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Canonical divisors and the additivity of the Kodaira dimension for morphisms or relative dimension one

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All schemes, varieties and morphisms are defined over the field of complex numbers $\mathbb{C}$.

The following conjecture due to Iitaka is a central problem of the classification theory of algebraic varieties ([21; p. 95], [22]):

**Conjecture $C_{n,m}$:** Let $\pi : V \to W$ be a surjective morphism of proper, regular varieties, $n = \dim(V)$ and $m = \dim(W)$. Assuming a general fibre $V_w = \pi^{-1}(w)$ is connected, we have the following inequality for the Kodaira dimension:

$$K(V) \geq K(W) + K(V_w).$$

$C_{2,1}$ is a corollary of Enriques’ and Kodaira’s classification theory of algebraic surfaces [21; p. 133]. Recently another proof has been given by K. Ueno [23]. I. Nakamura and K. Ueno solved $C_{n,m}$ for analytic fibre bundles $\pi : V \to W$. In this case, $V$ need not be algebraic and equality holds [21]. In [22], Ueno gave a proof of $C_{3,2}$ when $\pi : V \to W$ is a family of elliptic curves with locally meromorphic sections. Some other special cases of $C_{n,1}$ are treated in [21; p. 134] and [23].

In this paper we give an affirmative answer to $C_{n,n-1}$ ($\pi$ need not be equidimensional).

The case $C_{2,1}$ of “families of curves over a curve” is treated separately (3.7). The proof in this case is rather elementary. In addition we are able to give an explicite description of the canonical divisor of $V$ in terms of the Weierstrass points of the regular fibres and the local behavior of $\pi$ near the degenerate fibres (3.6). The resulting formulas for the square of the canonical divisor ((3.6) and (4.13)) generalize the formula given by Ueno (see [16; p. 188] and (4.13)).
(4.9)) for families of curves of genus 2. However, as we will see in §4, the “local contributions” are not completely determined by the local invariants of the degenerate fibres ([16], [19]).

In §1 we summarize some known results about the Kodaira dimension and give the reduction of $C_{n,n-1}$ to some statement $C'_{n,n-1}$ (1.6) about the “relative dualizing sheaf”. §2 deals with stable curves. We give a description of the relative dualizing sheaf of stable curves using Wronskian determinants (2.10). In §3 and §4 we handle the special case “families of curves over a curve”. The proof of $C'_{n,n-1}$ is given in the second half of this paper. §§5 contains the proof of some kind of “stable reduction theorem” for higher dimensional base schemes (5.1) and in §6 we use (5.1) and duality theory [6] to reduce the proof of $C_{n,n-1}$ to stable curves. This special case is handled in §7 and §8.

The proven result is slightly stronger than $C_{n,n-1}$ (see Remark 1.8).

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§1. Kodaira dimension and $L$-dimension

In this section, $X$ is assumed to be a proper, normal variety and $L$ to be an invertible sheaf on $X$.

1.1. Definition:

(i) We set $N(L, X) = \{m > 0; \dim H^0(X, L^m) \geq 1\}$.

(ii) For $m \in N(L, X)$, we denote by $\Phi_{m,L} : X \to \mathbb{P}^N$ the rational map given by $\Phi_{m,L}(x) = (\varphi_0(x), \ldots, \varphi_N(x))$ where $\varphi_0, \ldots, \varphi_N$ is a basis of $H^0(X, L^m)$.

(iii) The $L$-dimension of $X$ is

$$K(L, X) = \begin{cases} -\infty & \text{if } N(L, X) = \emptyset \\ \max \{\dim (\Phi_{m,L}(X)); m \in N(L, X)\} & \text{if } N(L, X) \neq \emptyset. \end{cases}$$

1.2. Definition: Let $X$ be regular and denote the canonical sheaf of $X$ by $\omega_X$. Then $K(X) = K(\omega_X, X)$ is the Kodaira dimension of $X$.

The reader is referred to Ueno [21] for a general discussion of $L$-dimension and Kodaira dimension. The proofs of the following statements can be found in chapters II and III of [21].
1.3. Properties of the $\mathcal{L}$-dimension:

(i) There exist positive real numbers $\alpha$, $\beta$ and $m_0$, such that for all $m \in \mathbb{N}(\mathcal{L}, X)$, $m \geq m_0$, we have:

$$\alpha \cdot m^{K(\mathcal{L}, X)} \leq \dim_c H^0(X, \mathcal{L}^m) \leq \beta \cdot m^{K(\mathcal{L}, X)}$$

(ii) Let $f : X' \to X$ be a surjective morphism of normal, proper varieties. Then $K(f^* \mathcal{L}, X') = K(\mathcal{L}, X)$.

(iii) Let $\mathcal{L}' \to \mathcal{L}^\otimes a$, $a > 0$, be a non trivial map of invertible sheaves on $X$. Then $K(\mathcal{L}', X) \leq K(\mathcal{L}, X)$.

1.4. Properties of the Kodaira Dimension: $X$ is assumed to be regular.

(i) $K(X)$ depends only on the field of rational functions $\mathbb{C}(X)$ (i.e., $K(X)$ is an invariant of the birational equivalence class of $X$).

(ii) For every $m \in \mathbb{N}(\omega_X, X)$ we denote the closure of the image of $X$ under $\Phi_{m,\omega_X}$ by $X_m$. The induced rational map from $X$ to $X_m$ we also denote by $\Phi_{m,\omega_X}$. Assume $K(X) \geq 0$. Then there exists a surjective morphism of regular projective varieties $f : X' \to Y'$ and $m_0 \in \mathbb{N}$, such that for every $m \in \mathbb{N}(\omega_X, X)$, $m \geq m_0$, the following conditions are fulfilled:

(a) $f$ is birationally equivalent to $\Phi_{m,\omega_X} : X \to X_m$ (see remark 1.5).

(b) $\mathbb{C}(Y')$ is algebraically closed in $\mathbb{C}(X)$ and $K(Y') = \dim(Y')$.

(c) There exist closed subvarieties $Z_i \subset Y'$, $Z_i \neq Y'$ for $i \in \mathbb{N}$, such that for every $y \in Y' - \bigcup_{i \in \mathbb{N}} Z_i$ the fibre $X'_y = f^{-1}(y)$ is irreducible, regular and of Kodaira dimension zero.

(iii) Let $g : X \to Y$ be a surjective morphism of proper, regular varieties and $X_y$ a general fibre of $g$. Then we have:

$$K(X) \leq K(X_y) + \dim(Y).$$

1.5. Remark: Two rational maps (or morphisms) $\pi_i : V_i \to W_i$, $i = 1, 2$, are called birationally equivalent, if there exist birational maps $\varphi : V_1 \to V_2$ and $\eta : W_1 \to W_2$ such that $\eta \cdot \pi_1 = \pi_2 \cdot \varphi$.

1.4(i) enables us to replace the morphism $\pi : V \to W$ in $C_{n,m}$ by any birational equivalent morphism. Of course, to prove $C_{n,m}$ we may always assume that neither $W$ nor $V_w$ is of Kodaira dimension $-\infty$. Let $C$ be a curve of genus $g$. Then

$$K(C) = \begin{cases} -\infty & \text{if } g = 0 \\ 0 & \text{if } g = 1 \\ 1 & \text{if } g \geq 2. \end{cases}$$
Hence we may assume, in order to prove $C_{n,n-1}$, that the general fibre of $\pi$ is a curve of genus $g \geq 1$. The only possible values of $K(\mathcal{V}_w)$ are 0 and 1.

1.6. **Statement $C'_{n,m}$**: Let $\pi_1 : V_1 \to W_1$ be a surjective morphism of regular, proper varieties with connected general fibre, $n = \dim (V_1)$ and $m = \dim (W_1)$. Then there exists a birationally equivalent morphism $\pi : V \to W$ of regular, proper varieties, such that for a general fibre $\mathcal{V}_w$ of $\pi$ we have the inequality

$$K(\omega_V \otimes \pi^* \omega_{W}^{-1}, V) \geq K(\mathcal{V}_w).$$

In §8 we are going to prove $C_{n,n-1}$. The special case $C_{2,1}$ is proven in §3. The connection with conjecture $C_{n,n-1}$ is given by:

1.7. **Theorem**: Assume that statement $C'_{r,r-1}$ is true for all $r \leq n$. Then $C_{n,n-1}$ is true.

**Proof**: Let $\pi : V \to W$ be a morphism, satisfying the assumptions of $C_{n,n-1}$. Remark 1.5 enables us to assume that $K(\omega_V \otimes \pi^* \omega_{W}^{-1}, V) \geq K(\mathcal{V}_w)$. We are allowed, of course, to exclude the trivial cases $K(\mathcal{V}_w) = -\infty$ or $K(W) = -\infty$. Choose $m \in \mathbb{N}$ such that $m \in \mathbb{N}(\omega_V, V) \cap \mathbb{N}(\omega_W, W)$ fulfills the condition 1.4(ii), and such that $(\omega_V \otimes \pi^* \omega_{W}^{-1})^\otimes_m$ has a nontrivial global section. We have an injection

$$\pi^* : H^0(W, \omega_W^\otimes_m) \to H^0(V, \omega_V^\otimes_m)$$

for every $l \in \mathbb{N}$.

