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A GEOMETRIC CHARACTERIZATION OF THE RADON–NIKODYM PROPERTY IN BANACH SPACES

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Abstract

It is shown that a Banach space $E$ has the Radon–Nikodym property (R.N.P.) if and only if every nonempty weakly-closed bounded subset of $E$ has an extreme point.

Notations

$E$, $\|\|$ is a real Banach space with dual $E'$. For sets $A \subset E$, let $c(A)$ and $\bar{c}(A)$ denote the convex hull and closed convex hull, respectively. If $x \in E$ and $\epsilon > 0$, then $B(x, \epsilon) = \{y \in E; \|x - y\| < \epsilon\}$. A subset $A$ of $E$ is said to be dentable if for every $\epsilon > 0$ there exists a point $x \in A$ such that $x \in \bar{c}(A \setminus B(x, \epsilon))$.

Suppose that $C$ is a nonempty, bounded, closed and convex subset of $E$. Let $M(C) = \sup\{\|x\|; x \in C\}$. If $f \in E'$, let $M(f, C) = \sup\{f(x); x \in C\}$, and for each $\alpha > 0$, let $S(f, \alpha, C) = \{x \in C; f(x) \geq M(f, C) - \alpha\}$. Such a set is called a slice of $C$.

**Lemma 1:** Let $C$ and $C_1$ be nonempty, bounded, closed and convex subsets of $E$, such that $C_1 \subset C$ and $C_1 \neq C$. Then there exist $x \in C$, $f \in E'$ and $\alpha > 0$ with $f(x) = M(f, C) > M(f, C_1) + \alpha$.

**Proof:** Without restriction, we can assume $M(C) \leq 1$. Take $x_1 \in C \setminus C_1$. By the separation theorem we have $f_1 \in E'$ and $\alpha_1 > 0$ with $f_1(x_1) > M(f_1, C_1) + \alpha_1$.

Let $\alpha = \alpha_1/3$. Using a result of Bishop and Phelps (see [1]), we

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LEMMA 2: Let $C$ be a nonempty, bounded, closed and convex subset of $E$. If for every $\epsilon > 0$, there exist convex and closed subsets $C_1$ and $C_2$ of $C$, such that $C = \partial(C_1 \cup C_2)$, $C_1 \neq C$ and $\text{diam } C_2 \leq \epsilon$, then $C$ is dentable.

PROOF: Take $\epsilon > 0$ and let $C_1, C_2$ be convex and closed subsets of $C$, such that $C = \partial(C_1 \cup C_2)$, $C_1 \neq C$ and $\text{diam } C_2 \leq \epsilon/2$. By Lemma 1, there exist $x \in C$, $f \in E'$ and $\alpha > 0$ with $f(x) = M(f, C) > M(f, C_1) + \alpha$.

Let $d = \text{diam } C$ and consider the set

$$Q = \left\{ \lambda y_1 + (1 - \lambda) y_2; \ y_1 \in C_1, y_2 \in C_2 \text{ and } \lambda \in \left[ \frac{\epsilon}{12d}, 1 \right] \right\}.$$ 

It follows immediately that $Q$ is a closed, convex subset of $C$ and $x \notin Q$. Suppose $z_1, z_2 \in C \setminus Q$. We find $z_1', z_2'$ such that $z_i' \in \partial(C_1 \cup C_2)$, $z_i' \notin Q$ and $\|z_i - z_i'\| < \epsilon/6$ ($i = 1, 2$). There exist $y_1 \in C_1$, $y_2 \in C_2$ and $\lambda_i \in [0, \epsilon/12d]$, with $z_i' = \lambda_i y_1 + (1 - \lambda_i) y_2$ ($i = 1, 2$). We obtain:

$$\|z_i - z_i'\| < \|z_i' - z_i\| + \frac{\epsilon}{3} \leq \|y_i - y_i'\| + \lambda_i \|y_1 - y_2\| + \lambda_2 \|y_1 - y_2\| + \frac{\epsilon}{3} \leq \epsilon.$$

This implies that $C \setminus Q \subset B(x, \epsilon)$ and therefore $\partial(C \setminus B(x, \epsilon)) \subset Q$. Because $x \notin Q$, we have that $x \notin \partial(C \setminus B(x, \epsilon))$, which proves the lemma.

