COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 35, nº 1 (1977), p. 91-97
<http://www.numdam.org/item?id=CM_1977__35_1_91_0>

COMPOSITIO MATHEMATICA, Vol. 35, Fasc. 1, 1977, pag. 91–97
Noordhoff International Publishing
Printed in the Netherlands

A FINITENESS THEOREM FOR THE BURNSIDE RING OF A COMPACT LIE GROUP

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Let G be a compact Lie group and let $A(G)$ be its Burnside ring [6]. We show that after inverting a finite number of primes the ring $A(G)$ is generated by idempotent elements. The following result (Theorem 1) about compact Lie groups is basic for our investigations.

Let H be a subgroup of G (subgroups will always be closed), let NH be its normalizer in G and denote NH/H by WH. If K is a compact Lie group let K_0 be its component of the unit element.

THEOREM 1: *There exists an integer b such that for each closed subgroups H of G the index $|WH:(WH)_0|$ is less than b.*

Let $A_c(G)$ be the integral closure of $A(G)$ in its total quotient ring.

THEOREM 2: *There exists an integer $n>0$ such that $nA_c(G)\subset A(G)$. The minimal such n is the least common multiple of the numbers WH where H runs through all subgroups such that WH is finite.*

The minimal integer $n(G)$ provided by Theorem 1 replaces the order of the finite group if one extends the general Artin induction theorem (see Dress [7], Theorem 2, p. 204) to compact Lie groups, whence its importance.

1. Normalizers

1.1. We prove in this section Theorem 1. The proof proceeds in three steps: We first reduce to the case that WH is finite; then we reduce to the case that H is finite; and finally show that for finite H

with finite WH the order of WH is uniformly bounded.

Let H be a closed subgroup of G (notation: $H < G$). Let $\mathrm{Aut}(H)$ be the automorphism group of H and $\mathrm{In}(H)$ the closed normal subgroup of inner automorphisms. The group $\mathrm{Aut}\, H/\mathrm{In}\, H$ is discrete. Mapping $n \in NH$ to the conjugation automorphism $c(n): h \mapsto nhn^{-1}$ of H induces a homomorphism $NH \to \mathrm{Aut}\, H/\mathrm{In}\, H$ with kernel $ZH \cdot H$, where ZH denotes the centralizer of H. Hence $NH/ZH \cdot H$ being a compact subgroup of the discrete group $\mathrm{Aut}\, H/\mathrm{In}\, H$ is finite. We conclude

LEMMA 1: *WH is finite if and only if $ZH/(ZH \cap H)$ is finite.*

LEMMA 2: *A compact Lie group contains only a finite number of conjugacy classes (K) where K is the centralizer of a closed subgroup.*

PROOF: Let G act on $M = G$ via conjugation $G \times M \to M: (g, m) \mapsto gmg^{-1}$. If $H < G$ then the fixed point set M^H is the centralizer ZH. A compact differentiable G-manifold has finite orbit type. Hence there exist finitely many conjugacy classes $(H_1), \ldots, (H_k)$ such that for any closed subgroup H $M^H = M^K$ and $(K) = (H_i)$ for a suitable i.

LEMMA 3: *For any $H < G$ the group $ZH \cdot H$ has finite index in its normalizer.*

PROOF: We have $Z(ZH \cdot H) < ZH < ZH \cdot H$; hence the assertion follows from Lemma 1.

If $n \in G$ normalizes H then also ZH and hence $ZH \cdot H$. We therefore have

$$NH/ZH \cdot H < N(ZH \cdot H)/ZH \cdot H.$$

Using Lemma 3 and the existence of an upper bound for the set

$$\{|WH| \,|\, H < G,\ WH \text{ finite}\} =: F(G)$$

we obtain

LEMMA 4: *There exists an integer c such that for all $H < G$ we have*

$$|NH/ZH \cdot H| < c.$$

Now we are able to obtain the first reduction of our problem. From the exact sequence $1 \to ZH/ZH \cap H \to WH \to NH/ZH \cdot H \to 1$ we see that $WH/(WH)_0 \to NH/ZH \cdot H$ has a kernel which is a quotient of $ZH/(ZH)_0$. Lemmas 2 and 4 then show that

$$\{|WH/(WH)_0| \,|\, H < G\}$$

is bounded. But note that Lemma 4 requires a bound for the set $F(G)$.

1.2. We show by induction over $|G/G_0|$ and dim G that $F(G)$ has an upper bound $a = a(G/G_0, \dim G)$. For finite G we can take $a = |G|$.