Using 1.3(i) we get $K(V) \cong K(W)$.

Assume now, that $K(\mathcal{V}_w) = 1$ and $K(V) = K(W)$. The injection $\pi^*$ gives a commutative diagram:

$$\begin{array}{ccc}
V & \xrightarrow{\Phi_{m,\omega_V}} & \mathbb{P}^M \\
\downarrow \pi & & \downarrow p \\
W & \xrightarrow{\Phi_{m,\omega_W}} & \mathbb{P}^N
\end{array}$$

where $p$ is a projection. The dimension of the images $V_m$ and $W_m$ are equal and $C(W_m)$ is algebraically closed in $C(W)$ (1.4(ii)) and therefore in $C(V)$. Hence $p$ induces a birational map from $V_m$ to $W_m$.

We can find $u \in W_m$ such that $(p \cdot \Phi_{m,\omega_V})^{-1}(u)$ and $\Phi_{m,\omega_W}(u)$ are
birationally equivalent to regular varieties $V''$ and $W''$ of Kodaira dimension zero (1.4(ii)(c)). Blowing up points of indeterminacy [8], we may assume that $\pi$ induces a surjective morphism $\pi'': V'' \to W''$. If we choose $u$ in general position, the dimension and the Kodaira dimension of a general fibre of $\pi''$ are both one. For $k \in \mathbb{N}(\omega_{W''}, W'') \neq \emptyset$ the sheaf $\omega_{W''}^k$ has a non-trivial section and hence (1.3(iii)) $0 = K(V'') \simeq K(\pi''_* \omega_{W''}^{-1} \otimes \omega_{V''}, V'')$. This is a contradiction to $C_{r',-1}^r$, where $r = \dim(V'')$.

1.8. REMARK: Let $K(W) \geq 0$. The argument used at the end of the proof gives the inequality $K(V) \geq \max(K(W) + K(V_w), K(\omega_V \otimes \pi_* \omega_{W}^{-1}, V))$. In §8 we are going to prove a slightly stronger statement than $C_{n,n-1}^n$. Let $g$ be the genus of $V_w$ and $M_{g_0}$ the coarse moduli scheme of regular curves of genus $g$. The smooth part of $\pi : V \to W$ induces a rational map $\varphi : W \to M_{g_0}$. We prove that

$$K(\omega_V \otimes \pi_* \omega_{W}^{-1}) \geq \max(K(V_w), \dim(\varphi(W))).$$

Hence we get in addition: $K(V) \geq \dim(\varphi(W))$ if $K(W) \geq 0$.

§2. Stable curves and Weierstrass sections

The main references for stable curves are [4], [12] and [5], [11] (genus one).

2.1. DEFINITION: Let $S$ be a scheme and $g \geq 1$.

(i) A pseudo-stable curve of genus $g$ over $S$ is a proper, flat morphism $\rho : C \to S$ whose geometric fibres are reduced, connected 1-dimensional schemes $C_s$ of genus $g$ (i.e., $g = \dim_C H^1(O_C)$) with at most ordinary double points as singularities.

(ii) A pseudo-stable curve is called stable, if any non singular rational component $E$ of a geometric fibre $C_s$ meets the other components of $C_s$ in more than 2 points.

Example: The only singular stable curve over $C$ of genus 1 is a rational curve with one ordinary double point.

An important advantage in considering stable curves is the existence of moduli schemes. For the definition of fine and coarse moduli schemes see [13; p. 99].

Popov gave a definition of level $\mu$-structure for stable curves of genus $g \geq 2$ in [18; p. 235]. We need only two basic properties:
2.2: 
(i) Let $K$ be a field of characteristic zero and $C$ a geometrically irreducible, regular curve over $K$ of genus $g \geq 2$. Then there exists a finite algebraic extension $K'$ of $K$, such that $C \times_K K'$ allows a level $\mu$-structure.

(ii) Let $C$ be a stable curve over $C$. Then there exists only a finite number of level $\mu$-structures of $C$ over $C$.

2.3. Theorem (Popp [18]):
(i) The coarse moduli space $M_g$ of stable curves of genus $g \geq 2$ exists in the category of algebraic spaces of finite type over $C$.

(ii) The fine moduli space $M_g^{(\mu)}$ of stable curves of genus $g \geq 2$ with level $\mu$-structure ($\mu \geq 3$) and a universal stable curve $\rho_g^{(\mu)} : Z_g^{(\mu)} \to M_g^{(\mu)}$ with level $\mu$-structure exist in the category of algebraic spaces of finite type over $C$.

(iii) $M_g$ and $M_g^{(\mu)}$ are proper over $\text{Spec}(C)$.

Knudson and Mumford obtained a stronger result (see [14]) which is, however, still unpublished:

2.4. Theorem: The coarse moduli space $M_g$ is a projective scheme over $C$.

For simplicity we are going to use this result. It would be possible, however, to avoid it by working in the category of algebraic spaces in §2, §5 and §7.

2.5. Corollary: $Z_g^{(\mu)}$ and $M_g^{(\mu)}$ are projective schemes.

Proof: $M_g^{(\mu)}$ is quasi-finite over $M_g$ (2.2(ii)) and $\rho_g^{(\mu)}$ is a projective morphism [4].

In the case $g = 1$ Deligne and Rapaport gave a definition of level $\mu$-structure in [5]. It includes the condition that the stable curve is a generalized elliptic curve [5; p. 178]. For our purpose it is enough to know that the properties 2.2 also hold in this case and that we have the theorem [5]:

2.6. Theorem: The coarse moduli scheme $M_1$ of stable curves of genus 1 and the fine moduli scheme $M_1^{(\mu)}$ of stable elliptic curves with level $\mu$-structure ($\mu \geq 3$) exist as projective curves. $M_1^{(\mu)}$ is a finite Galois cover of $M_1$. The open part of $M_1^{(\mu)}$ corresponding to the regular curves is affine.
2.7. **Lemma:** Let $S$ be a normal, proper variety. For $i = 1, 2$ let $\rho_i : C_i \to S$ be stable curves of genus $g \geq 2$. For some open set $U \subset S$ let $\rho_i^U : C_i^U \to U$ be the restriction of $\rho_i$. Then any $U$-isomorphism $f_U : C_1^U \to C_2^U$ can be extended to an $S$-isomorphism $f : C_1 \to C_2$.

**Proof:** The functor $\text{Isoms}_S(C_1, C_2)$ is represented by a scheme $I_S(C_1, C_2)$ which is finite over $S$ [4; p. 84]. $f_U$ induces a morphism $U \to I_S(C_1, C_2)$ over $S$, which can be extended to $S$ (see [24; II.6.1.13]).

A pseudo-stable curve $\rho : C \to S$ has an invertible dualizing sheaf $\omega_{C/S}$ [6].

2.8. **Lemma** [4]:

(i) $\omega_{C/S}$ is compatible with base change.

(ii) $\rho_* \omega_{C/S}$ and $R^1\rho_* O_C$ are locally free of rank $g$ and dual to each other.

(iii) Assume that $S$ is regular and $C_0 \subseteq C$ the open subscheme on which $\rho$ is smooth. Let $\Omega_{C/S}$ be the sheaf of relative differentials. Then $\omega_{C/S}|_{C_0} \cong \Omega^1_{C/S}|_{C_0}$.

In the second half of this section we are going to describe a divisor $D$ with $\omega_{C/S}^{\otimes (g+1)/2} \cong O_C(D)$.

Henceforth let $S$ be a normal scheme and $\rho : C \to S$ a pseudo-stable curve with smooth general fibre. Let $\mathcal{L}$ be an invertible sheaf on $C$, such that $\rho_* \mathcal{L}$ is locally free of rank $r > 0$. Let $S_0$ be the regular locus of $S$ and $C_0 \subseteq C$ the open part, lying over $S_0$, on which $\rho$ is smooth. The restriction of $\rho$ to $C_0$ is denoted by $\rho_0$.

