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SOME NOTES ON MARKUŠEVIČ BASES IN WEAKLY COMPACTLY GENERATED BANACH SPACES

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Abstract

Let X be a (non-separable) Banach space generated by a weakly compact subset. If X has Markuševič basis with norming coefficient space then so does every subspace. Extension of Markuševič bases from subspaces to the whole X and a renorming theorem for $X = C(K)$ is proved.

1. Introduction

In this paper some results on separable Banach spaces are generalized to the class of weakly compactly generated (WCG) Banach spaces. By Banach spaces X, $C(K)$ we will, in this introduction, understand (non-separable) WCG spaces.

In section 3 we show that $C(K)$ has Markuševič basis whose coefficient space is contained in span of K. Using the renorming technique of S. Trojanski and the results of E. Asplund and J. Moreau we observe that on $C(K)$ there exists an equivalent locally uniformly rotund norm whose dual norm on $[C(K)]^*$ is rotund and whose unit ball is pointwise closed. Thus on the unit sphere of this norm on $C(K)$ coincide the norm topology and the topology of pointwise convergence.

In section 4 is shown that every Markuševič basis of a subspace of X can be extended to a Markuševič basis of X. In the separable case it was proved in [6]. Further we show that if X has shrinking Markuševič basis then every shrinking Markuševič basis of a subspace of X can be extended to a shrinking Markuševič basis of X. Here it is not necessary to suppose explicitly that X is *WCG* because easily every space with shrinking Markuševič basis is *WCG* (cf. [20], [8]).

In section 5 it is proved that if X has Markuševič basis whose coefficient is norming then every closed subspace has also such a Markuševič basis. Thus the problem of the existence of Markuševič basis of X with norming coefficient space (cf. [17, p. 108] and [9, p. 688]) is reduced to $C(K)$ spaces.

The propositions rely on projectional resolutions of *WCG* spaces constructed by D. Amir and J. Lindenstrauss [1] with some refinements [7], [20].

2. Notation and definitions

If $\langle X, Y\rangle$ is a dual pair of vector spaces, then $w(X, Y)$ is the weak topology on X given by the duality $\langle X, Y\rangle$. For a normed space X, $w(X^*, X)$ (resp. $w(X, X^*)$) topology is denoted by w^* (resp. w)-topology. If $M \subset X$ and Y is a subspace of X^* (total on X), then $\operatorname{sp} M$ (resp. $w(X, Y) \operatorname{sp} M$) denotes the linear (resp. $w(X, Y)$ closed linear) span of M in X. Also we put $\overline{\operatorname{sp}}\, M = w(X, X^*) \operatorname{sp} M$, i.e. the norm closed span of M. A subspace $Y \subset X^*$ is called δ-norming on $(X, |\cdot|)$ if $\delta|x| \leq \{\sup f(x); f \in Y, |f| \leq 1\}$ for all $x \in X$. Evidently Y is 1-norming iff the closed unit ball of X is $w(X, Y)$ closed. If Y is δ-norming for some $\delta > 0$, we say that Y is norming.

A Banach space $(X, |\cdot|)$ is locally uniformly rotund (LUR) if whenever $|x_n| = |x| = 1$, $\lim |x_n + x| = 2$, then $\lim |x_n - x| = 0$. X is rotund if whenever $x, y \in X$, $|x| = |y| = \frac{1}{2}|x + y|$, then $x = y$. A topological space is called Eberlein compact, if it is homeomorphic to a weakly compact subset of a Banach space (in its w-topology). Banach space X is weakly compactly generated (WCG) if $X = \overline{\operatorname{sp}}\, C$ where $C \subset X$ is weakly compact. Let $Y \subset X$; (resp. $Y \subset X^*$); by dens Y (resp. w^* dens X^*) we mean the density of Y, i.e. the smallest cardinal number of a norm (resp. w^*)-dense subset of Y.

The restriction of a map f on a subset A is denoted by f/A. If F is a set of mappings then by F/A we mean the set $\{f/A; f \in F\}$.