Suppose that an upper bound $a(K/K_0, \dim K)$ is given for all K with $\dim K < \dim G$. Let $\Sigma(G) = \{H < G | WH \text{ finite}\}$. Suppose $H \in \Sigma(G)$ is not finite. We consider the projection $p: NH_0 \to NH_0/H_0 =: U$. Let V be the normalizer of H/H_0 in U. Then $WH = V/(H/H_0)$ and therefore $H/H_0 \in \Sigma(U)$. Since $\dim U < \dim G$ we obtain by induction hypothesis

$$|WH| \leq a(U/U_0, \dim U).$$

We show that the possible values for $|U/U_0|$ are finite in number. The group NH_0 is the normalizer of a connected subgroup. By [8], Ch. VII, Lemma 3.2, there are only a finite number of conjugacy classes of such subgroups. Hence for a given G the possible $|U/U_0|$ are bounded, say $|U/U_0| \leq m(G)$. We have inequalities

$$\begin{aligned} |U/U_0| &\leq |NH_0/N_0H_0||N_0H_0/(NH_0)_0| \\ &\leq |G/G_0| m(G_0), \end{aligned}$$

where N_0 means normalizer in G_0. By the classification theory of compact connected Lie groups there are only a finite number in each dimension. Hence there exists a bound for $|U/U_0|$ depending only on $|G/G_0|$ and $\dim G$. This proves the induction step as far as the non-finite H in $\Sigma(G)$ are concerned.

1.3. Let $H \in \Sigma(G)$ be finite. Let $\sigma(G)$ be the set of finite subgroups of G. We use the following classical theorem of Jordan.

LEMMA 5: *There exists an integer $j = j(|G/G_0|, \dim G)$ with the following properties: To each $H \in \sigma(G)$ there exists an abelian normal subgroup A_H of H such that $|H/A_H| \leq j$. Moreover the A_H can be chosen such that $H < K$ implies $A_H < A_K$.*

PROOF: Boothby and Wang [2]. Wolf [9]. In these references only connected groups are considered. The straightforward extension to non-connected groups we leave to the reader.

If $H \in \Sigma(G)$ is finite then also $K := NH$ is finite and by Lemma 1 $K \in \Sigma(G)$. We choose $j = j(|G/G_0|, \dim G)$ and A_H, A_K according to Lemma 5. We have

$$|K/H| \leq |K/A_K| \cdot |A_K/H \cap A_K| \leq j|A_K/H \cap A_K|.$$

Hence it is sufficient to find a bound for $|A_K/H \cap A_K|$. Consider the exact sequence $1 \leftarrow S \leftarrow H \leftarrow A_H \leftarrow 1$. The conjugation $c(a)$ with $a \in A_K$ is trivial on A_H, because $A_K > A_H$, and hence $c(a)$ induces an automorphism of S. Since $|S| \leq j$ this automorphism has order at most $J = j!$, i.e. $c(a^r)$ is the identity on S and A_H for a suitable $r \leq J$. The group of such automorphisms modulo the subgroups of inner automorphisms by elements of A_H is isomorphic to $H^1(S; A_H)$, with S acting on A_H by conjugation. Since this group is annihilated by $|S|$ we see that $c(a^s)$ is an inner automorphism by an element of A_H for a suitable $s \leq J|S| \leq jJ$. In other words: $a^s h^{-1} \in ZH$. Hence it is sufficient to find a bound for the order of $A_K \cap ZH/H \cap A_K \cap ZH$.

Let $U_1 = A_K \cap ZH$. By [3], Théorème 1, U_1 is contained in the normalizer NT of a maximal torus of G. Put $U = U_1 \cap T$. Then $|U_1/U| \leq |G/G_0||w(G_0)|$ where $w(G_0)$ denotes the Weyl group of G_0. We estimate the order of U. Since U is abelian we have $U < ZU$. Moreover $H < ZU$ by definition of ZH. Since U is contained in ZU it is contained in the center $C := CZU$ of ZU. The inclusion $H < ZU$ implies $C < NH$. Hence C is finite.

We proceed to show that for the order of a finite center $C(G)$ of G there exists a bound depending only on $|G/G_0|$ and $\dim G$. We let G/G_0 act by conjugation on $C(G_0)$. Then $C(G) \cap G_0$ is the fixed point set of this action. We have $C(G_0) = A \times T_1$, where A is a finite abelian group and T_1 a torus. The group A is the center of a semisimple group and therefore, by the classification theory of these groups, $|A|$ is bounded by a constant c depending only on $\dim G$. The exact cohomology sequence associated to the universal covering $0 \to \pi_1 T_1 \to V \to T_1 \to 0$ shows, that the fixed point set of the action of G/G_0 on $T_1 = C(G_0)_0$ is isomorphic to $H^1(G/G_0, \pi_1 T_1)$, hence its order is bounded by a constant d depending only on $|G/G_0|$ and the rank of T_1. Hence $|C(G)| \leq |G/G_0|cd$.

Finally we have to show that for the possible groups ZU the order $|ZU/(ZU)_0|$ is bounded.

U is contained in a maximal torus of G. Therefore ZU is a subgroup of maximal rank and $(ZU)_0$ a connected subgroup of maximal rank. By [4] there exist only finitely many conjugacy classes of connected subgroups of maximal rank. We have

$$|ZU/(ZU)_0| \leq |N(ZU)_0/(ZU)_0| \leq |G/G_0|\,|N_0(ZU)_0/(ZU)_0|.$$

There are only finitely many possibilities for normalizers $N_0(ZU)$ in G_0 of $(ZU)_0$.