Since $\rho_0$ is smooth, every point $x \in C_0$ has a neighbourhood $U$ such that the restriction of $\rho_0$ to $U$ factors $U \to \text{Spec}(A[t']) \to \text{Spec}(A) \to S_0$ where $g$ is etale. Let $t$ be a parameter on $U$, lying over $t'$, then $\Omega^1_{C_0/S_0}$ is generated at $x$ by $dt$. Let $\eta_1, \ldots, \eta_r$ be sections of $\rho_* \mathcal{L}$ in a neighbourhood of $\rho(x)$ and $\eta$ a generator of $\mathcal{L}$ in a neighbourhood of $x$. Locally we can write $\eta_i = f_i \cdot \eta$ for $i = 1, \ldots, r$. Define

$$[\eta_1, \ldots, \eta_r] = \det \begin{vmatrix} f_1 & \cdots & f_r \\ \cdots & \cdots & \cdots \\ \frac{df_1}{dt} & \cdots & \frac{df_r}{dt} \\ \frac{dt}{dt} & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \frac{d^{(r-1)}f_1}{dt^{(r-1)}} & \cdots & \frac{d^{(r-1)}f_r}{dt^{(r-1)}} \end{vmatrix} \cdot dt^{r(r-1)/2} \cdot \eta^r.$$

$[\eta_1, \ldots, \eta_r]$ is independent of the chosen $t$ and $\eta$ and defines a section...
of $L' \otimes (\Omega^1_{C/S} \otimes (r-1)/2)$ over some neighbourhood of $\rho^{-1}(\rho(x))$. Now assume, that $\eta_1, \ldots, \eta_r$ and $\eta'_1, \ldots, \eta'_r$ are two bases of $\rho_*L$ near $\rho(x)$ such that

$$\eta'_i = \sum_{j=1}^r a_{ij} \cdot \eta_j \quad \text{for } i = 1, \ldots, r.$$ 

Then

$$[\eta'_1, \ldots, \eta'_r] = \det |a_{ij}| \cdot [\eta_1, \ldots, \eta_r].$$

Hence $[\eta_1, \ldots, \eta_r] \cdot (\eta_1 \wedge \ldots \wedge \eta_r)^{-1}$ is independent of the chosen basis. Since $C - C_0$ is at least of codimension 2 in $C$ we are able to define:

2.9. DEFINITION: Denote $\rho_*L$ by $W(L)$. The global section $s(L) = [\eta_1, \ldots, \eta_r] \cdot (\eta_1 \wedge \ldots \wedge \eta_r)^{-1}$ of $W(L)$ over $C$ is called the Weierstrass section of $L$.

$\rho_*\omega_{C/S}$ is locally free of rank $g$ and hence we have a Weierstrass section $s(\omega_{C/S})$. If, for example, $S = \text{Spec}(C)$, then the divisor of this section is just the usual divisor of Weierstrass points.

$\omega_{C/S}$ and $s(\omega_{C/S})$ are compatible with base change, as long as the assumptions of this section are fulfilled.

For simplicity we use the following notation: Let $D$ be a divisor and $L$ an invertible sheaf such that $L \equiv O_C(D)$. Then we write $L \sim D$.

The general fibres of $\rho : C \to S$ are regular. We define $W_{C/S}$ to be the closure of the divisor of the Weierstrass points of the general fibres of $\rho$.

2.10. THEOREM (Arakelov [2; p. 1299]): Let $\rho : C \to S$ be a pseudo-stable curve of genus $g \geq 1$ and $S$ a normal scheme. Let $U = \{s \in S; s$ regular point and $\rho^{-1}(s)$ regular curve$\}$. Let $d$ be a divisor on $S$ such that $\Lambda^{g} \rho_*\omega_{C/S} \sim d$. Then there exists a positive divisor $E_{C/S}$ with support in $\rho^{-1}(S - U)$ such that

$$\omega_{C/S}^{\otimes (g+1)/2} \sim \rho^*d + W_{C/S} + E_{C/S}.$$ 

PROOF: For a (local) base $\omega_1, \ldots, \omega_g$ of $\rho_*\omega_{C/S}$ the section $[\omega_1, \ldots, \omega_g]$ does not vanish identically on a smooth fibre of $\rho$. Therefore, the divisor of this section is of the form $W_{C/S} + E_{C/S}$.

2.11. REMARKS:

(i) In the case $g = 1$, there are no Weierstrass points and it is possible to show that $E_{C/S}$ is the zero-divisor if the curve is stable [5; p. 175]. Hence in this case 2.10 reduces to $\rho^*\rho_*\omega_{C/S} \equiv \omega_{C/S}$. 

(ii) If dim (S) = 1, the support of $W_{PcS}$ is finite over $S$. This is unfortunately no longer true for dim (S) ≥ 2 and g ≥ 3.

§3. The canonical divisor of families of curves over a curve

In §3 and §4 we make the following assumptions:

3.1: Let $W$ be a regular, proper curve of genus $p$, let $V$ be a regular, proper surface and $\pi : V \to W$ a surjective morphism whose general fibre is a regular curve of genus $g \geq 1$.

It follows, that $\pi$ is flat. A fibre $V_w = \pi^{-1}(w)$ is "degenerate", if it is not reduced or if it has singularities. Let $\Delta = \{w \in W; V_w \text{ degenerate fibre}\}$ is a finite set of points. Let $\omega_{V/w} = \omega_V \otimes \pi^* \omega_w^{-1}$ be the dualizing sheaf of $\pi$ ([6] or §6). We want to describe $\omega_V$. If $g : V^* \to V$ is the proper birational morphism obtained by blowing up a closed point of $V$ and $F$ the exceptional divisor, it is well known, that $\omega_{V^*} = g^* \omega_V \otimes O_{V^*}(F)$. Therefore we may assume that for all $w \in \Delta$ $(\pi^{-1}(w))_{\text{red}}$ has only ordinary double points as singularities.

3.2. Local description ([11] for $g = 1$ and [19] for $g \geq 2$): For $w \in \Delta$ let $\rho : \Gamma \cong V \times_w \text{Spec}(\hat{O}_{w,w}) \to S = \text{Spec}(\hat{O}_{w,w})$ be the induced local family of curves ("\" denotes the completion with respect to the maximal ideal).

There exists a cyclic covering $S'$ of $S$ with Galois-group $\langle \sigma \rangle$ such that the normalization $\Gamma'$ of $\Gamma \times_S S'$ is birationally equivalent to a stable curve $\rho'' : \Gamma'' \to S'$. The group $\langle \sigma \rangle$ operates on $\Gamma''$, compatible with the operation on $S'$. Let's call the tuple $(\rho'' : \Gamma'' \to S', \sigma)$ a stable reduction of $\pi : V \to W$ at $w$. Denote the closed point of $S$ also by $w$ and the closed point of $S'$ by $w'$.

Assume that every multiplicity occurring in $\Gamma_w \cong V_w$ divides $n = \text{ord}(\sigma)$. The special fibre $\Gamma'_w$ of $\Gamma'$ is reduced and has singularities of the (analytic) type $u \cdot v - t'$, $r \in \mathbb{N}$ (see [12]). Hence the minimal desingularisation $\Gamma_d$ of $\Gamma'$ also has a reduced special fibre and is pseudo-stable over $S'$. The natural maps are denoted by:

\[
\begin{array}{c}
\Gamma_d \xrightarrow{h} \Gamma' \xrightarrow{f} \Gamma \xrightarrow{i} V \\
\rho_d \downarrow \quad \rho' \quad \rho \downarrow \quad \pi \\
S' \xrightarrow{f_0} S \xrightarrow{i_0} W
\end{array}
\]
Let $E_d = E_{\Gamma_dS'}$ be the divisor defined in 2.10. Let $l_d$ be the number of double points of the special fibre $(\Gamma_d)_w$. Define $\delta'_w = \sum (e_{C'} - n)C'$ where the sum is taken over the set of irreducible components $C'$ of $\Gamma'_w$ and $e_{C'}$ is the ramification index of $C'$ over $\Gamma$.

3.3. DEFINITION: Using the notation of 3.2, we define on $\Gamma$ (resp. on $V$):

$$E_w = n^{-1}f_{\ast}h_{\ast}E_d \quad (E_w = n^{-1}i_{\ast}f_{\ast}h_{\ast}E_d)$$

$$l_w = n^{-1}l_d$$

$$\delta_w = n^{-1}f_{\ast}\delta'_w \quad (\delta_w = n^{-1}i_{\ast}f_{\ast}\delta'_w).$$

$E_w$ may have rational coefficients and $l_w$ may be a rational number. The definition is independent of the chosen $S'$.

3.4. LEMMA:

(i) $\delta_w = \sum (1 - \text{mult}(C)) \cdot C$ where the sum is taken over all irreducible components $C$ of $\Gamma_w \equiv V_w$ and $\text{mult}(C)$ is the multiplicity of $C$ in $V_w$.

(ii) $E_w$, $l_w$, and $\delta_w$ are independent of the chosen stable resolution.

PROOF: The description of $\delta_w$ in (i) follows from [19, Lemma 7.2]. To prove (ii) we may assume that $\Gamma$ is already pseudo-stable over $S$. In this case, however, $\Gamma'$ is the fibre product $\Gamma \times_S S'$ and has singularities of the type $u \cdot v - t^n$ lying over every double point of $\Gamma_w$. The second statement follows from the compatibility of Weierstrass sections with base change and [3; Theorem 2.7].

3.5. REMARK: A stable resolution $(\rho'' : \Gamma'' \rightarrow S', \sigma)$ of $\pi$ in $w$ is called minimal if $\text{ord}(\sigma) = \text{ord}(\sigma_w)$ where $\sigma_w$ is the restriction of $\sigma$ to $\Gamma'_w$. Such a minimal stable resolution always exists [19]. It is unique up to isomorphism and determines $\rho : \Gamma \rightarrow S$. Lemma 3.4 just says that $E_w$, $\delta_w$, and $l_w$ depend only on the minimal stable resolution.

