R. Parthasarathy

A generalization of the Enright-Varadarajan modules

*Compositio Mathematica*, tome 36, n° 1 (1978), p. 53-73

<http://www.numdam.org/item?id=CM_1978__36_1_53_0>
A GENERALIZATION OF THE ENRIGHT-VARADARAJAN MODULES

R. Parthasarathy

For a semisimple Lie group admitting discrete series Enright and Varadarajan have constructed a class of modules, denoted $D_{p,a}$ (cf. [3]). Their infinitesimal description based on the theory of Verma modules parallels that of finite dimensional irreducible modules. The introduction of the modules $D_{p,a}$ in [3] was primarily to give an infinitesimal characterization of discrete series but we feel that [3] may well be a starting point for a fresh approach towards dealing with the problem of classification of irreducible representations of a general semisimple Lie algebra.

In order to give more momentum to such an approach we first construct modules which broadly generalize those in [3]. We briefly describe them now.

Let $g_0$ be any real semisimple Lie algebra, $g_0 = k_0 + p_0$ a Cartan decomposition and $\theta$ the associated Cartan involution. Let $g = k + p$ be the complexification. Let $U(g), U(k)$ be the enveloping algebras of $g, k$ respectively and let $U^k$ be the centralizer of $k$ in $U(g)$. For each $\theta$ stable parabolic subalgebra $q$ of $g$ we associate in this paper a class of irreducible $k$ finite $U(g)$ modules having the following property: Like finite dimensional irreducible modules and like the Enright-Varadarajan modules $D_{p,a}$, any member of this class comes with a special irreducible $k$-type occurring in it with multiplicity one, with an explicit description of the action of $U^k$ on the corresponding isotypic $k$-type. We obtain these modules by extending the techniques in [3].

To see in what way these modules are related to the $\theta$ invariant parabolic subalgebra $q$ we refer the reader to §2.

When our parabolic subalgebra $q$ is minimal in $g$ and when rank of $g = \text{rank of } k$, the class of $U(g)$ modules which we associate to this $q$ coincides with the class of modules $D_{p,a}$ of [3] (with a slight difference
in parametrization). On the other hand when \( q = g \) is the maximal parabolic subalgebra, the class we obtain is just the class of all finite dimensional irreducible representations of \( g \). If \( k \) has trivial center, the trivial one dimensional \( U(g) \) module is not equivalent to any of the modules \( D_{p_A} \) of \([3]\). This gap is bridged by the introduction of our class of \( U(g) \) modules for every intermediate \( \theta \) invariant parabolic subalgebra \( q \) between \( q = g \) and \( q = a \theta \) invariant Borel subalgebra of \( g \).

We have to point out that the knowledge of \([3]\) is a necessary prerequisite to read this paper. If an argument or construction needed at some stage of this paper is parallel to that in \([3]\) then instead of repeating them, we simply refer to \([3]\).

\[ \text{§1. } \theta\text{-stable parabolic subalgebras} \]

As in the introduction, \( g = k + p \) is the complexified Cartan decomposition arising from a real one \( g_0 = k_0 + p_0 \). Let \( \theta \) be the Cartan involution. Let \( b \) be the complexification of a fixed Cartan subalgebra \( b_0 \) of \( k_0 \). Then the centralizer of \( b \) in \( g \) is a \( \theta \) stable Cartan subalgebra \( h \) of \( g \). We can write

\[
(1.1) \quad h = b + a
\]

where \( a = p \cap h \). Let \( a_0 = a \cap g_0 \) and \( h_0 = h \cap g_0 \). Let \( \Delta \) be the set of roots of \((g, h)\). For \( \alpha \) in \( \Delta \), denote by \( g^\alpha \) the corresponding rootspace.