THEOREM 3: If the Banach space $E$ hasn’t the RNP, there exists a nonempty, bounded and weakly-closed subset of $E$ without extreme points.

PROOF: If $E$ hasn’t the RNP, there is a closed and separable subspace of $E$, which hasn’t the RNP (see [4]). Therefore we can assume $E$ separable.

Let $C$ be a non-dentable, convex, closed and bounded subset of $E$. By Lemma 2, there exists $\epsilon > 0$, such that if $C = \partial(C_1 \cup C_2)$, where $C_1, C_2$ are closed, convex and $\text{diam } C_2 \leq \epsilon$, then $C = C_1$. Suppose $C = \cup_{p \in \mathbb{N}^*} B_p$, where $B_p$ is the intersection of $C$ and a closed ball with radius $\epsilon/2$. By induction on $p \in \mathbb{N}^*$, we construct sequences $(N_p)_p$, $(V_p)_p$ and $(\alpha_p)_p$, where $N_p$ is a finite subset of $\mathbb{N}^*$, $V_p = \{(x_\omega, \lambda_\omega, f_\omega); \omega \in N_p\}$ a subset of $C \times [0, 1] \times E'$ and $\alpha_p > 0$, with the following properties:
(1) \(N_p\) is the projection of \(N_{p+1}\) on the \(p\) first co-ordinates \((p \in \mathbb{N}^*)\).

(2) \(\sum_{(w,i) \in N_p} \lambda_{(w,i)} = 1\) \((p \in \mathbb{N}^*, \omega \in N_p)\).

(3) \(\|x^i - \sum_{(w,i) \in N_{p+1}} \lambda_{(w,i)} x_{(w,i)}\| < (1/2^{p+1})\) \((p \in \mathbb{N}^*, \omega \in N_p)\).

(4) \(f_w(x_w) = M(f_{w}, C)\) \((p \in \mathbb{N}^*, \omega \in N_p)\).

(5) \(S(f_{(w,i)}, \alpha_{p+1}, C) \subseteq S(f_{(w,i)}, \alpha_{p}, C)\) \((p \in \mathbb{N}^*, (\omega, i) \in N_{p+1})\).

(6) \(S(f_{(w,i)}, \alpha_{p}, C) \cap B_1 = \emptyset\) \((p \in \mathbb{N}^*, \omega \in N_p)\).

(In (2) and (3), \(i\) is the summation index).

CONSTRUCTION:

(1) Take \(N_1 = \{1\}\) and \(\lambda_1 = 1\). Applying Lemma 1, we find \(x_1 \in C\), \(f_1 \in E'\) and \(\alpha_1 > 0\) such that \(f_1(x_1) = M(f_1, C)\) and \(S(f_1, \alpha_1, C) \cap B_1 = \emptyset\).

(2) Suppose we found \(N_p\), \(V_p\) and \(\alpha_p\).

Take \(\omega \in N_p\).

Let \(S = \{x \in C; \exists f \in E'\) such that \(f(x) = M(f, C)\)

> \sup f((C \setminus S(f_{(w,i)}, \alpha_{p}, C)) \cup B_{p+1})\}

By lemma 1, we obtain easily

\[C = \bar{c}((C \setminus S(f_{(w,i)}, \alpha_{p}, C)) \cup B_{p+1} \cup S)\).

Because \(\text{diam } B_{p+1} = \epsilon\), this implies

\[x_w \in C = \bar{c}((C \setminus S(f_{(w,i)}, \alpha_{p}, C)) \cup S)\]

Thus there are sequences \((a_m)_m\) in \(C \setminus S(f_{(w,i)}, \alpha_{p}, C)\), \((b_m)_m\) in \(c(S)\) and \((t_m)_m\) in \([0, 1]\), with \(x_w = \lim_{m \to \infty} (t_m a_m + (1 - t_m) b_m)\).

Because \(f_w(t_m a_m + (1 - t_m) b_m) \leq M(f_{(w,i)}, C) - t_m \alpha_p\), it follows that \(\lim_{m \to \infty} t_m = 0\) and thus \(x_w = \lim_{m \to \infty} b_m \in \bar{c}(S)\).

