytic spaces and non-projective varieties, then it would imply that all possible choices lead to the same Todd class.

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