\[ \text{(1.2) Lemma: Let } r_k \text{ be a Borel subalgebra of } k \text{ containing } b. \text{ Let } q \text{ be a } \theta \text{ stable parabolic subalgebra of } g \text{ containing } h \text{ and assume that } q \text{ contains } r_k. \text{ Then } q \text{ contains a } \theta \text{ stable Borel subalgebra } r \text{ of } g \text{ such that (i) } h \subseteq r \text{ and (ii) } r_k \subseteq r. \]

\[ \text{Proof. Let } u \text{ be the unipotent radical of } q. \text{ Define a } \theta \text{ invariant element } \mu \text{ of } h^X(= \text{Hom}_C(h, C)) \text{ by } \mu(H) = \text{trace}(ad(H)|u). \text{ Let } H'_\mu \text{ in } h \text{ be defined by } \lambda(H'_\mu) = (\lambda, \mu) \text{ for every } \lambda \text{ in } h^X. \text{ (Here and in the following the bilinear form is the nondegenerate one induced by the Killing form of } g). \text{ Then}
\]

\[
(1.3) \quad \theta(H'_\mu) = H'_\mu \text{ so } H'_\mu \in b.
\]

Let

\[
(1.4) \quad \Delta(q) = \{ \alpha \in \Delta|\alpha(H'_\mu) \geq 0 \}.
\]
Then one can see that

\begin{equation}
(1.5) \quad q = h + \sum_{a \in \Delta(q)} g^a .
\end{equation}

Let \( C_k \) be the open Weyl chamber in \( ib_0 \) for \( (k, b) \) defined by the Borel subalgebra \( r_k \). Since we assumed that \( r_k \subseteq q \), it follows from 1.5 that

\begin{equation}
(1.6) \quad H'_\mu \in \tilde{C}_k = \text{the closure of } C_k .
\end{equation}

Let \( \alpha \) be in \( \Delta \). If \( \alpha \) is identically zero on \( b \), it would follow that \( b \) is not maximal abelian in \( k \). Hence \( \alpha \) is not identically zero on \( b \). Let \( C'_k \) be the open subset of \( C_k \) got by deleting points of \( C_k \) where some \( \alpha \) belonging to \( \Delta \) vanishes. Then \( C'_k \) is the disjoint union

\begin{equation}
(1.7) \quad C'_k = \bigcup_{i=1}^{N} C'_{k,i}
\end{equation}

of its connected components and one has

\begin{equation}
(1.8) \quad \tilde{C}_k = \bigcup_{i=1}^{N} \tilde{C}'_{k,i} .
\end{equation}

Choose an index \( M \) between 1 and \( N \) such that

\begin{equation}
(1.9) \quad H'_\mu \in \tilde{C}'_{k,M} .
\end{equation}

Now choose an element \( X_j \) in \( C'_{k,j} \) and consider the weight space decomposition of \( g \) with respect to \( ad(X_j) \). We now define a Borel subalgebra \( r^j \) of \( g \) by

\begin{equation}
(1.10) \quad r^j = \text{the sum of the eigenspaces for } ad(X_j) \\
\text{with nonnegative eigenvalues} .
\end{equation}

If we define

\begin{equation}
(1.11) \quad P^j = \{ \alpha \in \Delta | \alpha(X_j) > 0 \}
\end{equation}

then clearly \( P^j \) is a positive system of roots in \( \Delta \) and \( r^j = h + \Sigma_{\alpha \in P^j} g^\alpha \). Since \( X_j \) belongs to \( k \) clearly both \( r^j \) and \( P^j \) are \( \theta \) stable. 1.9 implies that for every \( \alpha \) in \( P^M \), \( \alpha(H'_\mu) \) is nonnegative. Hence from 1.4 and 1.5

\begin{equation}
(1.12) \quad r^M \subseteq q .
\end{equation}
Also since $X_M$ belongs to $C_k$, (1.10) implies that

(1.13) \[ r_k \text{ is contained in } r^M. \] (q.e.d.)

(1.14) COROLLARY: Let $r_k$ be as in Lemma 1.2. Let $r$ be a $\theta$ stable Borel subalgebra of $g$ containing $r_k$. Then $r$ equals one of the $N$ Borel subalgebras $r^l$ of (1.10).

