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SERGIO ALBEVERIO
RAPHAEL HØEGH-KROHN

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THE ENERGY REPRESENTATION OF SOBOLEV-LIE GROUPS*

Sergio Albeverio and Raphael Høegh-Krohn

Abstract

A unitary representation of the Sobolev-Lie group of C_1-mappings from an orientable Riemann manifold M to a Lie group G with compact Lie algebra is constructed. The representation is given in terms of the energy function on $C_1(M, G)$ and provides a new type of representations of current algebras. The representation space can be realized in the L_2-space of a random field, which in the case where M is a closed interval of the real line reduces to the left Brownian motion on the Lie group G.

1. Introduction

In this paper we construct a unitary representation of the Sobolev-Lie group of C_1-mappings from an orientable Riemann manifold M to a Lie group G with compact Lie algebra. The representation is given in terms of the energy function on $C_1(M, G)$. Other types of representations of groups of mappings between manifolds have been considered before in several connections, e.g. for the representation of the "current groups" and "current algebras" of interest in quantum physics, see e.g. [1]–[5]. The best studied representations are "ultralocal" in the sense of unitary representations of G attached to each point of the original space M. These representations, factorizable in Araki's sense [7], are given by factorizable positive definite functions on the group G^M, hence are connected with the infinitely divisible laws of probability. For these aspects see particularly ([1], [6]–[11]. A general construction of this type has been

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developed from a different point of view by Vershik, Gelfand and Graev in a series of papers [12]–[4]. In this paper we shall construct a different class of unitary representations of the group $C_1(M, G)$, for the case where M is an arbitrary orientable Riemann manifold and G is a connected Lie group with compact Lie algebra. We call these representations "energy representations" in sofar as they are given in terms of the energy function T on $C_1(M, G)$

$$T(\phi) = \frac{1}{2}\int_M |d\phi(x)|^2 dx.$$

They provide a class of new representations of current algebras which could be called "local representations" to distinguish them from the "ultralocal representations" mentioned before.

Let us shortly summarize the content of the different sections.

In Section 2 we define the Sobolev-Lie group $H_1(M, G)$ of mappings from an orientable Riemann manifold M into a connected Lie group G with compact Lie algebra g. By construction $H_1(M, G)$ is the completion of $C_1^0(M, G)$ in a certain Sobolev metric d, where $C_1^0(M, G)$ is the set of mappings in $C_1(M, G)$ which are equal to the identity in G outside some compact in M. The Sobolev metric is defined by $d(\phi, \psi) = 2T(\phi^{-1}\psi)$ for any ϕ, ψ in $C_1^0(M, G)$.

In Section 3 we define the representation δ of the Sobolev-Lie group $H_1(M, G)$, given by the energy function $T(\phi)$. To describe the representation space for δ we take the free module over the complex numbers generated by the Sobolev-Lie group $H_1(M, G)$. Denoting the generator of this free module by e^ϕ with $\phi \in H_1(M, G)$, we introduce in the free module a positive sesquilinear form $(,)_{H_1}$ by setting

$$(\Phi, \Psi)_{H_1} = \sum_{i,j} \bar{\alpha}_i \beta_j e^{(\phi_i, \psi_j)}, \quad \text{with } \Phi = \sum \alpha_i e^{\phi_i},$$

$$\Psi = \sum_j \beta_j e^{\phi_j}, \quad (\phi, \psi) = T(\phi) + T(\psi) - T(\psi^{-1}\phi).$$

Then $H_1(M, G)$, suitably completed with respect to the seminorm given by this sesquilinear form, becomes an Hilbert space $E(M, G)$. This Hilbert space is the representation space for the unitary representation δ of the Sobolev-Lie group $H_1(M, G)$, defined as the linear continuous extension to all of $E(M, G)$ of the operation δ_0 defined by

$$\delta_0(\phi) e^\psi = e^{-1/2(\phi,\phi) + (\psi, \phi^{-1})} e^{\phi\psi}.$$

A continuity property of δ is also exhibited. All results of Sections 2 and 3 are also valid for the case where M is a manifold with boundaries.

The unitary representation δ of the Sobolev-Lie group $H_1(M, G)$ constructed in the preceding sections is then in Section 4, put in relation with the "Brownian motion representation", in the case where M is a closed interval $[0, t]$ of the real line. In this particular case one has namely that the representation space $E([0, t], G)$ is unitarily equivalent with $L_2(C([0, t], G), dw)$, where dw is the Wiener measure on the space of continuous paths in G, defined by left Brownian motion on G ([17]–|19|). Moreover the representation δ is unitarily equivalent with the representation of $H_1([0, t], G)$ induced by right translations on $L_2(C([0, T], G), dw)$, by the fact that dw is quasi invariant under right translations by the group $H_1([0, t], G)$. Thus the general construction of the preceding sections is an extension to the case of arbitrary orientable manifolds M of the representation of Sobolev-Lie groups given, when M is an interval of the real line, by the Brownian motion on G.