Take \(m_w \in \mathbb{N}^*, x_{(w,i)} \in S, \lambda_{(w,i)} \in [0, 1], f_{(w,i)} \in E'\) \((1 \leq i \leq m_w)\) and \(\beta_{\omega} > 0\), such that:

(1) \(\sum_{i=1}^{m_w} \lambda_{(w,i)} = 1\).

(2) \(\|x_w - \sum_{i=1}^{m_w} \lambda_{(w,i)} x_{(w,i)}\| < (1/2^{p+1})\).

(3) \(f_{(w,i)}(x_{(w,i)}) = M(f_{(w,i)}, C)\) \((1 \leq i \leq m_w)\).

(4) \(S(f_{(w,i)}, B_{\omega}, C) \subseteq S(f_{(w,i)}, \alpha_{p}, C)\) \((1 \leq i \leq m)\).

(5) \(S(f_{(w,i)}, B_{\omega}, C) \cap B_{p+1} = \emptyset\) \((1 \leq i \leq m_w)\).

Finally, let

\[N_{p+1} = \{(\omega, i); \omega \in N_p \text{ and } 1 \leq i \leq m_w\}\]

\[V_{p+1} = \{(x_{(w,i)}, \lambda_{(w,i)}, f_{(w,i)}); (\omega, i) \in N_{p+1}\}\]

\[\alpha_{p+1} = \min\{\beta_{\omega}; \omega \in N_p\}\].

We verify that this completes the construction. Now, for every \(p \in \mathbb{N}^*\) and \(\omega \in N_p\), we define

\[y_{\omega} = \lim_{p \to \infty} \sum_{i_{1}, \ldots, i_{p}} \lambda_{(w_{i_{1}}, \ldots, i_{p})} x_{(w_{i_{1}}, \ldots, i_{p})}\].
where for each \( \nu \in \mathbb{N}^\ast \) the summation happens over all integers \( i_1, \ldots, i_{\nu} \) satisfying \( (\omega, i_1, \ldots, i_{\nu}) \in N_{p+\nu} \). It is clear that these limits exist. Furthermore, we have for each \( p \in \mathbb{N}^\ast \) and \( \omega \in N_p \):

1. \( y_\omega = \sum_{(\omega,i) \in N_{p+1}} \lambda_{(\omega,i)} y_{(\omega,i)} \).
2. \( y_\omega \in S(f_\omega, \alpha_p, C) \).

(In (1) is \( i \) the summation index).

We will show that \( R = \{ y_\omega; p \in \mathbb{N}^\ast \text{ and } \omega \in N_p \} \) is the required set.

If \( z \in C \), there exists \( n \in \mathbb{N}^\ast \) such that \( z \in B_n \). By construction \( U = \bigcap_{\omega \in N_n} (E \setminus S(f_\omega, \alpha_n, C)) \) is a weak neighborhood of \( z \) and \( U \cap R \) is finite. Hence \( R \) is weakly closed and we also remark that \( R \) is discreet in its weak topology. It remains to show that \( R \) hasn’t extreme points.

Take \( p \in \mathbb{N}^\ast \) and \( \omega \in N_p \).

Then there is some \( n \in \mathbb{N}^\ast \) with \( y_\omega \in B_n \). Clearly, \( n > p \). Since \( y_\omega \in c(U \subset N_n (S(f_D, \alpha_n, C) \cap R)) \), and for each \( \Omega \in N_n \), we have \( S(f_D, \alpha_n, C) \cap B_n = \emptyset \), \( y_\omega \) is not an extreme point of \( R \).

This completes the proof of the theorem.

**Corollary 4:** A Banach space \( E \) has the RNP if and only if every bounded, closed and convex subset \( C \) of \( E \) contains an extreme point of its weak*-closure \( \tilde{C} \) in \( E'' \).

**Proof:** The necessity is a consequence of the work of Phelps (see [5]).

If now \( E \) does not possess the RNP, there exists a bounded, weakly closed subset \( R \) of \( E \) without extreme points. Clearly \( C = \text{c}(R) \) does not contain an extreme point of its weak*-closure.

**References**