This finishes the proof of Theorem 1.

2. The integral closure of $A(G)$

Let $\phi = \operatorname{Spec}(A(G) \otimes Q)$ be the prime ideal spectrum of $A(G) \otimes_Z Q =: A_0$ with Zariski topology. By [6], Theorem 4, the prime ideals of A_0 correspond bijectively to kernels of ring homomorphisms $A_0 \to Q$. Therefore ϕ is a totally disconnected compact Hausdorff space ([5], §4. Ex. 16; [1], Ch. 3. Ex. 11).

LEMMA 6: (1) *For each $a \in A_0$ the map $\varphi_a : \phi \to Q : \varphi \mapsto \varphi(a)$ is locally constant. If $a \in A(G)$ then $\varphi(a) \in Z$.* (2) *Let $C(\phi, Q)$ be the ring of locally constant functions $\phi \to Q$. The ring homomorphism $\alpha : A_0 \to C(\phi, Q) : a \mapsto \varphi_a$ is an isomorphism. The image $\alpha A(G)$ is contained in $C(\phi, Z)$.* (3) *The map $A(G) \to A_0 : a \mapsto a \otimes 1$ is the inclusion of A into its total quotient ring. The map $\alpha : A(G) \to C(\phi, Z)$ is the inclusion of $A(G)$ into the integral closure of $A(G)$ in $A_0 \cong C(\phi, Q)$.*

PROOF: (1) For $k \in Q$ the set $\varphi_a^{-1}(k)$ is closed in ϕ, by definition of the Zariski topology. Since A_0 is an absolutely flat ring, this set is also open by [5], §4. Ex. 16.b. Hence φ_a is continuous and there exist only a finite number of non-empty sets $\varphi_a^{-1}(k)$, because ϕ is compact. A homomorphism $A_0 \to Q$ is induced from a homomorphism $A(G) \to Z$, by [6] Theorem 4.

(2) Since the localizations of A_0 at its prime ideals are canonically isomorphic to Q, we can identify $C(\phi, Q)$ with the ring of sections of the structure sheaf of A_0. Then α corresponds to the canonical map of A_0 into this ring, hence α is an isomorphism.

(3) To form the total quotient ring we have to invert the elements which are not zero divisors. Hence A_0 is contained in the total quotient ring. If $x \in C(\phi, Q) \cong A_0$ is not a zero divisor then it is a locally constant function without zeros, hence a unit. Therefore $C(\phi, Q)$ is its own total quotient ring. Since a locally constant function $\phi \to Z$ takes only finitely many values the ring $C(\phi, Z)$ is generated by idempotent elements hence integral over any subring. Under the isomorphism α the ring $C(\phi, Z)$ corresponds to $\{a \in A_0 | \varphi \in \phi \Rightarrow \varphi(a) \in Z\}$. If $x \in A_0$ is integral over $A(G)$ then $\varphi(x)$ is integral over Z, hence $\varphi(x) \in Z$ and $\alpha(x) \in C(\phi, Z)$.

Let $A_c(G)$ denote the pre-image of $C(\phi, Z)$ under α. We recall that $A(G)$ is additively the free abelian group on homogeneous spaces G/H where H runs through a complete system of non-conjugate subgroups H of G with finite index in their normalizer ([6], Theorem 1). Let $\Sigma(G)$ denote the set of conjugacy classes (H) of subgroups H of G with finite $WH = NH/H$.

LEMMA 7: $A_c(G)$ is additively the free abelian group with basis $x_{(H)} := |WH|^{-1} G/H$, $(H) \in \Sigma(G)$.

PROOF: The elements $x_{(H)}$ are contained in $A_c(G)$: Suppose $\varphi \in \phi$ is given. Then there exists $(K) \in \Sigma(G)$ such that $\varphi(x \otimes r) = r\chi(x^K)$ where χ denotes the Euler characteristic and x^K the K fixed point set of any manifold representing x. But WH acts freely as a G-automorphisms group on G/H, hence also on G/H^K. Therefore $\chi(G/H^K)$ is divisible by $|WH|$ and we see that $\varphi(x_{(H)}) \in Z$ for all φ. By definition of $A_c(G)$ this means $x_{(H)} \in A_c(G)$.

The elements $x_{(H)}$ are obviously linearly independent over Z. We have to show that any $x \in A_c(G)$ is an integral linear combination of the $x_{(H)}$. In any case we have an expression $x = \Sigma\, r_H x_{(H)}$ with rational r_H. Take $(L) \in \Sigma(G)$ maximal with respect to inclusion such that $r_L \neq 0$. Then $\Sigma\, r_H |WH|^{-1} \chi(G/H^L) \in Z$. But $G/H^L = \phi$ for $(H) \neq (L)$, $r_H \neq 0$; and $\chi(G/L^L) = |WH|$. Therefore $r_L \in Z$. We apply the same argument to $x - r_L x_{(L)}$ and complete the proof by induction.

Lemma 7 and Theorem 1 give a proof of Theorem 2.

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(Oblatum 26–V–1976)

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