In Section 5 we realize isometrically the representation space $E(M, G)$ of the unitary representation δ as a subspace of the $L_2(\Omega, \beta, d\mu)$-space of square integrable functions of a random field $\psi \to \xi_\psi(\omega)$, $\psi \in H_1(M, G)$ associated with the pair (M, G). In the case where M is a closed interval of the real line this random field reduces to the Brownian motion process discussed in Section 4.

2. The group $H_1(M, G)$

If M and N are two Riemann manifolds, we denote by $C_1(M, N)$ the C_1-mappings from M to N that are constant outside a compact in M. Let $\phi \in C_1(M, N)$, then $d\phi(x) \in L(M_x, N_{\phi(x)})$, where M_x and $N_{\phi(x)}$ are the tangent spaces at x and $\phi(x)$ respectively. By assumption M_x and $N_{\phi(x)}$ are equipped with their Riemann metrics which indentify these spaces with their duals, so that if $A \in L(M_x, N_{\phi(x)})$ then the adjoint A' belongs to $L(N_{\phi(x)}, M_x)$. Hence the metrics in M_x and $N_{\phi(x)}$ induce a natural metric in $L(M_x, N_{\phi(x)})$ by $|A|^2 = \mathrm{tr}(AA') = \mathrm{tr}(A'A)$.

Let us now also assume that M is orientable, then there is a natural measure dx on M coming from its Riemann structure and we define the energy function T on $C_1(M, N)$ by

$$T(\phi) = \tfrac{1}{2} \int_M |d\phi(x)|^2 dx \tag{2.1}$$

We shall be interested in the case where N is a connected Lie group G with a compact Lie algebra g. We recall that a Lie algebra is said to be compact if $\mathrm{Int}(g)$ is compact, where $\mathrm{Int}(g)$ is the group of inner automorphisms of the Lie algebra g. Since G is connected, $\mathrm{Int}(g) = AdG$ (see [16], Ch. II, Section 5) where AdG is the adjoint representation of G in g. Since AdG is compact there are strictly positive definite invariant bilinear forms on g. We remark that any compact Lie algebra is the direct sum of a semisimple compact Lie algebra and its center, and that a semisimple Lie algebra over R is compact if and only if its Killing form is strictly negative definite (see [16], Ch. II, Section 6).

Now any strictly positive definite form on g defines obviously a unique Riemann structure on G which is invariant under left multiplications on G. Let now $B(X, Y)$ be any strictly positive definite form on g invariant under AdG, then the corresponding Riemann structure on G is left and right invariant. In what follows we shall consider G as a Riemann space with a left and right invariant Riemann structure induced by some fixed strictly positive definite form $B(X, Y)$ on g.

For $\phi \in C_1(M, G)$ we set

$$|\phi|^2 = 2T(\phi) = \int_M |d\phi(x)|^2 dx. \tag{2.2}$$

If $x_1, \dots, x_n$ are normal coordinates at $x \in M$ we denote $\nabla_i\phi(x) = (\partial/\partial x_i)\phi(x)$, and we have

$$\begin{aligned} |d\phi(x)|^2 &= B(\phi^{-1}(x)\nabla\phi(x), \phi^{-1}(x)\nabla\phi(x)) \\ &= \sum_{i=1}^{n} B(\phi^{-1}(x)\nabla_i\phi(x), \phi^{-1}(x)\nabla_i\phi(x)), \end{aligned} \tag{2.3}$$

with the notation $\phi^{-1}(x)$ for the inverse $(\phi(x))^{-1}$. Let now $\phi(x)$ and $\psi(x)$ be in $C_1(M, G)$, then we set $\phi\psi(x) = \phi(x)\psi(x)$, and this multiplication organizes $C_1(M, G)$ obviously to be a group. We see that

$$\begin{aligned} d(\phi\psi)(x) &= (d\phi(x))\psi(x) + \phi(x)d\psi(x) \\ &= dR(\psi(x))d\phi(x) + dL(\phi(x))d\psi(x), \end{aligned} \tag{2.4}$$

where $R(\sigma)$ and $L(\sigma)$ are the right and left multiplications in G by the element σ in G, and the sum is the sum as vectors in $G_{\phi(x)\psi(x)}$. Hence

$$(\phi\psi)^{-1}(x)\nabla(\phi\psi)(x) = Ad(\psi^{-1})(x))\phi^{-1}(x)\nabla\phi(x) + \psi^{-1}(x)\nabla\psi(x), \tag{2.5}$$

and by the invariance of B we therefore get

$$|d\phi\psi(x)|^2 = |d\phi(x)|^2 + |d\psi(x)|^2 + 2B(\phi^{-1}(x)\nabla\phi(x), \nabla\psi(x)\psi^{-1}(x)). \tag{2.6}$$

Let us now introduce the expression

$$(\phi, \psi) = \int_M B(\nabla\phi(x)\cdot\phi^{-1}(x), \nabla\psi(x)\cdot\psi^{-1}(x))dx, \tag{2.7}$$

for any ϕ and ψ in $H_1(M, G)$.