A system $\{x_i, x_i^*\}_{i \in I} \subset X \times X^*$ is Markuševič basis (M-basis) if $x_i^*(x_j) = \delta_{ij}$, $\overline{\operatorname{sp}}\,\{x_i\} = X$ and $\{x_i^*\}$ are total on X. M-basis $\{x_i, x_i^*\}$ is shrinking if $\overline{\operatorname{sp}}\,\{x_i^*\} = X^*$. By a coefficient space of M-basis $\{x_i, x_i^*\}$ we mean the (non closed) subspace $\operatorname{sp}\{x_i^*\}$. If $\{x_i, x_i^*\}_{i \in I'}$ is M-basis of the subspace $\overline{\operatorname{sp}}\,\{x_i\}_{i \in I'} \subset X$, then by an extension of this M-basis to X we mean an M-basis $\{x_i, x_i^*\}_{i \in I}$ of X such that $I' \subset I$.

$A \backslash B$ is the set theoretic difference $\{a \in A; a \notin B\}$.

3. *M*-bases and LUR norms in $C(K)$ spaces

We start with a lemma which is a modification of the fundamental finite dimensional Lemma 2 of [1]. We show that arbitrary linearly independent subset K can be preserved by the operators $T: Z \to C$. If we proceeded as in the proof of Lemma 2 in [6] and suitably restricting z_i to K, we would obtain only $T(Z \cap K) \subset \bigcup_{\alpha>0} \alpha K$ (because of the inequality $|\Sigma \lambda_i z_i| \geq |\lambda|$, $z_i \in K$ on page 43. To prove $T(Z \cap K) \subset K$ we will modify a little the proof and list it here for the sake of completeness.

LEMMA 1: Let X *be a linear space with two norms* $|\cdot|_1$, $|\cdot|_2$ *and let* $K \subset X$ *be a linear basis of* X. *Suppose that we are given* $\epsilon > 0$, m *elements* $f_1, \ldots, f_m$ *of* $(X, |\cdot|_2)^*$ *and a finite-dimensional subspace* $B \subset X$. *Then there exists an* $\aleph_0$*-dimensional subspace* $C \subset X$ *containing* B *such that, for every subspace* Z *of* X *with* $Z \supset B$ *and* $\dim Z/B < \infty$, *there is a linear operator* $T: Z \to C$ *with the properties* $|T|_1 \leq 1 + \epsilon$, $|T|_2 \leq 1 + \epsilon$, $Tb = b$ *for every* $b \in B$, $T(Z \cap K) \subset K$ *and* $|f_k(z) - f_k(Tz)| \leq \epsilon |z|_2$ *for every* $z \in Z$ *and* $k = 1, \ldots, m$.

PROOF: It is easy to see that we may suppose that $B = \mathrm{sp}\,(B \cap K)$ and also $Z = \mathrm{sp}\,(Z \cap K)$. Let r be a positive integer. Choose $b_1, \ldots, b_p \in B$ such that for every $b \in B$ we have:

(i) If $|b|_\alpha \leq r$ then there is $h (1 \leq h \leq p)$ such that $|b - b_h|_\alpha < r^{-1}$, $(\alpha = 1, 2)$.

Let n be an other integer and consider the Euclidean space R^n with the norm $|\lambda| = \Sigma_1^n |\lambda_1|$. Choose elements $\lambda^1, \ldots, \lambda^q$ of the unit sphere $S^n = \{\lambda \in R^n; |\lambda| = 1\}$ in R^n. Let us define on the set K^n the following $Q = 2n + 2pq + mn$ functions of $(x_1, \ldots, x_n) \in K^n$:

$$(1) \qquad |x|_\alpha, \left| b_h + \sum_{i=1}^{n} \lambda_i^j x_i \right|_\alpha, \; f_k(x_i) \text{ for } \alpha = 1, 2;$$

$$1 \leq h \leq p; \; 1 \leq j \leq q; \; 1 \leq k \leq m.$$

These functions can be regarded as a function $\varphi: K^n \to R^Q$. Taking in R^Q the metric ρ of maximal coordinate distance, we choose a sequence $\{x^t\}_t = \{x^{trn}\}$ for each r, n. Let $C \subset X$ be the subspace spanned by B and $\{x_i^{trn}\}$, $i = 1, \ldots, n$; t, r, $n = 1, 2, \ldots$.