3.6. THEOREM: We use the notations and assumptions made in 3.1 and 3.3. Let $W_p$ be the closure of the Weierstrass points of the general fibre of $\pi$. Then there exists a divisor $d$ on $W$ such that: (We denote the intersection numbers by $(\cdot)$ or $(\cdot)^2$)

(i) $\omega_{V_W}^{\otimes e(g+1)/2} \sim \pi^*d + W_p + \sum_{w \in \Delta} (E_w - \frac{1}{2}g(g + 1)\delta_w)$(up to torsion)

(ii) $(\omega_V)^2 = 8(p - 1)(g - 1) + (\omega_{V_W})^2$

(iii) $(\omega_{V_W})^2 = 12 \deg(d) - \sum_{w \in \Delta} (l_w + 2k(\delta_w) + (\delta_w)^2)$
= 12 \text{deg}(d) - \sum_{w \in d} [l_w + k(\delta_w) + 2 \cdot g^{-1} \cdot (g + 1)^{-1}(Wp \cdot \delta_w) + 2 \cdot g^{-1}(g + 1)^{-1}(E_w \cdot \delta_w)]

where \(k(D) = (\omega_V \cdot O_V(D))\).

(iv) Define \(\gamma = \chi(O_{Wp}) + (g^3 - g)(p - 1)\). Then we have

\[
[12 \cdot g^2(g + 1)^2 + 24 \cdot g(g + 1) - 8 \cdot (g^3 - 1)] \cdot \text{deg}(d) = -8 \cdot \gamma + \sum_{w \in d} [(g^2(g + 1)^2 + 2 \cdot g(g + 1) \cdot l_w + 4(Wp \cdot E_w) + (2 \cdot g^2 + 2 \cdot g + 4) \cdot k(E_w) + 4(Wp \cdot \delta_w) + (2 \cdot g^2 + 2 \cdot g + 4)(E_w \cdot \delta_w)]
\]

(v) \(\text{deg}(d) \geq 0\), and \(\text{deg}(d) = 0\) if and only if there exists a finite cover \(W'\) of \(W\) such that \(V \times_w W'\) is birationally equivalent to a trivial family of curves over \(W'\). In this case we may assume \(d = 0\).

3.7. COROLLARY: \(C_{2,1}'\) is true.

PROOF: If \(\text{deg}(d) > 0\), the divisor \(d\) is ample on \(W\) and hence \(C_{2,1}'\) follows from 3.6(i) and 1.3. If \(\text{deg}(d) = 0\) it follows from 3.6(v) and 1.3 that

\[K(\omega_{\nu\nu}, V) \geq K(O_V(Wp), V) \geq K(V_w).\]

3.8. REMARK: Using the Nakai-Moisezon criterion for ampltleness, 1.4(iii) and 3.6 we get more exact results:

\[K(V) = 2\] if \(g \geq 2\) and \(\text{deg}(d) > -(p - 1)(g + 1)g\),
\[K(V) \geq 1\] if \(g \geq 2\) and \(\text{deg}(d) = -(p - 1)(g + 1)g\) and
\[K(V) = 1\] if \(g = 1\) and \(\text{deg}(d) > -2 \cdot (p - 1)\).

PROOF OF THEOREM 3.6: Using the same kind of construction as in 3.2, we can find a Galois cover \(W'\) of \(W\) such that the desingularisation \(V_d\) of the normalisation \(V'\) of \(W' \times_w V\) is a pseudo-stable curve of genus \(g\). The natural morphisms are denoted by:

\[V_d \xrightarrow{h} V' \xrightarrow{l} V \quad \text{and} \quad W' \xrightarrow{h} W\]
Let \( n \) be the degree of \( W' \) over \( W \) and \( d' \sim h^g \pi_d \omega_{V/dW'} \). Define \( d = n^{-1}f_0d' \).

Statement 3.6(i) is true for pseudo-stable curves (2.10). We know [3; Theorem 2.7] that \( h_\pi \omega_{V/dW'} = \omega_{V/dW'} \) is an invertible sheaf. By definition of \( \delta_w \) and \( \omega_{V/W} \), we have:

\[
\omega_{V/W} = f^* \omega_{V/W} \boxtimes f^* O_V \left( \sum_{w \in \Delta} \delta_w \right)
\]

and 3.6(i) follows directly from the definition of the local terms. \( \omega_V = \omega_{V/W} \boxtimes \pi^* \omega_W \) and from 3.6(i) we get

\[
(\omega_{V/W} \cdot \pi^* \omega_W) = 2 \cdot g^{-1}(g + 1)^{-1}(O_V(W_p) \cdot \pi^* \omega_W) = 4(p - 1)(g - 1)
\]

and hence 3.6(ii).

3.9. Lemma: Assuming \( V \) is pseudo-stable, we have:

(i) \( \chi(O_V) = (p - 1)(g - 1) + \deg(d) \)

(ii) Let \( e(V) \) be the Euler number of \( V \). Then

\[
e(V) = 4(p - 1)(g - 1) + \sum_{w \in \Delta} l_w
\]

(iii) \( (\omega_{V/W})^2 = 12 \deg(d) - \sum_{w \in \Delta} l_w \).

Proof: The Leray spectral sequence \( H^q(W, R^p \pi^* O_V) \Rightarrow H^{q+p}(V, O_V) \) and \( \pi^* O_V = O_W \) gives us \( \chi(O_V) = (1 - p) - \chi(W, R^1 \pi^* O_V) \). Hence (i) follows from 2.8(ii) and the Riemann Roch formula for locally free sheaves on curves. Statement (ii) is proven in [9] and (iii) follows from Noether’s formula \( 12 \chi(O_V) = e(V) + (\omega_V)^2 \) and 3.6(i).

Back to the proof of the Theorem: In general, 3.6(iii) follows from 3.9(iii) and 3.8. To get 3.6(iv), one must simply compare the two equations for \( (\omega_{V/W})^2 \) you get from 3.6(i) and 3.6(iii). The term \( (W_p)^2 \) can be eliminated using the genus formula for curves on surfaces.

3.10. Lemma: Assume that \( V \) is pseudo-stable over \( W \) and that the Weierstrass points of the generic fibre of \( \pi \) are \( C(W) \)-rational. Then we have:

(a) \( W_p = \sum_{i=1}^r k_i D_i \), where \( D_i \) are prime divisors and the support of \( D_i \) is isomorphic under \( \pi \) to \( W \).

(b) \(-2\gamma = - \sum_{i=1}^r \frac{2(k_i^2 - k_i)}{g(g + 1) + 2k_i} \cdot \deg(d) + \sum_{i \neq j} \frac{k_i k_j (g^2 + g + 2)}{g(g + 1) + 2k_i} \cdot (D_i \cdot D_j)\)
(c) \(\deg (d) \geq 0\) and \(\deg (d) = 0\) if and only if \(\pi : V \rightarrow W\) is smooth and \((W_p)_{\text{red}}\) without singularities.

**Proof:** (a) is proven in [17; p. 1148].
(b) follows from 3.6(i) and the genus formula for curves on surfaces. To prove (c) substitute 3.10(b) in 3.6(iv) and check that the coefficients are positive for \(g \geq 1\).

**Proof of 3.6(v):** From the definition of \(d\) and 3.10, it follows, that \(\deg (d) \geq 0\). If \(\deg (d) = 0\) we know that \(\pi_d : V_d \rightarrow W'\) is smooth. From [17; Prop. 5] for \(g \geq 2\) or 2.6 for \(g = 1\), we find that we may assume, that \(V_d\) is trivial over \(W'\), and therefore \(d' = d = 0\).

§4. Calculation of the local terms

To calculate the square of the canonical divisor of \(V\) using 3.6, we have to know the local contributions. That means: let \(\pi : V \rightarrow W\) be as in 3.1 and let \(w \in \Delta\). The special fibre of \(\pi\) at \(w\) can be written \(V_w = \sum_{i=1}^r \nu_i C_i\), where \(\nu_i \in \mathbb{N} - \{0\}\) and \(C_i\) is a prime divisor of \(V\). We already know (3.4) that \(\delta_w = \sum_{i=1}^r (1 - \nu_i) C_i\). Hence we have to calculate \(E_w = \sum_{i=1}^r \mu_i C_i\), \((C_i \cdot W_p)\) and \(l_w\). The remaining term \(k(C_i)\) is determined by \(\chi(O_C)\) and the intersection theory of the fibre.

4.1: By assumption \((V_w)_{\text{red}}\) has only double points. Let \((\ , \ )\) denote the smallest common divisor of two natural numbers. Define \(\lambda (x) = (\nu_i, \nu_j)^2 \cdot \nu_i^{-1} \cdot \nu_j^{-1}\), if \(x \in C_i \cap C_j\) for \(i \neq j\), and \(\lambda (x) = 0\), if \(x\) is regular on \((V_w)_{\text{red}}\). Then it easily follows from [12; p. 7] and [19; §6] that \(l_w = \sum \lambda (x)\) where the sum is taken over all closed points of \(V_w\).

4.2. Remark: If we know for all \(j\), except \(j = i\), the multiplicity \(\mu_j\) of \(C_j\) in \(E_w\), we are able to express \(\mu_i\) in terms of \((C_i \cdot W_p)\) using 3.6(i) and the genus formula for curves on surfaces. Therefore the remaining problem is to calculate either the multiplicity in \(E_w\) or the intersection number for “enough” components. We may assume that \(p : \Gamma \rightarrow S\) is pseudostable and \(\Gamma_w = V_w\).

