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DECOMPOSABILITY OF EVALUATION FIBRATIONS AND THE BRACE PRODUCT OPERATION OF JAMES

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1. Introduction

Evaluation at the base point in the domain of a mapping space defines a class of Hurewicz fibrations over the target, one for each (path-)component in the mapping space. A study of these fibrations, called evaluation fibrations, was initiated in [5]. In this paper we shall deal exclusively with the evaluation fibration corresponding to the component of homotopically trivial maps.

Before describing the contents of the paper we fix some notation. Throughout, X denotes a connected CW-complex. All spaces are equipped with a base point. We shall assume that X is a simple space, i.e. the action of the fundamental group in the homotopy of X is trivial. Let S^m denote the m-sphere, $m \geq 1$, and denote by $G_0(S^m, X)$ that component in the space of free maps of S^m into X, which consists of the homotopically trivial maps. Denote by $\Omega_0^m X$ the corresponding component in the space of based maps of S^m into X, which contains the constant based map. All mapping spaces are equipped with the compact-open topology. We note that $\Omega_0^m X$ is just the identity component in the H-space of m-loops on X. Evaluation at the base point in S^m defines a Hurewicz fibration $p: G_0(S^m, X) \to X$, which we call the *neutral evaluation fibration* determined by S^m and X and denote by Ev (S^m, X). Since X is m-simple, Ev (S^m, X) has $\Omega_0^m X$ as fibre over the base point in X. Finally, we note that Ev (S^m, X) admits a canonical section $s: X \to G_0(S^m, X)$ defined by $s(x)(y) = x$ for $x \in X$ and $y \in S^m$.

Now a short description of the paper. In Section 2 we investigate how the brace product operation, introduced by James in e.g. [6] and

[7] for fibrations with a section, works in the class of neutral evaluation fibrations. In Theorem 2.1 we shall prove a formula relating brace products in $\mathrm{Ev}(S^m, X)$ to Whitehead products in X.

The motivation behind Section 3 is that $\mathrm{Ev}(S^m, X)$ at first sight looks like a principal fibration with a section so that one might expect it to be fibre homotopically trivial. Surprisingly enough, it turns out that this is very seldom the case. In fact, we shall describe a method, with Theorem 3.2 as the main result, which in many cases can be used to decide that the total space $G_0(S^m, X)$ in a given neutral evaluation fibration $\mathrm{Ev}(S^m, X)$ does not even decompose up to homotopy type as the product $X \times \Omega_0^m X$, i.e. it is not decomposable in the terminology of James [7]. The formula in Theorem 2.1 is used in the proof of Theorem 3.2.

2. Brace products in evaluation fibrations

First we introduce the brace product operation following James in e.g. [6] or [7]. Let

$$F \xrightarrow{i} E \underset{s}{\overset{p}{\rightleftarrows}} B$$

be a Serre fibration which admits a section s. Choose base points in the spaces F, E and B such that they are preserved by the maps i, p and s. The section s induces a splitting of the homotopy sequence for the fibration so that we get short split exact sequences

$$0 \to \pi_*(F) \xrightarrow{i_*} \pi_*(E) \underset{s_*}{\overset{p_*}{\rightleftarrows}} \pi_*(B) \to 0.$$

For any pair of homotopy classes $\alpha \in \pi_p(B)$ and $\beta \in \pi_q(F)$ with $p, q \geq 1$ we can form the Whitehead product $[s_*(\alpha), i_*(\beta)] \in \pi_{p+q-1}(E)$. Since $p_* \circ i_* = 0$, it follows that $p_*([s_*(\alpha), i_*(\beta)]) = 0$. By exactness of the homotopy sequence, and since i_* is a monomorphism, there exists therefore a unique element $\{\alpha, \beta\} \in \pi_{p+q-1}(F)$ such that $i_*(\{\alpha, \beta\}) = [s_*(\alpha), i_*(\beta)]$. This element is called the *brace product* of α and β.