Now let $\epsilon > 0$, $Z \supset B$, $\dim Z/B = n$ be given. If $B \cap K = \{b_1, \ldots, b_v\}$ and $Z \cap K = \{b_1, \ldots, b_v, z_1, \ldots, z_n\}$ then these are linear bases of B and Z respectively because of our assumptions on B and Z. Let P be the projection of Z onto B sending all z_i to zero and let K be such that $|P|_\alpha \le K$; $\alpha = 1, 2$. Now let the number u be such that $|\Sigma_1^n \lambda_i z_i|_\alpha \ge u|(\lambda_i)|$ (such u exists because all norms on R^n are equivalent). If $|z_i|_\alpha \le s$ for all $i = 1, \ldots, n$ and $\alpha = 1, 2$, we choose positive integer r such that $(2s+4)r^{-1} < \epsilon u(1+K)^{-1}$. Let $x = (x_1, \ldots, x_n) \in K^n$ be an element of the sequence defining C such that $\rho(\varphi(x), \varphi(z_1, \ldots, z_n)) < r^{-1}$. Define on Z

$$T\left(b + \sum_{i=1}^{n} \lambda_i z_i\right) = b + \sum_{i=1}^{n} \lambda_i x_i \qquad (b \in B)$$

We have $T(Z \cap K) = T\{b1, \ldots, b_v, z_1, \ldots, z_n\} = \{b_1, \ldots, b_v, x_1, \ldots, x_n\} \subset K$. Now we prove that $|T|_\alpha \le 1 + \epsilon$. It suffices to show that $|Tz|_\alpha = |b + \Sigma\, \lambda_i x_i| \le (1+\epsilon)|b + \Sigma\, \lambda_i z_i|_\alpha = (1+\epsilon)|z|_\alpha$ if $|\lambda| = \Sigma\, |\lambda_i| = 1$ and $z = b + \Sigma\, \lambda_i z_i \in Z$.

If $|b|_\alpha \ge r$ then $|z|_\alpha \ge r - s$ while

$$|Tz|_\alpha \le |z|_\alpha + |\Sigma\, \lambda_i z_i|_\alpha + |\Sigma\, \lambda_i x_i|_\alpha \le |z|_\alpha + s + (s+1) \le |z|_\alpha + \epsilon(r - s) \le (1+\epsilon)|z|_\alpha.$$

(We used the fact that $\| x_i|_\alpha - |z_i|_\alpha| \le r^{-1} \le 1$.)

If $|b|_\alpha \le r$, let $b_h \in B$ be r^{-1} approximation to b (according to (i)) and let $\lambda^j \in S^n$ be also r^{-1} approximation to $\lambda \in S^n$. We have

$$(2) \quad \left| b + \sum \lambda_i x_i \right|_\alpha - \left| b + \sum \lambda_i z_i \right|_\alpha \le 2|b - b_h|_\alpha + \left| b_h + \sum_i \lambda_i^j x_i \right|_\alpha - \left| b_h + \sum_i \lambda_i^j z_i \right|_\alpha + \left| \sum_i (\lambda_i^j - \lambda_i) x_i \right|_\alpha + \left(\sum_i (\lambda_i^j - \lambda_i) z_i \right|_\alpha \le 2r^{-1} + r^{-1} + (s+1)r^{-1} + sr^{-1} = (2s+4)r^{-1},$$

while

$$\epsilon |z|_\alpha \ge \epsilon |I - P|^{-1} \left| \sum \lambda_i z_i \right|_\alpha \ge \epsilon(1+K)^{-1} u |(\lambda_i)| \ge (2s+4)r^{-1}|(\lambda_i)|$$

Similarly

$$|f_k(z) - f_k(Tz)| = \left| f_k\left(\sum \lambda_i z_i\right) - f_k\left(\sum \lambda_i x_i\right)\right| \le r^{-1}|\lambda| \le \epsilon |z|_2$$

by (1) and (2).

