Locally convex spaces for which $\Lambda(E) = \Lambda[E]$ and the Dvoretsky-Rogers theorem

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The classical Dvoretsky-Rogers theorem states that if $E$ is a Banach space for which $\ell^1[E] = \ell^1(E)$, then $E$ is finite dimensional. This property still holds for any $\ell^p (1 < p < \infty)$ (see [5]).

Recently it has been shown (see [11]) that the result remains true when one replaces $\ell^1$ by any non-nuclear perfect sequence space $\Lambda$, having the normal topology $n(\Lambda, \Lambda^\perp)$. (This situation does not contain the $\ell^p (1 < p < \infty)$-case). The question whether the Dvoretsky-Rogers theorem holds for any perfect Banach sequence space $\Lambda$ is still open. (A partial, positive answer to this problem, generalizing the $\ell^p$-case for any $p$ is given in this paper.) It seemed however more convenient to us to tackle the problem from the “locally-convex view-point”.

The “locally-convex version” of the Dvoretsky-Rogers theorem was proved by A. Pietsch in [8] (for the definitions and notations see below):

If $E$ is a locally convex space then the following are equivalent:

(i) $E$ has property $(B)$ and $\ell^1[E] = \ell^1(E)$

(ii) $E'_\Lambda$ is nuclear (Remark that every Banach space $E$ has the property $(B)$).

An inspection of the proof shows that Pietsch actually proves that (i) is equivalent to

(iii) For every $A \in \mathcal{B}_E$ there exists a $B \in \mathcal{B}_E (A \subset B)$ such that the canonical mapping $\varphi_{AB} : E_A \to E_B$ is absolutely summing.

Since all the notions appearing in the ((i)$\iff$(iii))-version of the Dvoretsky-Rogers theorem have meaningfull generalizations when replacing $\ell^1$ by any perfect Banach sequence space $\Lambda$ one can ask if (i)$\iff$(iii) still holds in this generalized situation.

In this paper we show that the answer to this question is positive and we apply our result to some special cases.
§ 1. Preliminaries

All the classical notions and properties concerning locally convex spaces, as well as the elementary theory of sequence spaces, will be taken from [6]. They will be used without any further reference. The same will be done as far as nuclearity is concerned. Here we refer to [8].

We fix the following notations:
— If not specified $E$ will denote a complete locally convex Hausdorff space with topological dual space $E'$.
— $\mathcal{P}_E$ is a fundamental system of semi-norms determining the topology of $E$.
— $\mathcal{U}_E$ is a fundamental system of barrelled neighbourhoods of the origin in $E$.
— $\mathcal{B}_E$ is a fundamental system of closed, convex, bounded subsets of $E$.

For $B \in \mathcal{B}_E$ we denote by $E_B$ the Banach space $\bigcup_{n \in \mathbb{N}} n \cdot B$, normed by the gauge of $B$. This norm is denoted by $\| \cdot \|_B$.

— $\Lambda$ is a perfect sequence space with $\alpha$-dual space $\Lambda^\alpha$. We assume that $\Lambda$ is a Banach space for the strong topology $\beta(\Lambda, \Lambda^\alpha)$. Elements of $\Lambda$ are denoted by $\alpha = (\alpha_i)$.

— We consider the following generalized sequence spaces

$$\Lambda[E] = \{ (x_i) | x_i \in E, \ i = 1, 2, \ldots \ \text{and} \ \forall a \in E' : (x_i, a)_i \in \Lambda \}$$

and

$$\Lambda(E) = \{ (x_i) | x_i \in E, \ i = 1, 2, \ldots \ \text{and} \ \forall p \in \mathcal{P}_E : (p(x_i))_i \in \Lambda \}$$

A locally convex Hausdorff topology on $\Lambda[E]$ is given by the semi-norms:

$$\sup_{a \in U^\alpha} \|(x_i, a)_i\|_{\Lambda}, \ (x_i) \in \Lambda[E], \ U \in \mathcal{U}_E.$$ 

A locally convex Hausdorff topology on $\Lambda(E)$ is given by the semi-norms

$$\|p(x_i)\|_{\Lambda}, \ (x_i) \in \Lambda(E), \ p \in \mathcal{P}_E.$$ 

For $\Lambda = \ell^1$ the above spaces are studied in [8]. In their general form the spaces $\Lambda[E]$ and $\Lambda(E)$ are studied in [7] and in [1] and [10] respectively. Obviously $\Lambda(E)$ is continuously embedded in $\Lambda[E]$.