4.3. Twisting \(\omega_{FS}\): Let \(\Delta (\omega) = \sum_{i=1}^r n_i C_i\) be some positive divisor, \(\epsilon = \max \{n_i; i = 1, \ldots, r\}\) and define for \(j = 1, \ldots, \epsilon\) \(\Delta (j) = \sum_{i=1}^r \max (n_i - j, 0) \cdot C_i\), \(\mathcal{L} (j) = \omega_{FS} \otimes O (\Delta (j))\) and \(\mathcal{L}_w (j) = \mathcal{L} (j) \otimes O_{\Gamma_w}\).
Assumption (*):

(a) For \( j = 1, \ldots, \epsilon \) we have \( \dim_c H^0(\Gamma_w, \mathcal{L}_w(j)) = g \).

(b) For some component of \( \Gamma_w \) (let us say \( C_1 \)) the canonical map \( H^0(\Gamma_w, \mathcal{L}_w(0)) \to H^0(\Gamma_w, \mathcal{L}_w(0) \otimes_{\mathcal{O}_w} \mathcal{O}_{C_1}) \) is an isomorphism.

It follows for \( j = 1, \ldots, \epsilon \) that \( \rho_w \mathcal{L}(j) \) is locally free of rank \( g \).

Now let \( \eta_1^{(j)}, \ldots, \eta_g^{(j)} \) be a basis of \( \rho_w \mathcal{L}(j) \) and \( E(j) \) the part of the divisor of the section \( [\eta_1^{(j)}, \ldots, \eta_g^{(j)}] \) of \( \mathcal{W}(\mathcal{L}(j)) \) with support in the special fibre.

Let \( \varphi(j) : \mathcal{L}(j + 1) \to \mathcal{L}(j) \) be the canonical map and \( \varphi_w(j) : H^0(\Gamma_w, \mathcal{L}_w(j + 1)) \to H^0(\Gamma_w, \mathcal{L}_w(j)) \) the induced map. \( d(j) \) is defined to be the dimension of the image of \( \varphi_w(j) \).

4.4. Lemma: \( E(j) = E(j + 1) + g \cdot (\Delta(j) - \Delta(j + 1)) + (d(j) - g)\Gamma_w \)

Proof: Let \( E^* \) be the part of the divisor of the section \( \{ \varphi(j)(\eta_1^{(j+1)}), \ldots, \varphi(j)(\eta_g^{(j+1)}) \} \) of \( \mathcal{W}(\mathcal{L}(j)) \) with support in the fibre. Then \( E^* = E(j + 1) + g \cdot (\Delta(j) - \Delta(j + 1)) \). We may assume, that the first \( d(j) \) sections \( \eta_i^{(j+1)} \) generate the image of \( \varphi_w(j) \). The rest of the sections vanish on the special fibre with order 1 and the lemma follows.

4.5. Remarks:

(i) If the assumption (*) is fulfilled for some \( \Delta(0) \) and some \( C_1 \), we are able to calculate \( \mu_1 \) using 4.4. It is always possible to find such a divisor for every component, if the graph of \( \Gamma_w \) is simply connected and if the double points of \( \Gamma_w \) are in general position.

In this case \( E_w \) is uniquely determined by the isomorphism-class of \( \Gamma_w \).

This, however, is no longer true in general. A counterexample is given by a fibre \( \Gamma_w \) with three components and two double points, if one of the double points is a Weierstrass point of one component.

(ii) Under the assumptions and notations of 4.3, the isolated zeros of the Weierstrass section of \( \mathcal{L}_w(0) \) outside the singular points of \( V_w \) are (with multiplicity) intersection points of \( V_w \) and \( W_p \).

4.6. Examples:

(a) Assume \( V_w = C_1 \cup C_2 \cup \cdots \cup C_r \) where \( C_i \) is a regular rational curve for \( i = 2, \ldots, r \) and \( C_1 \) is a curve of genus \( g - 1 \). Assume that the double points are in general position on \( C_1 \).

For \( C_1 \) we can take \( \Delta(0) = 0 \). We get \( \mu_1 = 0 \) and there are \( g^3 - g^2 \) intersection points of \( C_1 \) and \( W_p \) outside the singular points of \( V_w \).
Hence $g^2 - g$ intersection points must lie on $C_2 \cup \cdots \cup C_r$. If $g = 2$, it is easy to show (using 4.3 and 4.4) that the remaining 2 intersection points lie on $C_{r+1}$, if 2 divides $r$.

(b) Assume $V_w = C_1 \cup \cdots \cup C_r \cup C_{r+1} \cup C_{r+2}$ where $C_1, \ldots, C_r$ are regular rational curves and $C_{r+i}$ is a regular curve of genus $g_i$, $i = 1, 2$ and assume that the double points are in general position.

Then $g = g_1 + g_2$. Take $\Delta(0) = \sum_{i=1}^{r} g_2 \cdot (r + 1) \cdot g_2 \cdot C_{r+2}$. From 4.4 we get $\mu_{r+1} = (r + 1) \cdot g_2$ and there are $g^2 \cdot g_1 - g_2$ intersection points of $C_{r+1}$ and $W_p$ outside the singular points of $V_w$. By symmetry and 4.2 we find:

$$E_w = (r + 1)g_2 \cdot C_{r+1} + (r + 1)g_1 \cdot C_{r+2} + \sum_{i=1}^{r} ((r + 1 - i)g_2 + ig_i) C_i.$$

4.7. The Case $g = 1$: ($\pi : V \to W$ as in 3.1) In this case we already know (2.11(i)) that $W_p = E_w = 0$ for stable curves, and hence 3.6 reduces.

Let $\eta_w$ be the maximal multiplicity occurring in $\pi^{-1}(W)$ and define $\delta_w = (1 - \eta_w) \cdot \eta_w^1 \cdot \pi^{-1}(W)$. Then:

$$\omega_{V/W} \sim \pi^*d - \sum_{w \in \Delta} \delta_w$$

and $\deg (d) = 12^{-1} \cdot \sum_{w \in \Delta} (l_w + (E_w \cdot \delta_w))$

4.8. The Case $g = 2$: Using the methods from 4.6, one is able to find $E_w$ for every stable curve of genus 2. It is important, that the components of a singular stable curve of genus 2 are at most of geometric genus 1 and hence “all points are in general position”. Let's make the following definition:
Let \((\rho'' : \Gamma'' \to S', \sigma)\) be a minimal stable resolution (3.5) of \(\pi\) in \(w\). Then let \(n_w = (\text{ord}(\sigma))^{-1} \cdot \# \{x \in \Gamma''_w; x \text{ singular and } \Gamma''_w - \{x\} \text{ connected}\}\) and \(m_w = (\text{ord}(\sigma))^{-1} \cdot \# \{x \in \Gamma''_w; x \text{ singular and } \Gamma''_w - \{x\} \text{ not connected}\}\).

Using that definition one gets \((E_w \cdot W_p) = 6m_w + n_w\) and \(k(E_w) = 2m_w\) if \(V_w\) is pseudo-stable. The fact that \(W_p\) has no singularities outside the degenerate fibres of \(\pi\) (every fibre is hyperelliptic) yields:

4.9. **Theorem** (Ueno; see [16]): Let \(\pi : V \to W\) be as in 3.1, \(g = 2\). Then we have \((\omega_V)^2 = 8(p - 1) + \sum_{w \in A} \eta_w\) where \(\eta_w\) depends only on the local invariants [19] of \(\pi\) in \(w\) and

\[
\eta_w = \frac{12}{5} \cdot m_w + \frac{9}{5} \cdot n_w - l_w - 2k(\delta_w) - (\delta_w)^2.
\]

4.10. **Example**: One possible fibre is of the form [19; Ex. 8.4]:

where all components are rational. Using 3.4(i) one gets \(k(\delta_w) = 0\) and \((\delta_w)^2 = -4\). The description in 4.1 yields \(l_w = \nu + 2\). The fibre of the minimal resolution (see [19; 8.4]) is simply connected and we have \(n_w = 0\) and \(m_w = \nu - \frac{7}{12}\). The local contribution in this example is therefore \(\eta_w = \frac{7}{3} \cdot \nu + \frac{1}{3}\).