Now the situation of Lemmas 3, 4 and 6 of [1] for WCG Banach space can also be modified such that some subsets $K \subset X^*$ may be preserved under $P^*: X^* \to X^*$. The following lemma corresponds to Lemma 6 of [1].

LEMMA 2: *Let $(X, \|\cdot\|)$ be a WCG Banach space generated by a weakly compact absolutely convex subset $C \subset X$. Let $K \subset X^*$ be w^**

compact subset of X^* *such that* sp K *is* 1-*norming and let one of the two following conditions be satisfied*: (a) K *is lineary independent*, (b) $K\backslash\{0\}$ *is lineary independent. Let* μ *be the first ordinal of cardinality* card X *and let* $\{x_\alpha;\ \alpha<\mu\}$ *be a dense subset of* X. *Then there is a "long sequence" of linear projections* $\{P_\alpha;\ \omega\le\alpha\le\mu\}$ *with* $\|P_\alpha\|=1$, $P_\alpha C\subset C$, $P_\alpha^*K\subset K$, $\operatorname{dens} P_\alpha X\le\bar{\bar{\alpha}}$, $P_\alpha P_\beta=P_\beta P_\alpha=P_\beta$ *whenever* $\beta<\alpha$, $\bigcup_{\beta<\alpha}P_{\cdot+1}X$ *is dense in* $P_\alpha X$ *for every* $\alpha>\omega$ *and* $x_\alpha\in P_{\alpha+1}X$.

PROOF: follows as in Lemmas 3, 4 and 6 of [1] with the following changes. As finite-dimensional lemma we use Lemma 1 (for $K\backslash\{0\}$ in case (b)) and we work on the vector space sp K with two norms: $\|\cdot\|$ and $|\cdot|$ where $|f|=\sup\{f(x);x\in C\}$. Both norms are w^*-lower semiconcontinuous and we may take the cluster points of operators T in the w^* topology with $TK\subset K$, $|T|=\|T\|=1$ and $T^*x_n=x_n$. We have canonical isometric imbedding $X\subset(\operatorname{sp}K)^*$. Now $T^*C\subset C$ because $|T|=1$ and thus $T^*X\subset X$, which implies that T is w^*-w^* continuous. This enables the construction of projection as in Lemma 4 of [1]. We use also the fact that if P is a projection $P:X\to X$, then $\operatorname{dens} PX=w^*\operatorname{dens} P^*X^*$.

The situation of Lemma 2 is hereditary on some complemented subspaces of X in the following sense:

LEMMA 3: *If* $(X,\|\cdot\|)$ *and* $K\subset X^*$ *are as in Lemma* 2 *and* P *is a continuous linear projection* $P:X\to X$ *such that* $P^*K\subset K$, *then* $(PX,\|\cdot\|)$ *and* $K'=K/PX\subset(PX)^*$ *satisfy again the assumptions of Lemma* 2, *i.e.* PX *is WCG*, sp K' *is* 1-*norming*, K' *is* w^* *compact and either* K' *or* K'-$\{0\}$ *is linearly independent.*

PROOF: Let $k_i\in K$, $k_i/PX\neq0$, $\Sigma_{i=1}^n\lambda_ik_i(Px)=0$ for all $x\in X$. Then $\Sigma\lambda_iP^*k_i=0$. But P^*k_i are different non-zero elements of K and thus linearly independent. Thus $\lambda_i=0$, which gives that K' or K'-$\{0\}$ is linearly independent. The other properties are quite evident.

The following is a refinement of some result of Trojanski [19]. We repeat it here explicitly.