PROOF: Since $r$ contains $b$, $r$ contains a Cartan subalgebra of $g$ containing $b$. $h$ is the unique Cartan subalgebra of $g$ containing $b$. Hence $r$ contains $h$. In the proof of Lemma 1.2 take $q = r$. Then it is seen $r = r^M$.

(q.e.d.)

Rather than starting with a Borel subalgebra $r_k$ of $k$ containing $b$, we want to start with an arbitrary $\theta$ invariant parabolic subalgebra of $g$ and recover the set up in Lemma 1.2. For this we prove the following lemma.

(1.15) LEMMA: Let $q$ be an arbitrary $\theta$ stable parabolic subalgebra of $g$. Then $q$ contains a Borel subalgebra of $k$.

PROOF: Let $Ad(g)$ be the adjoint group of $g$ and $Q$ the parabolic subgroup with Lie algebra $q$. Let $G''$ be the compact form of $Ad(g)$ with Lie algebra $k_0 + ip_0$. Note that $G''$ is $\theta$-stable. It is well known that $G'' \cap Q$ is a compact form of a reductive Levi factor of $Q$ (cf. [8, §1.2]). But $G'' \cap Q$ is $\theta$ stable since $G''$ and $Q$ are $\theta$ stable. Thus, going to the Lie algebra level, $q$ has a reductive Levi supplement which is $\theta$ stable. In this reductive Levi supplement we can surely find some $\theta$ stable Cartan subalgebra $h'$ of $g$. Then, as in the proof of Lemma 1.2, we can find an element $H'_\mu$ in $h'$ such that $\theta(H'_\mu) = H'_\mu$ and such that $q$ is the sum of the nonnegative eigenspaces of $ad(H'_\mu)$. Since $H'_\mu$ lies in $h' \cap k$, clearly it follows that $q$ contains a Borel subalgebra of $k$.

(q.e.d.)

(1.16) COROLLARY: Let $r$ by any $\theta$ stable Borel subalgebra of $g$. Then $r \cap k$ is a Borel subalgebra of $k$. 

§2. The objects $r$, $r'$, $P$, $P'$ and the choice of $P''$ associated with a $\theta$
stable parabolic subalgebra $q$

Now let $q$ be a $\theta$ stable parabolic subalgebra of $g$. By (1.15) we can
find a Borel subalgebra $r_k$ of $k$ contained in $q$. We fix a Cartan
subalgebra $b_0$ of $k_0$ contained in $r_k$. Let $a_0$ be the centralizer of $b_0$ in
$p_0$. Then $h_0 = b_0 + a_0$ is a $\theta$ stable Cartan subalgebra of $g_0$. Let
$h = b + a$ be its complexification. Note that $h \subseteq q$. By (1.2), we can
find a $\theta$ stable Borel subalgebra $r$ of $g$ such that $r_k \subseteq r$ and $r \subseteq q$. One
has then $h \subseteq r$. There is a unique Borel subalgebra $r'$ of $g$ contained in
$q$ such that

\begin{equation}
(2.1) \quad r \cap r' = h + u, \text{ where } u \text{ is the unipotent radical of } q.
\end{equation}

Since $\theta(r')$ has the same property, we have $\theta(r') = r'$. Let $r'_k = r' \cap k$. Then by (1.16), $r'_k$ is a Borel subalgebra of $k$. We observe that $r'_k$ is
the unique Borel subalgebra of $k$ such that

\begin{equation}
(2.2) \quad r_k \cap r'_k = b + u_k, \text{ where } u_k \text{ is the unipotent radical of } q_k (= q \cap k).
\end{equation}

We denote by $W_k$ the Weyl group of $(k, b)$ and by $W_g$ the Weyl
group of $(g, h)$. $W_k$ is naturally imbedded in $W_g$ as follows: if $s$
belongs to $W_k$ then $s$ normalizes $b$, hence also normalizes the cen-
tralizer of $b$ in $g$ which is precisely $h$. Thus $s$ belongs to $W_g$.