— For a normal bounded subset $R$ of $\Lambda$ and $B \in \mathcal{B}_E$, define:

$$[R, B] = \{ (x_i) | (x_i) \in \Lambda(E), \ x_i \in E_B, \ i = 1, 2, \ldots \ \text{and} \ (\|x_i\|_B)_i \in R \}$$
The space $E$ is said to be fundamentally $\Lambda$-bounded if the collection of all $[R, B]$ forms a fundamental system of bounded subsets of $\Lambda(E)$. The notion of fundamentally $\Lambda$-bounded space has been introduced in [10]. A fundamentally $\ell^1$-bounded space is exactly a space "having the property (B)" (see [8]).

An operator (i.e. a continuous linear mapping) from a Banach space $X$ to a Banach space $Y$ is called $\Lambda$-summing if for every $(x_i) \in \Lambda[X]$ the sequence $(f(x_i))_i$ is an element of $\Lambda(Y)$. For $\Lambda = \ell^1$ (resp. $\Lambda = \ell^p$, $1 < p < \infty$) such an operator is called absolutely summing (resp. $p$-summing).

§2. A generalized Dvoretsky-Rogers theorem

**Lemma 1:** If $D$ is a bounded subset of $\Lambda[E]$ then there exists $A \in \mathcal{B}_E$ such that $D \subseteq A[E_A]$ and

$$\sup_{(x_i) \in D} \|\langle x_i \rangle\|_{\Lambda[E_A]} \leq 1.$$

**Proof:** For $U \in \mathcal{U}_E$ we put

$$\sigma_u = \sup_{(x_i) \in D} \sup_{a \in U} \|\langle x_i, a \rangle\|_A.$$

Then $\sigma_u < \infty$ since $D$ is bounded in $\Lambda[E]$. If $A = \bigcap_{U \in \mathcal{U}_E} \sigma_u \cdot U$ then $A \in \mathcal{B}_E$. Take $(\lambda_i) \in A^\times$ such that $\|\langle \lambda_i \rangle\|_{A^*} \leq 1$. Then for $U \in \mathcal{U}_E$ and $a \in U$ we have:

$$\left| \sum_i \lambda_i x_i, a \right| = \left| \sum_i \lambda_i \langle x_i, a \rangle \right|$$

$$\leq \|\langle \lambda_i \rangle\|_{A^*} \cdot \|\langle x_i, a \rangle\|_A \leq \sigma_u,$$

for all $(x_i) \in D$. Hence $\Sigma_i \lambda_i x_i \in \sigma_u \cdot U$ for each $U \in \mathcal{U}_E$. So $\Sigma_i \lambda_i x_i \in A$ (or $\Sigma_i \lambda_i x_i \|_A \leq 1$) for all $(\lambda_i) \in A^\times$, $\|\langle \lambda_i \rangle\|_{A^*} \leq 1$ and all $(x_i) \in D$. I.e.

$$\left| \sum_i \lambda_i x_i, b \right| = \left| \sum_i \lambda_i \langle x_i, b \rangle \right| \leq 1$$

for all $(x_i) \in D$, all $(\lambda_i) \in A^\times$, $\|\langle \lambda_i \rangle\|_{A^*} \leq 1$ and all $b \in (E_A)'$ with $\|b\|_{(E_A)'} \leq 1$.

Since the unit ball in $A^\times$ is a normal subset of $A^\times$ we also have:

$$\sum_i |\lambda_i| \|\langle x_i, b \rangle\| \leq 1$$
under the same assumptions on \((x_i), (\lambda_i)\) and \(b\). Hence \((x_i) \in \Lambda[E_A]\) and \(\|x_i\|_{\Lambda[E_A]} \leq 1\) for all \((x_i) \in D\).

**Lemma 2:** If \(\Lambda(E) = \Lambda[E]\) and \(B\) is a bounded subset of \(\Lambda[E]\), then \(B\) is also bounded in \(\Lambda(E)\).