LEMMA 4: *Let* $(X,\|\cdot\|)$ *and* $K\subset X^*$ *be as in Lemma* 2. *Then there exists a transfinite sequence* $\{T_\alpha\}$ *of continuous linear projections* $T_\alpha:X\to X$ *satisfying the following conditions*

(i) *for each* $x\in X$ *and* $\epsilon>0$ *the set*

$$A(x,\epsilon)=\{\alpha;\|T_{\alpha+1}x-T_\alpha x\|\ge\epsilon(\|T_{\alpha+1}\|+\|T_\alpha\|)\}$$

is finite

(ii) *for each* $x \in X$

$$x \in Y_x = [\overline{\text{sp}}\ T_1 X \cup \bigcup_{\alpha \in \Lambda(x)} (T_{\alpha+1} - T_\alpha) X]$$

where $\Lambda(\mathrm{x}) = \bigcup_{\epsilon > 0} \Lambda(\mathrm{x}, \epsilon)$

(iii) $\operatorname{dens} (T_{\alpha+1} - T_\alpha) X \leq \operatorname{dens} T_1 X = \aleph_0$

(iv) $T^*_\alpha \operatorname{sp} K \subset \operatorname{sp} K$

(v) $T_\alpha T_\beta = T_\beta T_\alpha = T_\alpha$ *if* $\alpha < \beta$.

PROOF: follows exactly as the corresponding part of the proof of Theorem 1 of [19]. It remains only to observe (iv). Following [19] and using Lemmas 2 and 3, we put by induction $T_\alpha = S^{\alpha'}_{\alpha''}(P_{\alpha'+1} - P_{\alpha'}) + P_{\alpha'}$ where $S^{\alpha'}_{\alpha''} : (P_{\alpha'+1} - P_{\alpha'})X \to (P_{\alpha'+1} - P_{\alpha'})X$ and $\alpha = (\alpha', \alpha'')$. Thus $T^*_\alpha = (P^*_{\alpha'+1} - P^*_{\alpha'})(S^{\alpha'}_{\alpha''})^* + P^*_{\alpha'}$ where $(S^{\alpha'}_{\alpha''})^* : [(P_{\alpha'+1} - P_{\alpha'})X]^* \to [(P_{\alpha'+1} - P_{\alpha'})X]^*$ and $(S^{\alpha'}_{\alpha''})^* : \operatorname{sp} K' \to \operatorname{sp} K'$ where $K' = K/(P_{\alpha'+1} - P_{\alpha'})X$. Now we observe that $(P_{\alpha'+1} - P_{\alpha'})^* \operatorname{sp} K' = (P_{\alpha'+1} - P_{\alpha'})^* \operatorname{sp} K \subset \operatorname{sp} K$ (we denote $(P_{\alpha'+1} - P_{\alpha'}) : X \to (P_{\alpha'+1} - P_{\alpha'})X$ and $(P_{\alpha'+1} - P_{\alpha'}) : X \to X$ by the same letters and similarly for its dual). This shows that $T^*_\alpha : \operatorname{sp} K \to \operatorname{sp} K$.

PROPOSITION 1: *Let* $(X, \|\cdot\|)$ *and* $K \subset X^*$ *be as in Lemma* 2. *Then there is an M-basis* $\{x_i, x^*_i\}$ *of* X *such that* $\operatorname{sp}\{x^*_i\} \subset \operatorname{sp} K$.

PROOF: Let $\{T_\alpha\}$ be a transfinite sequence of projections satisfying (i)–(v) from Lemma 4. We can identify $(T^*_{\alpha+1} - T^*_\alpha)X^*$ with $[(T_{\alpha+1} - T_\alpha)X]^*$ by the canonical $w^* - w^*$ and norm–norm isomorphism. Every $(T_{\alpha+1} - T_\alpha)X$ is separable and thus there are M-bases $\{x^j_\alpha, f^j_\alpha\}_j$ of $(T_{\alpha+1} - T_\alpha)X$ such that $\operatorname{sp}\{f^j_\alpha\}_j \subset \operatorname{sp}(T_{\alpha+1} - T_\alpha)K$ (cf. e.g. [11, Theorem III.1]). As usually, we put these M-bases together (cf. e.g. [7]) to form M-basis $\{x^j_\alpha, f^j_\alpha\}_{j,\alpha} = \{x_i, x^*_i\}$ of X.