We will now define two distinguished elements of the Weyl group $W_k$. Let $t$ be the unique element of $W_k$ such that $t(P_k) = -P_k$. Next we
denote by $\tau$ the unique element of the Weyl group $W_k$ such that
$\tau(P_k) = P'_k$. The class of $U(g)$ modules associated to $q$ will be
parametrized by some subsets of $h^X$. We now prepare to describe
these. Let $\Delta_k$ be the set of roots for $(k, b)$. Whenever possible we will
denote elements of $\Delta_k$ by $\varphi$ while elements of $\Delta$ (= the roots of $(g, h)$)
will be denoted by $\alpha$. For a root $\varphi$ in $\Delta_k$, denote by $X_\varphi$ a nonzero root
vector in $k$ of weight $\varphi$. For $\alpha$ in $\Delta$, we denote by $E_\alpha$ a nonzero root
vector in $g$ of weight $\alpha$. Let $P$ and $P'$ be the sets of positive roots in
$\Delta$ defined respectively by $r$ and $r'$. Next let $P_k$ and $P'_k$ be the sets of
positive roots in $\Delta_k$ defined respectively by $r_k$ and $r'_k$. Let $\delta$ and $\delta'$
denote half the sum of the roots in $P$ and $P'$ respectively and let $\delta_k$
and $\delta'_k$ denote half the sum of the roots in $P_k$ and $P'_k$ respectively.
Let \( P'' \) be a \( \theta \) stable positive system of roots in \( \Delta \) such that if \( r'' \) is the corresponding \( \theta \) stable Borel subalgebra of \( g \) then

\[
(2.3) \quad r'' \supseteq r'_k \quad \text{and} \quad P'' \supseteq P' \cap -P.
\]

\[\text{(2.5) REMARK: If one takes } P'' = P' \text{ then (2.3) and (2.4) are clearly satisfied. If } q \text{ is a Borel subalgebra, then } P' = P \text{ and any } P'' \text{ which satisfies (2.3) also satisfies (2.4). If } q = g, \text{ then } P' = -P; \text{ the only candidate which satisfies (2.3) and (2.4) is } P'' = P'.\]

We can now describe the modules that we want to construct. As usual for \( \alpha \) in \( P \) denote by \( H_\alpha \) the element of \( ib_\alpha + a_\alpha \) such that \( \lambda(H_\alpha) = 2(\lambda, \alpha)/(\alpha, \alpha) \) for every \( \lambda \) in \( h^X \). Similarly for \( \varphi \) in \( P_k \), denote by \( H_\varphi^k \) the element of \( ib_\varphi \) such that \( \lambda(H_\varphi^k) = 2(\lambda, \varphi)/(\varphi, \varphi) \) for every \( \lambda \) in \( b^X \). (Note: The Killing form of \( g \) induces a nondegenerate bilinear form on \( b \) which in turn induces one on \( b^X \).

Let \( F(P'': q, r) \) be the set of all elements \( \mu \) in \( h^X \) with the following properties:

\[
(2.6) \quad \mu(H_\alpha) \text{ is a nonnegative integer for every } \alpha \text{ in } P''.
\]

\[
(2.7) \quad \mu(H_\varphi^k) \text{ is nonzero for every } \varphi \text{ in } P_k \text{ and } \mu(H_\varphi) \text{ is nonzero for every } \alpha \text{ in } P \cap -P'.
\]

\[\text{EXAMPLE: Suppose } \mu \text{ belonging to } h^X \text{ is such that } \mu(H_\alpha) \text{ is a positive integer for every } \alpha \text{ in } P''. \text{ Then one can show that } \mu \text{ belongs to } F(P'': q, r). \text{ The method of showing that } \mu(H_\varphi^k) \text{ is nonzero for every } \varphi \text{ in } P_k \text{ is found in the proof of (3.6).}\]

We now use some definitions and notations from [3, §§2, 5] (cf. also §§3, 5 here). Let \( U_k \) be the centralizer of \( k \) in \( U(g) \). Let \( \mu \in F(P''': q, r) \). Our aim is to construct a \( k \)-finite irreducible \( U(g) \) module, denoted \( D_{P''': \mu}(\mu) \) in which the irreducible \( k \) type with highest weight \(-t(\tau_\mu + \tau_\delta - \tau_\delta_k - \delta_k)\) (cf. 3.7) occurs with multiplicity one and such that on the corresponding isotypical \( U(k) \) submodule, elements of \( U_k \) act by scalars given by the homomorphism \( \chi_{P,-\mu-\delta} \) (cf. §5).