**Proof:** Remark that \(B \subset \Lambda[E]\) is bounded if and only if for every \(a \in E'\) the set

\[
\{(x, a) | (x) \in B\}
\]

is bounded in \(\Lambda\) (see [7]). Also \(B \subset \Lambda(E)\) is bounded if and only if for each \(p \in \mathcal{P}_E\) the set

\[
\{(p(x_i)), (x_i) \in B\}
\]

is bounded in \(\Lambda\) (see [1]). Finally a subset \(A\) of \(\Lambda\) is bounded if and only if for each \(\beta \in \Lambda^*\) the set

\[
\left\{ \sum_i |\alpha_i \beta_i| |\alpha \in A\right\}
\]

is bounded in \(\mathbb{R}\). Taking these facts in mind, the proof proceeds exactly as in [8] Theorem 2.1.2.

**Theorem:** The following are equivalent

(i) \(E\) is fundamentally \(\Lambda\)-bounded and \(\Lambda(E) = \Lambda[E]\).

(ii) For every \(A \in \mathcal{B}_E\) there exists a \(B \in \mathcal{B}_E\) \((B \supset A)\) such that the canonical injection \(\varphi_{AB} : E_A \rightarrow E_B\) is \(\Lambda\)-summing.

**Proof:** (i) \(\Rightarrow\) (ii): Take \(A \in \mathcal{B}_E\) and put

\[
D = \{(x_n) | (x_n) \in \Lambda[E_A], \|x_n\|_{\Lambda[E_A]} \leq 1\}
\]

The continuous injection \(i_A : E_A \rightarrow E\) extends canonically to a continuous injection \(\tilde{i}_A : \Lambda[E_A] \rightarrow \Lambda[E]\) ([2] Prop. 28). Hence \(D\) is bounded in \(\Lambda[E]\).

Since \(\Lambda[E] = \Lambda(E)\), \(D\) is also bounded in \(\Lambda(E)\) (lemma 2). Since \(E\) is fundamentally \(\Lambda\)-bounded, there exists \(B \in \mathcal{B}_E\) such that the set

\[
\{(\|x_n\|_B)_n | (x_n) \in D\}
\]

is bounded in \(\Lambda\). In particular we have \(D \subset \Lambda(E_B)\). So \((x_n) \in \Lambda[E_A]\) implies \((x_n) \in \Lambda(E_B)\) and (ii) is proved.

(ii) \(\Rightarrow\) (i): Let \(D\) be a bounded subset of \(\Lambda(E)\). Then \(D\) is bounded in \(\Lambda[E]\) and, by lemma 1, \(D\) is bounded in \(\Lambda[E_A]\) for some \(A \in \mathcal{B}_E\). By (ii) there exists \(B \in \mathcal{B}_E\) such that \(\varphi_{AB} : E_A \rightarrow E_B\) is \(\Lambda\)-summing.
Then $D$ is a bounded subset of $\Lambda(E_B)$ since the extended mapping $\varphi_{AB}: \Lambda[E_A] \to \Lambda(E_B)$ is continuous ([2] Prop. 28). So $E$ is fundamentally $\Lambda$-bounded. For the second half of (i) suppose $(x_i) \in \Lambda[E]$. Then by lemma 1 there exists $A \in \mathcal{B}_E$ such that $(x_i) \in \Lambda[E_A]$. (Consider $\{(x_i)\}$ as a bounded subset of $\Lambda[E]$). By (ii) $(x_i) \in \Lambda(E_B)$ for some $B \in \mathcal{B}_E$. Finally the canonical injection $i_B: E_B \to E$ induces an injection $\tilde{i}_B: \Lambda(E_B) \to \Lambda(E)$ and the conclusion follows.

§3. Examples and special cases

1. Locally convex spaces for which $\Lambda(E) = \Lambda[E]$.

Recall that a sequence $(e_i)$ in $E$ is called a Schauder basis for $E$ if every $x \in E$ can be written uniquely as $x = \sum \alpha_i e_i$ and if the coefficient functionals $f_k: x \to \alpha_k$ are continuous. The basis $(e_i)$ is strong if $\sum p(e_i)p_B(f_i) < \infty$ for every $p \in \mathcal{P}_E$ and every $B \in \mathcal{B}_E$ ($p_B$ denotes the seminorm on $E'$, $\beta(E', E)$ corresponding to $B$). For the connection between the existence of a strong basis with the nuclearity of the space, as well as for examples of spaces having a strong basis we refer to [3].