PROPOSITION 2: *Let* $(X, \|\cdot\|)$ *and* $K \subset X^*$ *be as in Lemma* 2. *Then there exists one to one imbedding* $T : X \to c_0(\Gamma)$ *which is* $w(X, \operatorname{sp} K) - w$ *continuous on bounded subsets and* $\|T\| = 1$.

PROOF: We follow Dyer [5]. Let $\{x_i, x^*_i\}_{i \in \Gamma}$ be an M-basis of X with $\operatorname{sp}\{x^*_i\} \subset \operatorname{sp} K$ and $\|x^*_i\| = 1$. We define $Tx = \{x^*_i(x)\}$. Evidently T is continuous with respect to $w(X, \operatorname{sp} K)$ topology on X and the topology of coordinate convergence on $c_0(\Gamma)$. But the latter coincides with weak topology on bounded subsets.

PROPOSITION 3: *Let $(X, \|\cdot\|)$ and $K \subset X^*$ be as in Lemma 2. Then X has an equivalent LUR norm which is lower $w(X, \operatorname{sp} K)$ semicontinuous and its dual norm on X^* is rotund.*

PROOF: First we construct LUR norm $|||\cdot|||$ on X which is lower $w(X, \operatorname{sp} K)$ semicontinuous. Let $\|\cdot\|$ be the lower $w(X, \operatorname{sp} K)$ semicontinuous norm on X, i.e. the closed unit ball is $w(X, \operatorname{sp} K)$ closed. Now we use Propositions 1 and 2, Lemma 7 from [10] and proceed as in [19] to obtain LUR norm $|||\cdot|||$ on X which is lower $w(X, \operatorname{sp} K)$ semicontinuous on bounded subsets. Let a be such that $S''' = \{x;\ |||x||| \leq 1\} \subset S'' = \{x; \|x\| \leq a\}$. Thus S''' is $w(X, \operatorname{sp} K)$ closed in S'', but because S'' is also $w(X, \operatorname{sp} K)$ closed, we obtain that S''' is $w(X, \operatorname{sp} K)$ closed in X.

By Lemma 11 of [10] there is another equivalent norm on X which is lower $w(X, \operatorname{sp} K)$ semicontinuous and its dual norm on X^* is rotund. Now we combine these two norms by the averaging procedure of E. Asplund ([2] and [3]), similarly as in the proof of Theorem 1 in [10] and using some results of J. Moreau [16], to obtain the desired norm.

COROLLARY 1: *Let K be an Eberlein compact. Then on $C(K)$ there exists an equivalent LUR norm $|||\cdot|||$ the dual norm of which is rotund and the unit ball $\{x;\ |||x||| \leq 1\}$ of which is pointwise closed. Thus on the unit sphere $\{x;\ |||x||| = 1\}$ coincide the norm and pointwise topology.*

4. Extension of M-bases in WCG spaces

The following lemma is implicitly contained in [20].

LEMMA 5: *Let $\{(x_i, g_i)\}_{i \in I} \subset X \times X^*$ be a biorthogonal system such that $\{g_i\}$ are total over $L = \overline{\operatorname{sp}}\{x_i\} \subset X$. Let $P: X \to X$ be a continuous linear mapping and denote $PX \cap \{x_i\} = \{x_i;\ i \in M\}$. Suppose that*
(a) $PL = \overline{\operatorname{sp}}\{x_i;\ i \in M\}$,
(b) $P^*g_i = g_i$ *for all* $i \in M$.
Then $Px_i = 0$ *for all* $i \notin M$.

PROOF: Let $i \notin M$. If $j \notin M$ then $g_j(Px_i) = 0$ because of (a). If $j \in M$ then also $g_j(Px_i) = (P^*g_j)(x_i) = g_j(x_i) = 0$ using (b).

Definition: Let $\{x_i, x_i^*\}_{i\in I}$ be an M-basis of its closed linear span in X and let $P: X \to X$ be a projection. We will say that the Projection P agrees with the M-basis $\{x_i, x_i^*\}$ if, for all i, either $Px_i = x_i$ or $Px_i = 0$.