4.11. **Remark**: The fact that for \(g > 2\) the local contributions \(E_w\) are not completely determined by the local invariants, defined in [19], of the degenerate fibres can be explained in the following way: Let \(M_g^*\) be a fine moduli scheme of stable curves of genus \(g\) with some suitable additional structure (for example: The Hilbert scheme of three canonical embedded stable curves [4] or \(M^{(\alpha)}_g\)) (see 2.3)) and \(\rho : Z_g^* \to M_g^*\) the corresponding universal curve. Let \(M_{g_o}^*\) be the open subscheme of \(M_g^*\) corresponding to regular curves. Then \(M_g^* - M_{g_o}^*\) is the union of irreducible closed subschemes \(S_0, \ldots, S_r\) of codimension 1 \((r = \frac{1}{2}g\) or \(r = \frac{1}{3}(g - 1)\)) with the following property [4]: \(S_0 \times_{M_g^*} Z_g^*\) is a family of irreducible curves with one double point. \(S_i \times_{M_g^*} Z_g^*\) has two components \(Z_i^{(1)}\) and \(Z_i^{(2)}\) which are families of curves of genus \(i\) and \(g - i\).
The arguments of 4.4 and 4.6 give the description of \( \omega = \omega_{Z_*/M^*} \):

\[
\omega^{\otimes \left( g+1 \right)} \otimes \left( \rho^* \wedge \rho_* \omega \right)^{-1} \sim W_{Z_*/M^*} + \sum_{i=1}^{r} ((g-i) \cdot Z_i^{(1)} + i \cdot Z_i^{(2)}).
\]

Let \( \pi : \Gamma \to S \) be a pseudo-stable curve over a local scheme. The corresponding divisor \( E_{T/S} + W_{T/S} \) is then the pullback of the right hand side, but \( W_{T/S} \) is not the pullback of \( W_{Z_*/M^*} \). One of the reasons is that for \( g > 2 \) the support of \( W_{Z_*/M^*} \) is not finite over \( M_g^* \).

4.12. ADDENDUM: The situation is much better if we restrict ourselves to families of curves with hyperelliptic general fibre. In this case, the number \( \gamma \) occurring in 3.6(iv) is zero and hence \( \deg(d) \) depends only on the local behaviour of the family near the degenerate fibres.

Now let \( H_{g_0}^{(\mu)} \) be the subscheme of \( M'_g^{(\mu)} \) for some \( g > 2 \) corresponding to hyperelliptic regular curves, \( H = H_{g_0}^{(\mu)} \) the closure in \( M'_g^{(\mu)} \) and \( \rho : C \to H \) the corresponding family of curves. If \( \pi : V \to W \) is a family of stable curves with hyperelliptic general fibre and level \( \mu \)-structure let \( \varphi : W \to H \) be the induced morphism. \( \deg(d) \) is nothing but the intersection number of \( \varphi(W) \) and \( \wedge \rho_* \omega_{C/H} \). In this case 3.6(iv) means that \( \wedge \rho_* \omega_{C/H} \) is numerically equivalent to a divisor with support in \( H - H_{g_0}^{(\mu)} \). Therefore Theorem 4.9 can be generalised:

4.13. THEOREM: Let \( \pi : V \to W \) be as in 3.1, \( g \geq 2 \), and assume that the general fibre is a hyperelliptic curve. Then we have: \( (\omega_{V})^2 = 8(p-1)(g-1) + \sum_{w \in \Delta} \eta_w \) where \( \eta_w \) depends only on the local invariants [19] of \( \pi \) in \( w \).

The numbers \( \eta_w \) can be calculated using 3.6, 4.1, 4.6(a) (the remaining 2 intersection points of multiplicity \( \frac{1}{2}(g^2 - g) \) lie on \( C_{1r+1} \) if 2 divides \( r \)) and 4.6(b).

§5. “Stable reduction” for higher dimensional base-schemes

In this section we are going to prove the following theorem:

5.1. THEOREM: Let \( \pi_1 : V_1 \to W_1 \) be a surjective morphism of proper, regular varieties such that the general fibre of \( \pi_1 \) is a connected curve of genus \( g \geq 1 \). Then there exists the following commutative diagram of morphisms of proper varieties:
having the following properties:

(i) \( \pi : V \to W \) is a surjective morphism of regular varieties with connected general fibre and is birationally equivalent (1.5) to \( \pi_1 : V_1 \to W_1 \).

(ii) \( h : V' \to V \) and \( g : W' \to W \) are flat covers and \( V' \) is birational equivalent to \( W' \times_W V \). The only singularities of \( W' \) and \( V' \) are quotient singularities [20].

(iii) \( \pi_s : V_s \to W' \) is a stable curve of genus \( g \) with level \( \mu \)-structure, \( \mu \geq 3 \), and \( f : V' \to V_s \) is a birational morphism.

(iv) Every morphism and every scheme occurring in the diagram is projective.

**Proof:** The smooth fibres of \( \pi_1 \) induce a rational map \( \varphi_1 : W_1 \to M_g \) (see §2). \( M_g \) is proper and hence after eliminating the points of indeterminacy [8] and replacing the pullback of \( V_1 \) by a regular model [8], we may assume that \( \varphi_1 \) is a morphism. Using Chow’s lemma we may also assume that \( \pi_1, W_1 \) and \( V_1 \) are projective.

For \( \mu \geq 3 \) we are able (2.2(i)) to find a finite Galois cover \( g_1 : W_1 \to W_1 \) such that the generic fibre of \( V_1 \times_{W_1} W'_1 \) over \( W'_1 \) has a level \( \mu \)-structure. Let \( \Delta(W'_1/W_1) \) be the ramification locus of \( W'_1 \) in \( W_1 \). By “purity of the branch locus” \( \Delta(W'_1/W_1) \) is of codimension one. Using “embedded resolution of singularities” [8] we find a sequence of monoidal transformations \( \eta : W \to W_1 \) such that \( \eta^{-1}(\Delta(W'_1/W_1)) \) has regular components and at most normal crossings as singularities. Let \( W' \) be the normalization of \( W \times_{W_1} W'_1 \) and \( g : W' \to W \) the induced morphism. Since \( \Delta(W'/W) \subseteq \eta^{-1}(\Delta(W'_1/W_1)) \), it follows from [20; Lemma 2] that \( g \) is flat and \( W' \) has at most quotient singularities.

Let \( M_g^{(\mu)} \) be the fine moduli scheme of stable curves with level \( \mu \)-structure (2.4). Then the generic fibre of \( W' \times_{W_1} V_1 \) induces a rational map \( \varphi' : W' \to M_g^{(\mu)} \) which is compatible with the morphism \( \varphi = \varphi_1 \cdot \eta : W \to M_g \). Since \( M_g^{(\mu)} \) is finite over \( M_g \) it follows that \( \varphi' \) is a morphism [24; II, 6.1.13] and induces a stable curve \( \pi_s : V_s \to W' \).

The morphism \( \pi_s \) is projective [414] and hence \( V_s \) is projective. Let \( G \) be the Galois group of \( W' \) over \( W \). By definition of \( V_s \) we have an operation of \( G \) on the generic fibre of \( \pi_s \).
5.2. **Lemma**: The operation of $G$ on the generic fibre of $\pi_s$ extends to an operation of $G$ on $V_s'$, compatible with the operation of $G$ on $W'$.

Since $V_s$ is projective, the quotient exists. The universal property of quotients gives us a morphism $V_s/G \to W'/G \cong W$. If we look only on the generic fibres, we have just made an extension of the base field and then divided by the Galois group of this extension. Therefore $V_s/G$ is birationally equivalent to $V_s$. Using resolution of singularities, Chow’s lemma and embedded resolution of singularities again, we find a regular projective variety $V$ and a projective birational morphism $V \to V_s/G$, such that: Let $V'$ be the normalization of $V \times_W W'$ and $h : V' \to V$ the induced morphism, then $\Delta(V'/V)$ has regular components and only normal crossings as singularities. Hence $V'$ is also the normalization of $V_s \times_{V_s/G} V$.

**Proof of Lemma 5.2**: Every $\sigma \in G$ induces an isomorphism $\sigma$ of $W'$ and there exists a $\sigma$ invariant open subscheme $U \subseteq W'$ such that

\[
\pi_s^{-1}(U) \xrightarrow{\sigma'} \pi_s^{-1}(U) \xrightarrow{\sigma \cdot \pi_s} U
\]

is commutative. We denote by $\sigma'$ the isomorphism induced by $\sigma$ on $\pi_s^{-1}(U)$. For $g \geq 2$ it follows from 2.7 that $\sigma'$ can be extended to an isomorphism of $V_s$. For $g = 1$ we know that $M_1^{(\mu)}$ is a curve and it follows directly that there is an isomorphism $\sigma''$ of $M_1^{(\mu)}$ such that

\[
W' \xrightarrow{\sigma''} M_1^{(\mu)}
\]

is commutative. Therefore in this case the Lemma follows directly from the definition of $V_s$. 

5.3. **Corollary:** Under the assumptions and with the notations of 5.1

(i) $V'$, $W'$ and $V_s$ have rational singularities [20] and are Gorenstein schemes [6; V §9].

(ii) $R^i f_* O_{V'} = 0$ for $i \neq 0$ and $f_* O_{V'} \cong O_{V_s}$.