Then

$$(\phi, \psi) = \tfrac{1}{2}[|\phi|^2 + |\psi|^2 - |\psi^{-1}\phi|^2] \tag{2.8}$$

or

$$(\phi, \psi) = T(\phi) + T(\psi) - T(\psi^{-1}\phi). \tag{2.9}$$

We observe that since

$$d\phi^{-1}(x) = -\phi^{-1}(x)d\phi(x)\phi^{-1}(x) \tag{2.10}$$

we have, by the left and right invariance of the Riemann structure on G, that

$$|\phi^{-1}| = |\phi|. \tag{2.11}$$

Then we have

$$(\phi, \psi) = (\psi, \phi). \tag{2.12}$$

Moreover we have that

$$\begin{aligned}(\phi, \psi) &= -\int_M B(\psi(x)\nabla\psi^{-1}(x), \nabla\phi(x)\phi^{-1}(x))dx \\ &= -\int_M B(d\psi^{-1}(x), \psi^{-1}(x)(d\phi(x))\phi^{-1}(x))dx.\end{aligned} \tag{2.13}$$

By the Schwarz inequality and the invariance of B we then get

$$|(\psi, \phi^{-1})| \le |\psi| \cdot |\phi^{-1}|, \tag{2.14}$$

which by (2.6) yields

$$|\phi\psi|^2 = |\phi|^2 + |\psi|^2 - 2(\psi, \phi^{-1}). \tag{2.15}$$

This together with (2.14) and (2.11) gives us

$$|\phi\psi| \le |\phi| + |\psi|. \tag{2.16}$$

Let now $C_1^0(M, G)$ be the set of mappings in $C_1(M, G)$ which are equal to the identity in G outside a compact in M. We shall see that $d(\phi, \psi) = |\phi^{-1}\psi|$ is a metric on $C_1^0(M, G)$. From (2.11) we have

$$d(\phi, \psi) = d(\psi, \phi). \tag{2.17}$$

From (2.16) we get

$$d(\phi, \psi) \le d(\phi, \chi) + d(\chi, \psi). \tag{2.18}$$

If $|\phi| = 0$ then $d\phi(x) = 0$ for all x, so that $\phi(x)$ is constant, hence, if M is non compact $\phi(x) = e$. Therefore

$$d(\phi, \psi) = 0 \Rightarrow \phi = \psi. \tag{2.19}$$

Hence $d(\phi, \psi)$ is a metric on $C_1^0(M, G)$ and we define $H_1(M, G)$ as the completion of $C_1^0(M, G)$ in the metric d. $H_1(M, G)$ is then a complete metric group.

We say that $\phi \in H_1(M, G)$ has support in an open set $U \subset M$ if it is the limit in the metric d of elements $\phi_n \in C_1^0(M, G)$ which are equal to the identity e in G outside a compact subset of U. It is then obvious that if supp $\phi_i \subset U_i$ and $U_1 \cap U_2 = \phi$ then $\phi_1\phi_2 = \phi_2\phi_1$. We shall refer to this property by saying that the group $H_1(M, G)$ is local.

3. The unitary representation δ of $H_1(M, G)$

From the definition

$$(\phi, \psi) = \int B(\nabla\phi(x) \cdot \phi^{-1}(x), \nabla\psi(x) \cdot \psi^{-1}(x))dx \tag{3.1}$$

it follows easily that, for any set of complex numbers $\lambda_1, \ldots, \lambda_n$ and elements $\phi_1, \ldots, \phi_n$ in $H_1(M, G)$, we have

$$\sum_{i,j} \bar{\lambda}_i\lambda_j(\phi_i, \phi_j) \geq 0 \tag{3.2}$$

and hence

$$\sum_{i,j} \bar{\lambda}_i\lambda_j e^{(\phi_i, \phi_j)} \geq 0. \tag{3.3}$$

Consider now the free C-module generated by the group $H_1(M, G)$, and denote the generators of this module by e^{ϕ}, $\phi \in H_1(M, G)$.

Then for $\Phi = \Sigma_{i=1}^{m} \alpha_i e^{\phi_i}$ and $\Psi = \Sigma_{j=1}^{n} \beta_j e^{\psi_j}$ we define the inner product

$$(\Phi, \Psi) = \sum_{i,j} \bar{\alpha}_i\beta_j e^{(\phi_i, \psi_j)} \tag{3.4}$$

We have then that (,) is a positive sequilinear form on the free C-module, and hence that $|\Phi| = (\Phi, \Phi)^{1/2}$ is a seminorm. Hence the set of elements in the free C-module with norm zero is a linear subspace, and after dividing out by this linear subspace we then get a pre-Hilbert space. We denote by $E(M, G)$ the completion of this pre-Hilbert space.