The neutral evaluation fibration $\mathrm{Ev}(S^m, X)$,

$$\Omega_0^m X \xrightarrow{i} G_0(S^m, X) \underset{s}{\overset{p}{\rightleftarrows}} X,$$

with its canonical section s of constant maps, turns out to behave well under the brace product operation. As base point in $\Omega_0^m X$, respectively $G_0(S^m, X)$, we use always the constant map of S^m into X with the base point in X as value. Base points are then clearly preserved by the maps i, p and s.

Before we can state our main theorem on brace products in $\mathrm{Ev}\,(S^m, X)$, we have to recall the definition of the adjoint isomorphism. For each $i \geq 1$ this is the isomorphism

$$H : \pi_i(\Omega_0^m X) \to \pi_{i+m}(X)$$

defined as the composition of natural isomorphisms

$$\pi_i(\Omega_0^m X) = \pi(S^i, \Omega_0^m X) \underset{\simeq}{\to} \pi(S^i \wedge S^m, X) \underset{\simeq}{\to} \pi(S^{i+m}, X) = \pi_{i+m}(X).$$

Here $\pi(\cdot,\cdot)$ denotes a homotopy set of based maps. $\vee$ and $\wedge$ between based spaces denotes respectively wedge product and smash product.

THEOREM 2.1: *Given homotopy classes* $\alpha \in \pi_p(X)$ *and* $\beta \in \pi_q(\Omega_0^m X)$. *Form the brace product* $\{\alpha, \beta\} \in \pi_{p+q-1}(\Omega_0^m X)$ *in* $\mathrm{Ev}\,(S^m, X)$ *and the Whitehead product* $[\alpha, H(\beta)] \in \pi_{p+q+m-1}(X)$ *in* X. *Then*

$$H(\{\alpha, \beta\}) = [\alpha, H(\beta)].$$

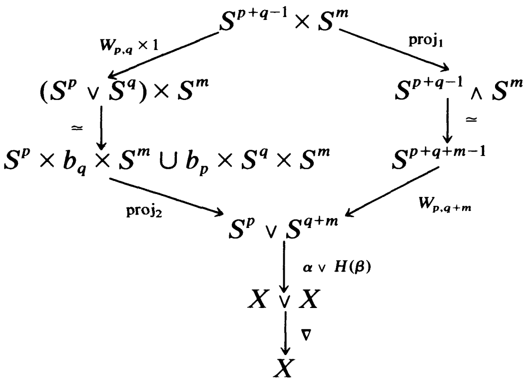
PROOF: Define the Whitehead product map $W_{p,q} : S^{p+q-1} \to S^p \vee S^q$ as follows. Let D^i, respectively ∂D^i, denote the i-cell, respectively its boundary sphere. Then we have obvious identifications

$$S^{p+q-1} = \partial(D^{p+q}) = \partial(D^p \times D^q) = D^p \times \partial D^q \cup \partial D^p \times D^q.$$

By pinching ∂D^p and ∂D^q to a point in both factors we get the canonical projection $W_{p,q} : S^{p+q-1} \to S^p \vee S^q$. Similarly, we can define the Whitehead product map $W_{p,q+m} : S^{p+q+m-1} \to S^p \vee S^{q+m}$.

Denote by b_p, b_q and b_m respectively the base point in S^p, S^q and S^m and let $\mathrm{proj}_2 : S^p \times b_q \times S^m \cup b_p \times S^q \times S^m \to S^p \vee S^{q+m}$ be the map, which projects $S^p \times b_q \times S^m$ onto S^p and projects $b_p \times S^q \times S^m$ onto S^{q+m} by identifying $b_p \times b_q \times S^m \cup b_p \times S^q \times b_m$ to a point. Let $\alpha \vee H(\beta) : S^p \vee S^{q+m} \to X \vee X$ denote a map obtained by choosing representatives for the homotopy classes involved and let $\nabla : X \vee X \to X$ denote the folding map.