**Proof:** (ii) follows from (i) and [20; Lemma 1]. We know from [20; Prop. 1] that $V'$ and $W'$ have rational singularities. To prove that $V_s$ has rational singularities we may assume (using “flat base change”) that $W'$ is regular. Then from [4] or the deformation theory of ordinary double points of curves, it follows that the completion of a singular local ring of $V_s$ has the form

$$\mathbb{C}[[t_1, \ldots, t_r, u, v]]/(u \cdot v - g(t_1, \ldots, t_r)), g(t_1, \ldots, t_r) \in \mathbb{C}[[t_1, \ldots, t_r]]$$

and the rationality follows from [20; Prop. 2]. $V_s$ is locally a complete intersection over $W'$ and hence it remains to show that $V'$ and $W'$ are Gorenstein schemes. Both are flat, finite covers of regular schemes. The question is local and hence it is enough to consider the following situation: Let $A$ be a local Gorenstein ring, $B$ a local ring and a free, finite $A$-module [1; p. 60]. There exists $n_0$ such that $\text{Ext}^i_A(A/\mathfrak{m}_A, A) = 0$ for $i \geq n_0$ [10; p. 163]. Hence $\text{Ext}^i_A(A/\mathfrak{m}_A, B) = 0$ and from [7; p. 164] we get $\text{Ext}^i_B(B \otimes A/\mathfrak{m}_A, B) = 0$ for $i \geq n_0$. Since $B$ is free over $A$ we get $\text{Ext}^i_B(B/\mathfrak{m}_B, B) = 0$ for $i \geq n_0$ and $B$ is a Gorenstein ring.

§6. **Duality theory**

In order to compare the Kodaira dimensions of the varieties occurring in 5.1 we need some results of Grothendieck duality theory. Let $f : X \to Y$ be a projective embeddable morphism of noetherian schemes of finite Krull dimension. This is the case if $Y$ is a projective variety and if $f$ is a projective morphism [6; p. 206].

Let $D_{qc}(X)$ be the derived category of quasi-coherent sheaves on $X$ [6; p. 85] and $D^+_{qc}(X)$ (resp. $D^-_{qc}(X)$) the full subcategory of complexes bounded below (resp. above). Then there exists a functor [6; p. 190] $f^! : D^+_{qc}(Y) \to D^+_{qc}(X)$ with the following properties:

6.1: (We denote the derived functors $Rf_*$, $Lf^*$ and $\otimes$)

(i) For every composition $X \to Y \to Z$ of projective embeddable morphisms there is an isomorphism of functors $(g \cdot f)^! \cong f^! g^!$.
(ii) For every flat base extension $u : Y' \to Y$ there is an isomorphism $v^*f^! = g^!u^*$ where $v$ and $g$ are the two projections of $X \times_Y Y'$.

(iii) Let $D_{qc}^+(Y)_{\text{td}}$ be the subcategory of $D_{qc}(Y)$ generated by the bounded complexes of finite Tor-dimension [6; p. 97]. There is a functorial isomorphism $f'(F) \otimes Lf^*(G) \cong f'(F \otimes G)$ for $F \in D_{qc}^+(Y)$ and $G \in D_{qc}^b(Y)_{\text{td}}$ [6; p. 194].

(iv) Let $F \in D_{qc}^+(X)$ and $G \in D_{qc}^+(Y)$. There exists a duality isomorphism [6; p. 210]:

$$\Theta_f^i: \text{Ext}^i_c(F', f^!G) \to \text{Ext}^i_c(Rf_*F^-, G^-)$$

(v) Assume that $f$ is a flat morphism of Gorenstein schemes of relative dimension $m$. Then there exists an invertible sheaf $\omega_{X/Y}$ such that $f'(O_Y)$ is isomorphic in $D_{qc}^+(X)$ to the complex $\omega_{X/Y}[m]$ (see 6.2(i)) [6; p. 298 and 388].

(vi) Under the assumptions of (v), $\omega_{X/Y}$ is compatible with arbitrary base change [6; p. 388].

6.2. REMARK:

(i) Let $[m] : D_{qc}(Y) \to D_{qc}(Y)$ be the functor defined by $(G[m])^! = G^{\sim +m}$, then $[m]$ is compatible with derived functors. Let $G$ be an invertible sheaf on $Y$, considered as the trivial complex having $G$ at the $0^\text{th}$ place. Then $Lf^*(G[m]) \cong (Lf^*G)[m] \cong (f^*G)[m]$ and for $F \in D_{qc}^+(Y)$ we have $F^* \otimes G[m] \cong (F^* \otimes G)[m] \cong (F' \otimes G)[m]$ where $(F' \otimes G)' = F' \otimes G$. Under these conditions the isomorphism in 6.1(iii) reduces to $(f'(F') \otimes f^*(G))[m] \cong f'(F' \otimes G)[m]$.

(ii) If $Y = \text{Spec}(R)$ and $X$ is a regular and projective variety, then $\omega_{X/Y}$, defined in 6.1(v), is the usual canonical sheaf on $X$.

We extend the definition of $\omega_{X/Y}$ given in 6.1(v).

6.3. DEFINITION: Let $f : X \to Y$ be a surjective and projective embeddable morphism of irreducible noetherian schemes whose general fibre is of dimension $m$. If $f'(O_Y) \equiv G[m]$ in $D_{qc}^+(X)$ for an invertible sheaf $G$, we say that the dualizing sheaf of $f$ exists and denote $\omega_{X/Y} = G$.

Statement $C_{nm}$ just says, that the Kodaira dimension of the general fibre is smaller than the $L$-dimension of the dualizing sheaf.

6.4. LEMMA: Let $h : X \to S$ and $g : Y \to S$ be surjective and projective embeddable morphisms of irreducible noetherian schemes of finite Krull dimension and let $l$ (resp. $m$) be the dimension of a general fibre
of $h$ (resp. $g$). Assume that the dualizing sheaves $\omega_{X/S}$ and $\omega_{Y/S}$ exist. Let $f : X \to Y$ be a surjective and projective embeddable morphism over $S$. Then $\omega_{X/Y}$ exists and is isomorphic to $\omega_{X/S} \otimes f^*\omega_{Y/S}^{-1}$.

**Proof:** We have (6.1(i)) $h^! = f^! g^!$ and from 6.2(i) we get: $h^!(O_S) \equiv f^!(g^!(O_S)) \equiv f^!(O_Y \otimes_{O_Y} g^!(O_S)) \equiv f^!(O_Y \otimes_{O_Y} f^*g^!(O_S))$ or $\omega_{X/S}[l] \equiv f^!(O_Y) \otimes_{O_Y} f^*(\omega_{Y/S}[m])$. Hence $f^!(O_Y) \equiv (\omega_{X/S} \otimes f^*\omega_{Y/S}^{-1})[l-m]$.

6.5. **Corollary:** Using the notations from 6.4, we assume that $l = m$, $f_* (O_X) \equiv O_Y$ and $R^1 f_* (O_X) = 0$ for $i \neq 0$. Then there exists an injection $f^*\omega_{Y/S} \to \omega_{X/S}$.

**Proof:** The duality isomorphism (6.1(iv)) for $i = 0$ gives you: $\text{Hom}_X (O_X, f^! O_Y) \equiv \text{Hom}_{D^b_c(Y)} (Rf_* O_X, O_Y) \equiv \text{Hom}_Y (O_Y, O_Y)$. Therefore there is a non trivial morphism $O_X \to f^! O_Y \equiv \omega_{X/S} \otimes f^*\omega_{Y/S}^{-1}$.

6.6. **Corollary:** Using the notations from 6.4, we assume $l = m$ and $f$ finite and birational. Then there exists an injection

$$\omega_{X/Y} \to O_X.$$ 

**Proof:** We know that $R^i f_* G = 0$ for $i \neq 0$ and any invertible sheaf $G$ on $X$. The duality isomorphism for $i = 0$ gives $\text{Hom}_X (\omega_{X/Y}, \omega_{X/Y}) \equiv \text{Hom}_Y (f_* \omega_{X/Y}, O_Y)$. Therefore $0 \neq \text{Hom}_Y (f_* \omega_{X/Y}, f_* O_X) \equiv \text{Hom}_X (f^* f_* \omega_{X/Y}, O_X)$. Since $f$ is affine and birational there is a sheaf $\epsilon$ with support in codimension 1 such that $0 \to \epsilon \to f^* f_* \omega_{X/Y} \to \omega_{X/Y} \to 0$ is exact. We have $\text{Hom}_X (\epsilon, O_X) = 0$ and hence $0 \neq \text{Hom}_X (f^* f_* \omega_{X/Y}, O_X) \equiv \text{Hom}_X (\omega_{X/Y}, O_X)$.