We shall see that there is a unitary representation δ of the group $H_1(M, G)$ in the Hilbert space $E(M, G)$. For any generating element e^{ψ} of the free C-module over $H_1(M, G)$ we define

$$\delta_0(\phi)e^{\psi} = e^{-1/2|\phi|^2 + (\psi, \phi^{-1})} e^{\phi\psi}. \tag{3.5}$$

Then we extend $\delta_0(\phi)$ by linearity to the whole free C-module. Let $\Psi = \Sigma_i \beta_i e^{\psi_i}$, then

$$\delta_0(\phi)\Psi = \sum_i \beta_i e^{-1/2|\phi|^2+(\psi_i,\phi^{-1})} e^{\phi\psi_i} \tag{3.6}$$

and

$$|\delta_0(\phi)\Psi|^2 = \sum_{i,j} \bar{\beta}_i\beta_j e^{-|\phi|^2+(\psi_i,\phi^{-1})+(\psi_j,\phi^{-1})} e^{(\phi\psi_i,\phi\psi_j)}. \tag{3.7}$$

By (2.8) and (2.15) we have

$$\begin{aligned}(\phi\psi_i, \phi\psi_j) &= \tfrac{1}{2}[|\phi\psi_i|^2 + |\phi\psi_j|^2 - |(\phi\psi_j)^{-1}\phi\psi_i|^2] \\ &= |\phi|^2 - (\psi_i, \phi^{-1}) - (\psi_j, \phi^{-1}) + \tfrac{1}{2}[|\psi_i|^2 + |\psi_j|^2 \\ &\quad - |\psi_j^{-1}\psi_i|^2] = |\phi|^2 - (\psi_i, \phi^{-1}) - (\psi_j, \phi^{-1}) \\ &\quad + (\psi_i, \psi_j).\end{aligned} \tag{3.8}$$

Hence

$$|\delta_0(\phi)\Psi|^2 = \sum_{i,j} \bar{\beta}_i\beta_j e^{(\psi_i,\psi_j)} = |\Psi|^2. \tag{3.9}$$

Thus $\delta_0(\phi)$ is a linear isometry defined on the free C-module, which therefore maps the linear subspace of norm zero vectors into itself. Hence there is a natural mapping on the pre-Hilbert space we get by dividing out by the linear subspace of norm zero vectors. This mapping being again an isometry, it extends by continuity to an isometry $\delta(\phi)$ on the Hilbert space $E(M, G)$, which for any $\psi \in H_1(M, G)$ satisfies

$$\delta(\phi)e^{\psi} = e^{-1/2|\phi|^2+(\psi,\phi^{-1})} e^{\phi\psi}. \tag{3.10}$$

Let now $\eta \in H_1(M, G)$, then

$$\delta(\eta)\delta(\phi)e^{\psi} = e^{-1/2|\phi|^2+(\psi,\phi^{-1})-1/2|\eta|^2+(\phi\psi,\eta^{-1})} e^{\eta\phi\psi} \tag{3.11}$$

and

$$\delta(\eta\phi)e^{\psi} = e^{-1/2|\eta\phi|^2+(\psi,(\eta\phi)^{-1})} e^{\eta\phi\psi}. \tag{3.12}$$

By (2.15) we have

$$\begin{aligned}\tfrac{1}{2}|\eta\phi|^2(\psi, (\eta\phi)^{-1}) &= \tfrac{1}{2}|\eta\phi\psi|^2 - \tfrac{1}{2}|\psi|^2 \\ &= \tfrac{1}{2}|\eta|^2 + \tfrac{1}{2}|\phi\psi|^2 - (\phi\psi, \eta^{-1}) - \tfrac{1}{2}|\psi|^2 \\ &= \tfrac{1}{2}|\eta|^2 + \tfrac{1}{2}|\phi|^2 - (\psi, \phi^{-1}) - (\phi\psi, \eta^{-1}),\end{aligned} \tag{3.13}$$

which proves the identity of (3.11) and (3.12). Hence by linearity and continuity we have that δ is a representation of $H_1(M, G)$

$$\delta(\eta) \cdot \delta(\phi) = \delta(\eta\phi). \tag{3.14}$$

We obviously have that $\delta(e) = 1$, where e is the constant function $e(x) = e$, the identity in G. This, together with (3.14), gives that $\delta(\phi^{-1}) = \delta(\phi)$. Hence $\delta(\phi)$ has an inverse and it is therefore unitary. We have thus proven the following theorem.

THEOREM 3.1: *δ as defined by* (3.10) *is a unitary representation of $H_1(M, G)$ on $E(M, G)$.* ■

Consider now the space $C(M, G)$ of continuous mappings from M into G. Since G is a compact Riemann manifold in the left and right invariant Riemann structure given by B, there is a metric d_B on G such that $d_B(\sigma, \tau)$ is the length of the shortest geodesics from σ to τ. d_B induces a natural metric on $C(M, G)$ given by

$$d_0(\phi, \psi) = \sup_{x \in M} d_B(\phi(x), \psi(x)). \tag{3.15}$$

It is immediate that d_0 is a metric on the group $C(M, G)$ in which $C(M, G)$ is a complete metric space and the topology induced on $C(M, G)$ by d_0 is that of uniform convergence.