\[\text{(2.8) REMARK: Fix } q \text{ and } r. \text{ For any compatible choice of } P'' \text{ and for any element } \mu \text{ in } F(P'': q, r), \text{ we will show (cf. 3.6) that (i) } -\mu - \delta(H_\alpha) \text{ is a nonnegative integer for every } \alpha \text{ in } P \cap -P' \text{ and (ii) } \tau_\mu + \tau_\delta - \tau_\delta_k - \delta_k(H_\varphi^k) \text{ is a nonnegative integer for every } \varphi \text{ in } P_k. \text{ Now define } \overline{F}(q, r) \text{ to consist of all } \mu \text{ in } h^X \text{ satisfying (i) and (ii) above. In general } \overline{F}(q, r) \text{ properly contains } U_{P'}F(P'': q, r). \text{ Our constructions}\]
and proofs in §§3, 4, 5 go through perfectly well for any $\mu$ in $\overline{F}(q, r)$ and so we do have a $k$-finite irreducible $U(g)$ module in which the irreducible $k$ type with highest weight $-t(\mu + \tau \delta - \tau \delta_k - \delta_k)$ occurs with multiplicity one and such that on the corresponding isotypical $U(k)$ submodule elements of $U^k$ act by scalars given by $\chi_{\tau \mu - \mu - \delta}$. We have restricted ourselves to the subsets $F(P^\prime: q, r)$ rather than all of $\overline{F}(q, r)$ only because condition (ii) is the definition of $F(q, r)$ is quite incomprehensible.

§3

Choose and fix an element $\mu$ in $F(P^\prime: q, r)$ as in §2 (cf. (2.6) and (2.7)). For facts about Verma modules that we will be using we refer to [1, 2, 5, 6].

Let $M$ be any $U(g)$ module. Let $Q$ be a subset of $\Delta_k$. An element $v$ of $M$ is said to be $Q$ extreme if $X_\varphi \cdot v = 0$ for every $\varphi$ in $Q$. For $\lambda$ in $b^X$, $v$ is called a weight vector of weight $\lambda$ with respect to $b$ if $H \cdot v = \lambda(H) \cdot v$ for all $H$ in $b$. By $J(M)$ we denote the set of all $\lambda$ in $b^X$ for which there exists a nonzero weight vector of weight $\lambda$ in $M$, which is $P_k$ extreme where $P_k$ is the positive system of roots in $\Delta_k$ defined in §2. For $\varphi$ in $\Delta_k$, $M$ is said to be $X_\varphi$ free if $X_\varphi \cdot v = 0$ implies $v = 0$. For a subalgebra $s$ of $g$, $M$ is said to be $s$-finite if every vector of $M$ lies in a finite dimensional $s$ submodule of $M$. For any $\eta$ in $\pi_k$, let $m(\eta)$ denote the subalgebra of $g$ spanned by the elements $X_\eta, X_{-\eta}$ and $H_\eta^k$. For the notion of $U(k)$ module of ‘type $P_\nu$’ we refer to [3, §2].

Let $P_0$ be a positive system of roots of $\Delta$ and let $\lambda \in h^X$. The Verma module $V_{g,P_0,\lambda}$ of $U(g)$ is defined as follows: It is the quotient of $U(g)$ by the left ideal generated by the elements $H - \lambda(H), (H \in h)$ and $E_\alpha (\alpha \in P_0)$. The Verma modules of $U(k)$ are defined similarly. We will suppress $g$ and write $V_{P_0,\lambda}$ for the Verma module $V_{g,P_0,\lambda}$.