Thus Lemma 5 says that if P is projection moreover then P agrees with the M-basis $\{xi, g_i/L\}$.

The following lemma is a modification of Lemma 4 from [1].

Lemma 6: *Let $(X, |\cdot|)$ be a Banach space generated by a weakly compact absolutely convex subset K. Let $\{x_i, x_i^*\}_{i\in I}$ be an M-basis of its closed linear span $\overline{\mathrm{sp}}\{x_i\} \subset X$. Let $\mathfrak{M}$ be an infinite cardinal number; Y, a subspace of X with* dens $Y \leq \mathfrak{M}$; *and F, a subspace of X^* with w^** dens $F \leq \mathfrak{M}$. *Then there exists a linear projection $P: X \to X$ which agrees with the M-basis $\{x_i, x_i^*\}$ and $|P| = 1$, $Py = y$ for every $y \in Y$, $P^*f = f$ for every $f \in F$, $PK \subset K$, and* dens $PX \leq \mathfrak{M}$.

If, moreover, $\overline{\mathrm{sp}}\{x_i\} = X$ and a closed subspace $L \subset X$ is given, then the projection P may be constructed so that also $PL \subset L$.

Proof: Suppose the first alternative $\overline{\mathrm{sp}}\{x_i\} \neq X$ and put $L = \overline{\mathrm{sp}}\{x_i\}$ and $\mathrm{sp}\, K = N$. There is $Y' \subset N$ such that $Y \subset \bar{Y}'$ and dens $Y' =$ dens Y. Thus we may assume that $Y \subset N$. The proof now follows as in Lemmas 3 and 4 in [1], but using as the starting finite-dimensional lemma Lemma 2 of [7]. In Lemma 3 of [1] we thus obtain the existence of $T: X \to X$ with the additional properties $TL \subset L$ and $TK \subset K$. Now we proceed quite similarly as in the proof of Lemma 4 of [1] to construct the projection P, which has also the properties (a), (b) from Lemma 5. Indeed, if $\mathfrak{M} = \aleph_0$ we choose Y_n and T_n (from the proof of Lemma 4 of [1]) with the additional properties $Y_n \cap L = \overline{\mathrm{sp}}\,(Y_n \cap \{x_i\})$ and $T_n^* g_i = g_i$ for all i such that $x_i \in Y_{n-1}$; ($g_i \in X^*$ are arbitrary fixed extensions of $x_i^* \in L^*$). We have

$$PL \subset \overline{\cup\, T_n L} \subset \overline{\cup\,(Y_n \cap L)} = \overline{\mathrm{sp}}\,[(\cup\, Y_n) \cap \{x_i\}] \subset \overline{\mathrm{sp}}\,(PX \cap \{x_i\}) \subset PL.$$

Thus all these sets agree, which gives (a) and $PL = \overline{\mathrm{sp}}\,[(\cup\, Y_n) \cap \{x_i\}] = \overline{\mathrm{sp}}\{x_i; i \in M\}$. This easily implies (b).

If $\overline{\mathrm{sp}}\{x_i\} = X$ and a closed subspace $L \subset X$ is given, we proceed quite similarly. Y_n and T_n in Lemma 4 of [1] are now chosen with the additional properties: $Y_n = \overline{\mathrm{sp}}\,(Y_n \cap \{x_i\})$, $T_n L \subset L$ and $T_n^* x_i^* = x_i^*$ for all i such that $x_i \in Y_{n-1}$. Then also $PX = \overline{\mathrm{sp}}\,(\cup\, Y_n \cap \{x_i\}) = \overline{\mathrm{sp}}\{x_i; i \in M\}$ and also (b) from Lemma 6 follows easily.

If $\mathfrak{M} > \aleph_0$ we proceed again similarly as in [1] but take all projections P_α such that they agree with the M-basis $\{x_i, x_i^*\}$.