We also have that the group $C(M, G) \cap H_1(M, G)$ is dense in $C(M, G)$ and $H_1(M, G)$ respectively, where both these groups are considered in their metric topology. Furthermore the group $C(M, G) \cap H_1(M, G)$ is a complete metric space with metric $d_0 + d$. As for the continuity of the representation δ we have now the following theorem:

THEOREM 3.2: *The restriction of δ to $C(M, G) \cap H_1(M, G)$ is continuous as a mapping from $C(M, G) \cap H_1(M, G)$ with metric topology given by $d_0 + d$ into $E(M, G)$ with the strong topology.*

PROOF: Since the adjoint representation of G on it Lie algebra g is continuous and by assumption AdG is compact, we have that $Ad\sigma$ is a uniformly continuous function from G into the set of orthogonal matrices on g. Hence if $\| \ \|_B$ is the natural norm in the algebra of linear transformations on g induced by the B-norm on g, we therefore have that

$$\|Ad(\sigma) - 1\| \le K d_B(\sigma, e) \tag{3.16}$$

where K is some fixed constant and $d_B(\sigma, e)$ is the Riemann distance from σ to e. Consider now for ϕ and ψ in $H_1(M, G)$ and η in $C(M, G) \cap H_1(M, G)$

$$(\phi, \eta\psi) = \int_M B(\nabla\phi \cdot \phi^{-1}, \nabla(\eta\psi) \cdot (\eta\psi)^{-1}) dx$$

$$=\int_M B(\nabla\phi\cdot\phi^{-1},\eta(\nabla\psi\cdot\psi^{-1})\eta^{-1}+\nabla\eta\cdot\eta^{-1})dx. \tag{3.17}$$

Then

$$\begin{aligned}|(\phi,\eta\psi)-(\phi,\psi)|&\leq\left|\int B(\nabla\phi\cdot\phi^{-1},Ad\eta(x)(\nabla\psi\cdot\psi^{-1}(x))\right.\\&\quad\left.-\nabla\psi\cdot\psi^{-1}(x))dx\right|+\left|\int B(\nabla\phi\cdot\phi^{-1},\nabla\eta\cdot\eta^{-1})dx\right|\\&\leq\int|\nabla\phi\cdot\phi^{-1}(x)|_B|(Ad(\eta(x))-1)\nabla\psi\cdot\psi^{-1}(x)|_B dx+|\phi|\cdot|\eta|\\&\leq\sup_x\|Ad\eta(x)-1\|_B|\phi|\,|\psi|+|\eta|\,|\phi|\\&\leq K\sup_x d(\eta(x),e)\cdot|\phi|\,|\psi|+|\eta|\,|\phi|,\end{aligned}$$

where $|\;|_B$ is the norm in g induced by B. Hence

$$|(\phi,\eta\psi)-(\phi,\psi)|\leq K|\phi|\,|\psi|d_0(\eta,e)+|\eta|\;|\phi|. \tag{3.18}$$

Since $\delta(\eta)$ is unitary we have that

$$\|\delta(\eta)\Phi-\Phi\|^2=2\|\Phi\|^2-2\operatorname{Re}(\Phi,\delta(\eta)\Phi) \tag{3.19}$$

and with $\Phi=\Sigma_i\alpha_i e^{\phi_i}$ we have

$$(\Phi,\delta(\eta)\Phi)=\sum_{i,j}\bar{\alpha}_i\alpha_j e^{-1/2|\eta|^2+(\phi_i,\eta^{-1})+(\phi_j,\eta^{-1})+(\phi_i,\eta\phi_j)}. \tag{3.20}$$

Since $|(\eta^{-1},\phi)|\leq|\eta^{-1}|\cdot|\phi|$ we have by (3.18) that $(\Phi,\delta(\eta)\Phi)$ converges to $\|\Phi\|^2$ as $d(\eta,e)+d_0(\eta,e)$ tends to zero. By uniform boundedness we get the same result for any $\Phi\in E(M,G)$. This proves the theorem. ■

We shall also need to consider manifolds with boundaries. So let M be a manifold with a boundary $\partial M\neq\emptyset$ and let now $C_1(M,\partial M,G)$ be the space of differentiable maps ϕ from M into G such that $\phi(x)=e$ on ∂M. $C_1(M,\partial M,G)$ has a natural group structure and on the subgroup of elements ϕ with $|\phi|$ finite we see that $d(\phi,\psi)=|\phi^{-1}\psi|$ is a metric. The completion of this subgroup in the metric $d(\phi,\psi)$ is denoted $H_1(M,\partial M,G)$. Similarly we also introduce the Hilbert space $E(M,\partial M,G)$. It is then an immediate consequence of the proofs of Theorem 3.1 and Theorem 3.2 that these theorems extend to the case of manifolds with boundaries, where we introduce the notation $C(M,\partial M,G)$ for the group of continuous mappings that satisfy the boundary conditions.