We want to apply these results to the situation described in §5. Remember, in the conclusion of 5.1 we got a diagram of projective morphisms of projective Gorenstein schemes (5.3):

$$
\begin{array}{ccc}
V & \to & V' \\
\downarrow h & & \downarrow f \\
W & \to & V_s \\
\downarrow \pi & & \downarrow \pi' \\
& \leftrightarrow & \\
W' & & \\
\end{array}
$$

From 6.1(v) and 6.4 we know that for every morphism of this diagram the dualizing sheaf exists. The following proposition reduces the proof of $C_{n,n-1}'$ to the case of stable curves.
6.7. **PROPOSITION:** Using above notations we have:

\[ K(\omega_{V'/W'}, V_s) \leq K(\omega_{V'/W'}, V') \leq K(\omega_{V/W}, V). \]

**PROOF:** Using 5.3(ii) it follows that the assumptions of 6.5 are fulfilled for

\[ V' \xrightarrow{f} V_s \xrightarrow{\pi_s} W' \]

and hence the first inequality follows from 1.3(ii) and (iii). The left hand side of the diagram can be written as:

\[ V \xrightarrow{h_1} V_1 = V \times_W W' \xleftarrow{h_2} V' \]

Where \( h = h_1 \cdot h_2 \). Since \( g \) is a flat morphism of Gorenstein schemes (5.3) we know from 6.1(vi) that \( \omega_{V'/V} \cong p^* \omega_{W'/W} \). The assumptions of 6.6 are fulfilled by \( h_2 \) and we get: \( \omega_{V'/V} \cong \pi'^* \omega_{W'/W} \cong \omega_{V'/V_1} \to O_{V'} \) and therefore \( \omega_{V'/W} \cong \omega_{V'/V} \otimes \pi'^* \omega_{W'/W} \otimes h^* \omega_{V/W} \to h^* \omega_{V/W} \). Now the second inequality holds by 1.3(ii) and (iii).

§7. The dualizing sheaf for stable curves

For \( g \geq 1 \) and \( \mu \geq 3 \) let \( M = M_g^{(\mu)} \) be the fine moduli scheme of stable curves with level \( \mu \)-structure and \( \rho: Z \to M \) the corresponding universal curve. Let \( M_0 \subseteq M \) be the open subscheme corresponding to the regular curves, \( \omega = \omega_{Z/M} \) and \( \mathcal{D} = \wedge^g \rho_* \omega \).

7.1. **PROPOSITION:** \( K(\mathcal{D}, M) = \dim(M) \) and for some \( m > 0 \) we have (see 1.1) \( \Phi_{m,\mathcal{D}}|_{M_0} \) is a finite morphism of \( M_0 \) on a subscheme of \( \mathbb{P}^N \).

**PROOF OF 7.1 FOR \( g = 1 \):** We know (2.6) that \( M \) is a curve and that
\(\rho : Z \to M\) is not smooth. From 3.6 we get \(\deg D > 0\) and hence \(D\) is ample on \(M\).

7.2. REMARK: As far as the author knows, proposition 7.1 for \(g \geq 2\) will follow from the announced proof of the projectivity of \(M_g\) by Knudsen and Mumford (see [14]). In \(\S 3\) we have already seen, that \(D\) behaves like an ample sheaf: Every curve in \(M\) whose general point is contained in \(M_0\) has a positive intersection number with \(D\). It seems reasonable that \(D\) is a good candidate (after some correction along the boundary of \(M_0\)), if you are looking for an ample sheaf on \(M\).

Nevertheless, we give the outline of a proof of the weaker statement of 7.1, using analytic methods which can be found in [2] and [15]. Namikawa constructed in [15] a holomorphic morphism from \(M_g\) into a projective space, which is an injection on \(M_{g0}\). We just have to show that the induced morphism of \(M\) is given by global sections of \(D^{\otimes m}\) for some \(m\). This follows, however, from the methods used by Arakelov to prove [2; Theorem 1.1]:

**Proof of 7.1 for \(g \geq 2\):** Henceforth we will use the complex topology of \(M\) and \(Z\). For sufficiently small \(U \subseteq M_0\) we can find cycles \(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_g\) and \(\tilde{\beta}_1, \ldots, \tilde{\beta}_g\) in \(R^1\rho_* Z\) such that for every \(t \in U\) the induced cycles (via duality) \(\alpha_{it}, \ldots, \alpha_{gt}, \beta_{it}, \ldots, \beta_{gt}\) in \(H_1(Z_t, Z)\) have the property:

\[
(\alpha_i \cdot \alpha_j) = (\beta_i \cdot \beta_j) = 0 \text{ for } 1 \leq i, j \leq g \text{ and } (\alpha_{it}, \beta_{it}) = \delta_{ij}.
\]

We may assume that \(\rho_* \omega\) is free over \(U\). Let \(\omega_1, \ldots, \omega_g\) be a basis of \(\rho_* \omega\) on \(U\) and define:

\[
\Omega_1(t, \omega_1, \ldots, \omega_g) = \left| \int_{\beta_{it}} (\omega_j)_{1 \leq k, t \leq g} \right| \quad \text{and}
\]

\[
\Omega_2(t, \omega_1, \ldots, \omega_g) = \left| \int_{\alpha_{it}} (\omega_j)_{1 \leq i, j \leq g} \right|.
\]

Both are holomorphic in \(t\), and by replacing \(\omega_1, \ldots, \omega_g\) by another basis of \(\rho_* \omega\) we may assume that \(\Omega_2(t, \omega_1, \ldots, \omega_g)\) is the unit matrix for all \(t \in U\). Then \(\Omega_1(t) = \Omega_2(t, \omega_1, \ldots, \omega_g)\) is called the period matrix of the fibre \(Z_t\).

For \(m > 0\) let \(s_1, \ldots, s_r\) be a basis of the vector space of Siegel modular forms of weight \(m\) and define \(\eta = (\omega_1 \wedge \cdots \wedge \omega_g)^{\otimes m}\). Then \(s_i(\Omega_1(t)) \cdot \eta\) defines a section of \(D^{\otimes m}\) on the open set \(U\).

If we choose a second system of cycles \(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_g, \tilde{\beta}_1, \ldots, \tilde{\beta}_g\) and a basis of \(\rho_* \omega\), normalized as above, it follows easily that the section
s_i(\Omega_i(t)) \cdot \eta remains unchanged. Hence we get global sections \gamma_1, \ldots, \gamma_r of \mathcal{D} on M_0 which don't vanish simultaneously. The corresponding holomorphic morphism \( M_0 \to \mathbb{P}^{r-1} \) is the same as constructed in [15]. Hence it factors through the coarse moduli scheme \( M_{g_0} \), and the induced morphism \( M_{g_0} \to \mathbb{P}^{r-1} \) is an embedding.

It remains to show that the sections \( \gamma_1, \ldots, \gamma_r \) extend to sections of \( \mathcal{D}^{\otimes m} \) over \( M \) and are not simultaneously zero on the boundary of \( M_0 \). Let \( U' \) be a small neighbourhood of a point \( y \in M - M_0 \) in \( M \). Since the sections \( \gamma_i \) are locally obtained from sections of the corresponding sheaf on the Hilbert scheme of three canonically embedded stable curves [15] we may assume that \( U' \) is regular. Using Hartog's theorem it is enough to show that the \( \gamma_i \) extend to holomorphic sections of \( \mathcal{D}^{\otimes m} \) along a general line through \( y \). This, however, is proven in [2; proof of 1.1]: Arakelov applies the same construction to abelian varieties over a curve, but [2; Lemma 1.4] gives the connection to the case we consider.

§8. The proof of \( C'_{n,n-1} \) and 1.8

Let \( \pi_1 : V_1 \to W_1 \) be a surjective morphism of proper, regular varieties, \( n = \dim (V_1) \) and \( n - 1 = \dim (W_1) \), such that the general fibre of \( \pi_1 \) is a connected curve of genus \( g \geq 1 \).

Choose a diagram of morphisms of proper schemes as in 5.1. Let \( \varphi : W \to M_g \) be the rational map induced by the general fibres of \( \pi \) (or \( \pi_1 \)). Let \( \varphi_1 : W' \to M_{g_1}^{\mu} \) be the morphism corresponding to \( \pi_1 : V_1 \to W' \). Then \( \dim (\varphi(W)) = \dim (\varphi_1(W')) \). We have to prove (1.8):

\[
(8.1) \quad K(\omega_{V/W}, V) \geq \max (K(\omega_{W}), \dim (\varphi_1(W'))) .
\]

8.2. LEMMA: (8.1) follows from \( K(\omega_{V'/W'}, V_1) \geq \dim (\varphi_1(W')) \).

PROOF: The above inequality and 6.7 give us \( K(\omega_{V'/W'}, V) \geq \dim (\varphi_1(W')) \) and hence we have only to consider the case \( g \geq 2 \) and \( \dim (\varphi_1(W')) = 0 \). This, however, means that there is a regular curve \( C \) of genus \( g \) over \( \overline{C} \) such that \( V_1 \cong C \times_{\text{Spec}(\overline{C})} W' \) and 1.3(ii), 1.5 and 6.7 give us the inequality we need.

Now \( \varphi_1 \ast \omega_{V'/W'} \) and hence \( \wedge^2 \varphi_1 \ast \omega_{V'/W'} \) are compatible with base change. Therefore (using the notation of 7.1) \( \wedge^2 \varphi_1 \ast \omega_{V'/W'} = \varphi_1^* \mathcal{D} \). Since \( \varphi_1 \) maps an open subscheme of \( W' \) into \( M_0 \), 7.1 yields \( K(\wedge^2 \varphi_1 \ast \omega_{V'/W'}, W') \geq \dim (\varphi_1(W')) \). In 2.10 we proved that there is an
injection
\[ \pi^* \otimes \pi_* \omega_{V_j/W} \cong \omega_{V_j/W}^{(g+1)/2}, \]
and 1.3(iii) and (ii) prove \( K(\omega_{V_j/W}, V_s) \geq \dim (\varphi_s(W')). \)

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