Consider then the following diagram in which proj_1 is a canonical projection and the maps marked $\simeq$ are natural identifications,



This diagram is commutative and hence the composition of maps from top to bottom on the left in the diagram is identical with that on the right. The maps of S^{p+q-1} into $G_0(S^m, X)$ induced from these compositions of maps are therefore also identical and hence they represent the same homotopy class in $\pi_{p+q-1}(G_0(S^m, X))$. Now it follows almost by definition that considered as a map of S^{p+q-1} into $G_0(S^m, X)$, the composition on the left represents the homotopy class $i_*(\{\alpha, \beta\}) = [s_*(\alpha), i_*(\beta)]$ and the composition on the right represents the homotopy class $i_*(H^{-1}([\alpha, H(\beta)]))$. Since i_* is a monomorphism we get then $\{\alpha, \beta\} = H^{-1}([\alpha, H(\beta)])$, which proves the theorem.

REMARK 2.2: A weaker result related to Theorem 2.1 was obtained by Federer ([4], p. 356).

3. Decomposability of evaluation fibrations

Let

$$F \xrightarrow{i} E \underset{s}{\overset{p}{\rightleftarrows}} B$$

be a Serre fibration with section s. Following James [7] we say that this fibration is *decomposable* if E and $B \times F$ have the same homotopy type. Clearly a fibration is decomposable if it is fibre homotopically trivial.

The purpose of this section is to investigate the neutral evaluation fibration $\mathrm{Ev}\,(S^m, X)$,

$$\Omega_0^m X \xrightarrow{i} G_0(S^m, X) \underset{s}{\overset{p}{\rightleftarrows}} X,$$

w.r.t. decomposability.

First we observe that the spaces $G_0(S^m, X)$ and $X \times \Omega_0^m X$ have the same homotopy groups. This follows easily, since the section s splits the homotopy sequence for $\mathrm{Ev}\,(S^m, X)$. We remark that $(m+1)$-simplicity of X is needed here in order to get this statement for π_1, see Abe [1].

Next we mention the following well-known positive result.

THEOREM 3.1: *Suppose that X is an H-space with a strict unit element. Then the neutral evaluation fibration* $\mathrm{Ev}\,(S^m, X)$ *is fibre homotopically trivial, in particular decomposable.*

PROOF: If we take the unit element for the multiplication on X as base point in X, and if we use the multiplication properly, it is easy to construct a map $\theta : X \times \Omega_0^m X \to G_0(S^m, X)$, which commutes with projections onto X, and which is the identity map on the fibre over the base point in X. Hence θ is a fibre homotopy equivalence by a fundamental theorem of Dold ([3], Theorem 6.3). This proves the theorem.

The following theorem provides a method to obtain non-decomposability results.

THEOREM 3.2: *Suppose that there exist homotopy classes $\alpha \in \pi_n(X)$ and $\beta \in \pi_{m+k}(X)$ with m, n, $k \geq 1$ such that the Whitehead product $[\alpha, \beta] \neq 0$. Suppose also that the set of Whitehead products $[\pi_n(X), \pi_k(X)] = 0$.*

Then the neutral evaluation fibration $\mathrm{Ev}\,(S^m, X)$ *is not decomposable.*

PROOF: Let $\beta' \in \pi_k(\Omega_0^m X)$ be the unique element such that $\beta = H(\beta')$. By Theorem 2.1 we have then $H(\{\alpha, \beta'\}) = [\alpha, H(\beta')] = [\alpha, \beta] \neq 0$. Hence $\{\alpha, \beta'\} \neq 0$ and then also $[s_*(\alpha), i_*(\beta')] \neq 0$.

Consequently, the set of Whitehead products $[\pi_n(G_0(S^m, X)), \pi_k(G_0(S^m, X))] \neq 0$. In passing we mention that this will also follow from the theorem of Federer ([4], p. 356).

On the other hand, the set of Whitehead products $[\pi_n(X \times \Omega_0^m X), \pi_k(X \times \Omega_0^m X)] = 0$. This follows, since the two sets of Whitehead products $[\pi_n(X), \pi_k(X)]$ and $[\pi_n(\Omega_0^m X), \pi_k(\Omega_0^m X)]$ vanishes, the first one by assumption and the second one since $\Omega_0^m X$ is an H-space.