4. The Brownian motion on a Lie group of compact type

That a Lie group G is of compact type just means that its Lie algebra g is compact in the sense of the previous sections. In this section we consider the Brownian motion on such a Lie group. So let $\eta(t)$ be the left invariant Brownian motion on G which starts at the identity e in G and satisfies the stochastic differential equation.

$$d\eta(t) = \eta(t)d\xi(t), \tag{4.1}$$

where $\xi(t)$ is the standard Brownian motion (or Wiener process) in the Lie algebra g equipped with the invariant inner product given by the strictly positive definite form $B(\xi_1, \xi_2)$. Hence $\eta(t)$ is the non anticipating solution of the stochastic integral equation corresponding to (4.1) with initial condition $\eta(0) = e$, where $d\xi \to \eta \cdot d\xi$ is the mapping from $g = G_e$ to G_η, from the tangent space at e to the tangent space at η, induced by the left translation by η on the group G. So that if $L(\eta) \cdot \sigma = \eta \cdot \sigma$ then, according to our notation, we have $\eta d\xi = dL(\eta) \cdot d\xi$.

Stochastic differential equations of the type (4.1) have been studied in the well known work of Ito [19], McKean [18], Gangolli [17], and the existence and uniqueness of solutions is established. The solution $\eta(t)$ has the property that it starts afresh at its stopping times T in the sense that, conditional on $T < \infty$, the future $\eta^+(t) = \eta(t)^{-1}\eta(t+T)$, $t \geq 0$ is independent of the past $\eta(s): s \leq T+$, and identically in law to the original motion $\eta(t)$: $t \geq 0$ starting at e. This is an analog of the independent increments of the one dimensional Brownian motion, and we shall shortly refer to this property by saying that $\eta(t)$ has left independent increments. It follows from the uniqueness of solutions of (4.1) and the fact that the standard Brownian motion $\xi(t)$ in g is invariant under time reflection, that the backward process $\eta^*(t) = \eta(-t)$ is identical in law with $\eta(t)$. Moreover since $d/dt(\phi^{-1}(t)) = -\phi^{-1}(t)\dot{\phi}(t)\phi^{-1}(t)$, where $\dot{\phi}(t) = (d/dt)\phi(t)$, if ϕ is C_1, we have that $d\eta^{-1}(t) = -\eta^{-1}(t)d\eta \cdot \eta^{-1}(t)$ and since $\xi(t)$ and $-\xi(t)$ are identical in law we get that $\zeta(t) = \eta^{-1}(t)$ satisfies the differential equation

$$d\zeta(t) = -d\xi(t) \cdot \zeta(t) \tag{4.2}$$

where $d\xi \cdot \zeta = dR(\zeta) \cdot d\xi$. So that the mapping $\sigma \to \sigma^{-1}$ on G takes the Brownian motion with left independent increments into the Brownian motion with right independent increments. It follows from the uniqueness of solutions of (4.1) and the fact that B is invariant under AdG that $\eta(t)$ is a left and right invariant process on G, i.e. if $\eta(t)$ is the solution of (4.1) with $\eta(0) = e$, then $\sigma\eta(t)$ as well as $\eta(t) \cdot \sigma$ for

$\sigma \in G$ are identical in law with the solution of (4.1) that starts at σ, for $t = 0$.

Let $d\sigma$ be the Haar measure on G, then since $\eta(t)$ is left invariant it must leave $d\sigma$ invariant, the Haar measure being unique up to a constant. Hence $\eta(t)$ induces a symmetric Markov semigroup in $L_2(G, d\sigma)$, which we denote by $e^{t\Delta}$, so that for $f \in L_2(d\sigma)$

$$(e^{t\Delta}f)(\sigma) = Ef(\eta(t) \cdot \sigma) \tag{4.3}$$

where $\eta(t)$ is the Brownian motion such that $\eta(0) = e$. It follows then easily that Δ is the standard self-adjoint Laplacian in $L_2(G, d\sigma)$ given by the invariant form B. In the case G is a simple compact Lie group we have that B is proportional to the Killing form, in which case Δ is proportional to the Casimir operator. Consider now the left Brownian motion on G given by the stochastic differential equation

$$d\eta(t) = \eta(t)d\xi(t) \tag{4.4}$$

starting at e for $t = 0$, where $\xi(t)$ is the standard Brownian motion on its Lie algebra g equipped with the inner product given by $4 \cdot B(\xi_1, \xi_2)$. It follows from the proof of the existence of solutions of (4.4) that almost all paths $\eta(s)$, $0 \leq s \leq t$ are continuous paths on G and in fact there is a measure dw on the space of continuous paths $C([0, t), \{0\}, G) \equiv C([0, t), G)$, where dw is a regular measure in the metric topology given by the supremum metric $d_0(\phi, \psi)$ on the metric group $C([0, t), G)$. It follows from a Sobolev inequality in R^1 that $H_1([0, t), \{0\}, G) \equiv H_1([0, t), G)$ is actually a subgroup of $C([0, t), G)$.