**Proposition 1:** If $E$ has a strong basis $(e_i, f_i)$ then $\Lambda(E) = \Lambda[E]$ whenever $e^i \subset \Lambda$.

**Proof:** Take $(y_n) \in \Lambda[E]$. Since $E$ is semi-reflexive ([3] Prop. 4), its strong dual space $E'_{\beta}$ is barrelled. Hence, by the Banach-Steinhaus theorem, the linear mapping

$$g: E'_{\beta} \to \Lambda : a \to (\langle y_n, a \rangle)_n$$

is continuous. i.e. $\exists B \in \mathcal{B}_E, \exists K > 0$ such that

$$\|g(f_i)\|_\Lambda = \|(\langle y_n, f_i \rangle)_n\|_\Lambda \leq Kp_B(f_i), \ i = 1, 2, \ldots$$

For $p \in \mathcal{P}_E$ we then have:

$$p(y_n) = p\left(\sum_i \langle y_n, f_i \rangle e_i\right) \leq \sum_i |\langle y_n, f_i \rangle|p(e_i)$$

So for $\beta \in \Lambda^\times$ we obtain:

$$\sum_n |\beta_n|p(y_n) \leq \sum_n \sum_i |\beta_n||\langle y_n, f_i \rangle|p(e_i)$$

$$\leq \sum_i p(e_i)\|\beta\|_{\Lambda^\times} \cdot \|(\langle y_n, f_i \rangle)\|_\Lambda$$
\[ \leq \|\beta\|_{A^*} \cdot K \cdot \sum_{i} p(e_i)p_B(f_i) < \infty, \]

which implies that \((y_n) \in \Lambda(E)\).

2. \textit{Spaces having property (ii) in the Theorem.}
For convenience we’ll call them \(\Lambda\)-spaces.

\textbf{PROPOSITION 2: Under each of the following conditions \(E\) is a \(\Lambda\)-space:
\begin{itemize}
  \item[(a)] \(E_\beta^\prime\) is nuclear
  \item[(b)] \(E\) has a strong basis and is fundamentally \(\Lambda\)-bounded
  \item[(c)] \(E\) is a Frechet space or a DF-space with a strong basis
  \item[(d)] \(E\) is a Frechet space or a DF-space and \(\Lambda(E) = \Lambda[E]\).
\end{itemize}

\textbf{PROOF:}
(a) Every absolutely summing map between Banach spaces is \(\Lambda\)-summing (see [4]), then apply Pietsch's result mentioned in the introduction, and the theorem.
(b) From Prop. 1.
(c) Every Frechet space (and every DF-space) is fundamentally \(\Lambda\)-bounded (see [10]). Then apply (b)
(d) As in (c).

\textbf{REMARK: Proposition 2(a) can also be interpreted as follows: If \(E\) has property (B) and \(l_1(E) = l_1[E]\) then for every \(\Lambda\), \(E\) is fundamentally \(\Lambda\)-bounded and \(\Lambda(E) = \Lambda[E]\).}

3. \textit{The relation to nuclearity.}
If \(\Lambda\) is such that sufficiently many compositions of \(\Lambda\)-summing maps provide an absolutely summing map then every \(\Lambda\)-space \(E\) has a strong dual space \(E_\beta^\prime\) which is nuclear. It is shown in [9] that this is the case whenever \(\Lambda = \ell^p\) \((1 < p < \infty)\). We therefore obtain:

\textbf{PROPOSITION 3: The following are equivalent:
\begin{itemize}
  \item[(i)] \(E_\beta^\prime\) is nuclear
  \item[(ii)] \(E\) is fundamentally \(\ell^p\)-bounded and \(\ell^p(E) = \ell^p[E]\) \((\text{for some} \ 1 \leq p < \infty)\).
\end{itemize}

\textbf{COROLLARY: IF \(E\) is a Frechet space or a DF-space then \(E\) (and \(E_\beta^\prime\)) is nuclear if and only if \(\ell^p(E) = \ell^p[E]\) \((\text{for some} \ 1 \leq p < \infty)\).}

(This result contains Grothendieck's result mentioned in the introduction).
REFERENCES


(Oblatum 20–V–1976)