$G_0(S^m, X)$ and $X \times \Omega_0^m X$ must therefore have different homotopy types and the theorem is proved.

We restrict now our attention to the case $X = S^n$, the n-sphere. First we note the following special case of Theorem 3.2.

THEOREM 3.3: *Denote by* $\iota_n \in \pi_n(S^n)$ *the homotopy class represented by the identity map on* S^n *and suppose that there exists a homotopy class* $\beta \in \pi_{m+k}(S^n)$ *with* $1 \leq k < n$ *such that the Whitehead product* $[\iota_n, \beta] \neq 0$.

Then the neutral evaluation fibration $\mathrm{Ev}(S^m, S^n)$ *is not decomposable.*

Next we give some concrete applications of Theorem 3.3.

COROLLARY 3.4: *Let* $1 \leq m < n$. *Then the neutral evaluation fibration* $\mathrm{Ev}(S^m, S^n)$ *is decomposable if and only if* $n = 3, 7$.

For $n = 3, 7$ *the fibration is even fibre homotopically trivial.*

PROOF: For $n = 3, 7$, S^n is an H-space with a strict unit element and hence $\mathrm{Ev}(S^m, S^n)$ is fibre homotopically trivial by Theorem 3.1.

For $n \neq 3, 7$, the Whitehead product $[\iota_n, \iota_n] \neq 0$ by Adams [2]. Writing $n = (n - m) + m$ we see that ι_n can be used as the element β in Theorem 3.3. Hence $\mathrm{Ev}(S^m, S^n)$ is not decomposable for $n \neq 3, 7$.

COROLLARY 3.5: *Consider the neutral evaluation fibration* $\mathrm{Ev}(S^n, S^n)$ *for* $n \geq 1$.

(1) *For* $n = 1, 3, 7$, $\mathrm{Ev}(S^n, S^n)$ *is fibre homotopically trivial.*

(2) *For* $n \geq 8$, $n \neq 11, 27$ *and* $n \not\equiv 15 \bmod 16$, $\mathrm{Ev}(S^n, S^n)$ *is not decomposable.*

PROOF:

(1) Follows from Theorem 3.1.

(2) For $n \geq 8$, let $\sigma_n \in \pi_{n+7}(S^n)$ be the element represented by suspensions of the Hopf map $S^{15} \to S^8$. By results of Mahowald [9] and Kristensen and Madsen [8], it follows that $[\iota_n, \sigma_n] \neq 0$ for $n \neq 11$, 27 and $n \not\equiv 15 \bmod 16$. Hence for these values of n, the element σ_n can be used as the element β in Theorem 3.3. This proves (2).

REMARK 3.6: Corollary 3.5 is certainly not the best possible result for the evaluation fibrations $\mathrm{Ev}(S^n, S^n)$. The author is indebted to Siegfried Thomeier for showing him his tables over Whitehead products. Using these tables in connection with Theorem 3.3 it is possible to prove that $\mathrm{Ev}(S^n, S^n)$ is not decomposable if $n \neq 1, 2, 3, 7$ and $n \not\equiv 15 \bmod 16$.

Of the remaining cases we mention in particular that it is unknown whether $\mathrm{Ev}(S^2, S^2)$ is decomposable or not. Since the set of Whitehead products $[\pi_3(S^2), \pi_2(S^2)] = 0$, this problem cannot be solved using Theorem 3.3.

Using Theorem 3.3 and all known information on Whitehead products we can prove that $\mathrm{Ev}(S^m, S^n)$ is not decomposable in many special cases. However, we cannot get the complete solution to the following problem using this method.

PROBLEM: Let $1 \leq n \leq m$. Determine when the neutral evaluation fibration $\mathrm{Ev}(S^m, S^n)$ is decomposable.

It is a tempting conjecture to suggest that $\mathrm{Ev}(S^m, S^n)$ is decomposable if and only if $n = 1, 3, 7$.

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