By utilizing equation (4.1) and the corresponding stochastic integral equation one proves that the Wiener measure dw on $C([0, t), G)$ is quasi invariant under right multiplication by elements in $H_1([0, t), G)$. In fact if dw_ψ is the image of dw under the mapping $\eta(x) \to \eta(x)\psi(x)$, for $\psi \in H_1([0, t), G)$, then one has

$$\alpha(\eta, \psi) = \frac{dw_\psi}{dw}(\eta) = e^{-2|\psi|^2} e^{-4\int_0^t B(\dot{\psi}\psi^{-1}, \eta^{-1}d\eta)}, \tag{4.5}$$

where the Radon–Nikodym derivative $\alpha(\eta, \psi)$ is a positive integrable function with integral 1.

Let now $L_2(C([0, t), G), dw) = L_2(C, dw)$ be the corresponding L_2-space. Since dw is quasi invariant under right translations by the group $H_1(([0, t), G)$ there is a natural unitary representation of this group in $L_2(C, dw)$ given by

$$(V(\psi)f)(\eta) = \alpha(\eta, \psi)^{1/2} f(\eta\psi). \tag{4.6}$$

We now compute $(V(\phi)1, V(\psi)1)$ and find

$$(V(\phi)\cdot 1,\ V(\psi)\cdot 1) = e^{-1/2|\phi|^2 - 1/2|\psi|^2} e^{(\phi,\psi)}, \tag{4.7}$$

thus

$$(e^{+1/2|\phi|^2} V(\phi)\cdot 1,\ e^{+1/2|\psi|^2} V(\psi)\cdot 1) = e^{(\phi,\psi)}. \tag{4.8}$$

Consider now the linear mapping from $E([0, t), G)$ into $L_2(C, dw)$ defined by

$$e^{\psi} \to e^{+1/2|\psi|^2} V(\psi) 1. \tag{4.9}$$

It follows from (4.8) that (4.9) defines an isometry from $E([0, t), G)$ into $L_2(C, dw)$. We shall see that this isometry is actually a unitary equivalence between $E([0, t), G)$ and $L_2(C, dw)$. Let $X(\tau)$ be arbitrary in $L_2([0, t), G)$, where g is the Lie algebra of G equipped with the Euclidean norm given by B. Then the differential equation $\dot\psi(\tau) = X(\tau)\psi(\tau)$ has a unique solution with initial value $\psi(0) = e$, and obviously this solution ψ is in $H_1([0, t], G)$. By (4.4)

$$e^{\psi} \to e^{-1/2|\psi|^2} e^{-2\int_0^t B(\dot\psi\psi^{-1}, \eta^{-1} d\eta)} = e^{-1/2|\psi|^2} e^{-2\int_0^t B(X(\tau), \xi(\tau)) d\tau} \tag{4.6}$$

which obviously generates the full σ-algebra of $\eta(s)$, $0 \le s \le t$, since $\eta(s)$ is given in terms of $\xi(s)$ by a non anticipating stochastic integral. We summarize this in the following theorem.

THEOREM 4.1: *Let $\eta(s)$ be the standard left Brownian motion on G relative to the form 4. $B(\xi_1, \xi_2)$ on its Lie algebra g. Then the mapping*

$$e^{\psi} \to e^{-1/2|\psi|^2} e^{-2\int_0^t B(\dot\psi\psi^{-1}, \eta^{-1} d\eta)}$$

defined for $\psi \in H_1([0, t), G)$ extends by linearity to a unitary equivalence between $E([0, t), G)$ and $L_2(C, dw)$ which takes the unitary representation $\delta(\eta)$ into the unitary right translation $V(\eta)$ on $L_2(C, dw)$, which is the L_2-space with respect to the Wiener measure.

5. The random field associated with the pair (M, G)

Let now M be a Riemann manifold, G as before a Lie group of compact type and B a positive definite invariant bilinear form on its Lie algebra g. Let $H_1(M, G)$ be the corresponding Sobolev-Lie group. To any element A in the group algebra of $H_1(M, G)$, i.e. $A = \Sigma_{i=1}^n \alpha_i\psi_i$ with $\alpha_i \in C$ and $\psi_i \in H_1(M, G)$, we may associate a g^c-valued one-form $a(x)$, which is square integrable over M and given by

$$a(x) = \sum_{i=1}^n \alpha_i d\psi_i \cdot \psi_i^{-1}(x). \tag{5.1}$$

We also write this as $\Sigma_{i=1}^{n}\alpha_i\nabla\psi_i(x)\cdot\psi_i^{-1}(x)$. ($g^c$ is the complexification of g). Now the linear space of g^c-valued square integrable one-forms has a natural Hilbert structure given by

$$(a, c)=\int_M\sum_{ij}\bar{\alpha}_i\beta_j B(\nabla\psi_i\psi_i^{-1},\nabla\phi_j\phi_j^{-1})dx \tag{5.2}$$

where dx is the Riemann measure on M, $C=\Sigma_{jc}^{m}\beta_j\phi_j$ and c is the one form corresponding to C.