PROPOSITION 4: *Let $\{x_i, x_i^*\}_{i\in I}$ be an M-basis of a subspace of a WCG Banach space X. Then the M-basis $\{x_i, x_i^*\}$ can be extended to an M-basis of X.*

PROOF: is by induction on dens X. If X is separable then Proposition 4 reduces to Theorem 1 of [6]. If dens $X > \aleph_0$, we construct a transfinite sequence $\{T_\alpha\}$ having properties (i)–(iii) and (v) from Lemma 4 and such that all T_α agree with M-basis $\{x_i, x_i^*\}$. Then we have

$$\{x_i\}_{i\in I} = \bigcup_\alpha [(T_{\alpha+1} - T_\alpha)X \cap \{x_i\}] \cup (T_1 X \cap \{x_i\})$$

because of the monotony of T_α's and the density of $\cup\, T_\alpha X$ in X. Now, if we extend the M-bases $(T_{\alpha+1} - T_\alpha)X \cap \{x_i\}$ to M-bases of $(T_{\alpha+1} - T_\alpha)X$ and put them together (cf. e.g. proof of Proposition 5 in [7]), they form an M-basis of X which extends $\{x_i, x_i^*\}$.

PROPOSITION 5: *Let $\{x_i, x_i^*\}$ be a shrinking M-basis of a subspace of X and let in X exists a shrinking M-basis. Then the M-basis $\{x_i, x_i^*\}$ can be extended to a shrinking M-basis of X.*

PROOF: If X has a shrinking M-basis then it is WCG and has an equivalent Fréchet differentiable norm $|\cdot|$ (cf. e.g. [8]). Then the usual decomposition of X by transfinite sequence of projections $\{P_\alpha\}$, $|P_\alpha| = 1$ has the property that $\cup_{\beta<\alpha} P^*_{\beta+1} X^*$ is dense in $P^*_\alpha X^*$ (cf. e.g. [8, Lemma 3]). Thus the system $\{T_\alpha\}$ constructed in Lemma 4 has also the property that $\cup_{\beta<\alpha} T^*_{\beta+1} X^*$ is dense in $T^*_\alpha X^*$. Now we proceed as in the preceeding proof.

5. Heredity of the existence of norming *M*-basis in WCG spaces

PROPOSITION 6: *Let X be a WCG Banach space which has an M-basis whose coefficient space is δ-norming. Then every closed subspace $L \subset X$ has also an M-basis with δ-norming coefficient space.*

PROOF: Let $\{x_i, x_i^*\}$ be an M-basis of X with δ-norming coefficient space. Similarly as in the proof of Proposition 4 and using the second part of Lemma 7 we construct a transfinite system of projections $\{T_\alpha\}$ with properties (i)–(iii), (v) from Lemma 4, $T_\alpha L \subset L$ and such that all T_α agree with the M-basis $\{x_i, x_i^*\}$. Evidently $x_i \in (T_{\alpha+1} - T_\alpha)X \Leftrightarrow x_i^* \in (T_{\alpha+1}^* - T_\alpha^*)X^*$. Thus the sets $C_\alpha = (T_{\alpha+1}^* - T_\alpha^*)X^* \cap \{x_i^*\}$ are at most countable. By Theorem III.1 of [11] there is (for each α) M-basis $\{x_{\alpha n}, x_{\alpha n}^*\}$ of $(T_{\alpha+1} - T_\alpha)L$ whose coefficient space $\operatorname{sp}\{x_{\alpha n}^*\}_n$ contains the countable set $C_\alpha/(T_{\alpha+1} - T_\alpha)L$. Denote $f_{\alpha n} = (T_{\alpha+1}^* - T_\alpha^*)x_{\alpha n}^*/L$. As usually $\{x_{\alpha n}, f_{\alpha n}\}_{\alpha,n}$ now form the M-basis of L whose coefficient space contains $\cup C_\alpha/L = \{x_i^*/L\}$ (we used the fact that T_α agree with $\{x_i, x_i^*\}$). Evidently $\cup C_\alpha$ is δ-norming on L.

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