Let us now consider the free C-module over the group algebra $\mathscr{A}$ of $H_1(M, G)$. We denote the elements in this C-module by

$$\sum_{i=1}^{n}\lambda_i e^{A_i} \tag{5.3}$$

where $\lambda_i\in C$ and $A_i\in\mathscr{A}$. We introduce the Hilbert norm given by

$$\left\|\sum_{i=1}^{n}\lambda_i e^{A_i}\right\|^2=\sum_{ij}\bar{\lambda}_i\lambda_j e^{(A_i,A_j)} \tag{5.4}$$

where $(A_i, A_j)=(a_i, a_j)$ and a_j is the one-form corresponding to A_i. Since (a_i, a_j) is the Gram matrix given by vectors $a_1,\ldots,a_n$ in a Hilbert space it is non negative definite and thus $e^{(A_i,A_j)}$ is non negative definite, so after dividing out by the vectors of norm zero we have that (5.4) defines a Hilbert space which we denote by $F(M, G)$.

Let now $\psi\in H_1(M, G)$ and $t\in R$, then we define the operator $U(t\psi)$ on $F(M, G)$ by

$$U(t\psi)e^A=e^{-t^2/2\|\psi\|^2}e^{it\,\mathrm{Im}(\psi,A)}e^{A+it\psi}. \tag{5.5}$$

It is easily verified that $U(t\psi)$ are strongly continuous unitary groups as functions of t and they commute for different ψ_a in $H_1(M, G)$. Hence by the spectral theorem applied to the weakly closed C^*-algebra generated by $U(t\psi)$ for $t\in R$ and $\psi\in H_1(M, G)$ we have that there is a probability space $(\Omega,\mathscr{B}, d\omega)$ and a mapping $\psi\to\xi_\psi$ from $H_1(M, G)$ into $\mathscr{B}$-measurable functions on Ω such that

$$(e^0, U(t\psi)e^0)=E(e^{it\xi}\psi) \tag{5.6}$$

and the mapping

$$U(t\psi)e^0=e^{-t^2/2|\psi|^2}e^{it\psi}\to e^{it\xi_\psi} \tag{5.7}$$

extends to a unitary isomorphism between the cyclic Hilbert space generated by $U(t\psi)$ acting on e^0 and $L_2(\Omega, d\omega)$. We shall call the mapping $\psi\to\xi_\psi$ the random field associated with the pair (M, G) and we note that by construction we have that the σ-algebra generated by the functions $\xi_\psi(\omega)$ is equal to $\mathscr{B}$.

It follows from what is above that $\{\xi_{\psi_1}, \ldots, \xi_{\psi_n}\}$ are jointly Gaussian distributed with mean zero and covariance matrix (ψ_i, ψ_j), since

$$(e^0, U(t_1\psi_1) \ldots U(t_n\psi_n)e^0) = E(e^{i\sum_{j=1}^{n} t_j\xi_{\psi_j}}). \tag{5.8}$$

From this it follows that for $\alpha_i \in C$ we have that $e^{\Sigma \alpha_i \xi_{\psi_i}}$ is integrable with

$$E(e^{\Sigma \alpha_i \xi_{\psi_i}}) = e^{\Sigma \alpha_i \alpha_j (\psi_i, \psi_j)}. \tag{5.9}$$

Especially we see that

$$e^{-1/2\|A\|^2} e^A \to e^{\Sigma \alpha_i \xi_{\psi_i}} \tag{5.10}$$

with $A = \Sigma \alpha_i \psi_i$, extends to a unitary mapping of $F(M, G)$ in $L_2(\Omega, d\omega)$. That its image is all of $L_2(\Omega, d\omega)$ follows by the construction of $L_2(\Omega, d\omega)$. We see that $E(M, G)$ thus becomes the subspace of $L_2(\Omega, d\omega)$ spanned by functions of the form e^{ξ_ψ}. We summarize this in the following theorem.

THEOREM 5.1: *There exists a probability space* $(\Omega, \mathcal{B}, d\omega)$ and a mapping from $H_1(M, G)$ into the space of measurable functions on Ω, $\psi \to \xi_\psi(\omega)$, such that $\mathcal{B}$ is the smallest σ-algebra generated by the random variables ξ_ψ and

$$e^{-1/2|\psi|^2} e^{\psi} \to e^{\xi_\psi}$$

extends by linearity to an isometry of $E(M, G)$ into $L_2(\Omega, d\omega)$.

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