We have the inclusions $h \subset r \subset q$ (cf. §2). Let $\pi$ be the set of simple roots for $P$. The parabolic subalgebras of $g$ containing $r$ are in one to one correspondence with subsets of $\pi$. The subset of $\pi$ corresponding to $q$ is got as follows: Let $\sigma$ in $h^X$ be defined by $\sigma(H) = \text{trace}(ad H)|u$. Then

$$\pi(q) = \{ \alpha \in \pi | (\sigma, \alpha) = 0 \}$$

From standard facts about parabolic subalgebras (cf. [8, §1.2]) we know that elements of $P \cap -P'$ are of the form $\Sigma m_\alpha \alpha$ where $m_\alpha$ are nonnegative integers and $\alpha_\alpha$ are in $\pi(q)$. For $\alpha$ in $\Delta$ the element $s_\alpha$ of
$W_g$ is the reflection corresponding to $\alpha$. It is given by $s_\alpha(\lambda) = \lambda - 2(\lambda, \alpha)/(\alpha, \alpha) \cdot \alpha$. We now define a $U(g)$ module $W_1$ by

$$W_1 = V_{P, -\mu - \delta}$$

considered as a $U(k)$ module it has some nice properties.

(3.4) Lemma: $W_1$ considered as a module for $U(k)$ is a weight module with respect to $b$; i.e. $W_1$ is the sum of the weight spaces with respect to $b$. Denoting also $-\mu - \delta$ the restriction of $-\mu - \delta$ to $b$, all the weights are of the form $-\mu - \delta - \Sigma n_i \varphi_i$ where $\varphi_i$ are elements of $P$ and $n_i$ are positive integers. Finally the weight spaces are finite dimensional and the weight space corresponding to $-\mu - \delta$ is one dimensional.

Proof: Since as a $U(g)$ module $W_1$ is the sum of weight spaces with respect to $h = b + a$, the first statement is clear. Since no root $\alpha$ in $\Delta$ is identically zero on $b$, we can pick up an element $H$ in $b$ such that for every $\alpha$ in $P, \alpha(H)$ is real and positive. As a $U(g)$ module, the weights of $W_1$ with respect to $h$ are of the form $-\mu - \delta - \Sigma m_i \alpha_i$ ($\alpha_i \in P, m_i$ nonnegative integers). By considering the action of $H$ it is clear that weight spaces of $W_1$ with respect to $b$ are finite dimensional and the weight space of $b$ with weight $-\mu - \delta$ is one dimensional. Finally since $P$ is $\theta$ stable the restriction to $b$ of the weights with respect to $h$ are of the form $-\mu - \delta - \Sigma n_i \varphi_i$ where $\varphi_i$ are in $P$ and $n_i$ nonnegative integers.

(q.e.d.)

(3.5) Corollary: The $U(k)$ submodule of $W_1$ generated by the unique weight vector in $W_1$ of weight $-\mu - \delta$ is isomorphic to the $U(k)$ Verma module $V_{k, P, -\mu - \delta}.$ $W_1$ is $X_\varphi$ free for every $\varphi$ in $P_k$.

Proof: Let $v_1$ be the nonzero weight vector in $W_1$ of weight $-\mu - \delta \cdot v_1$ is killed by every element of $[r, r]$ hence in particular by every element of $[r_k, r_k]$. On the other hand let $\bar{r}$ be the unique Borel subalgebra of $g$ such that $\bar{r} \cap r = h$ and let $n(\bar{r})$ be the unipotent radical of $\bar{r}$. If $\bar{r}_k = \bar{r} \cap k$, then $\bar{r}_k$ is the unique Borel subalgebra of $k$ such that $\bar{r}_k \cap r_k = b$. Let $U(n(\bar{r}))$ and $U(n(\bar{r}_k))$ denote the corresponding enveloping algebras considered as subalgebras of $U(g)$. One knows that $W_1$ is $U(n(\bar{r}))$ free, [2]. Hence in particular it is $U(n(\bar{r}_k))$ free. The corollary now follows from [2, 7.1.8].

(q.e.d.)
There is an ascending chain of $U(k)$ Verma modules containing $V_{k,P_k,-\mu-\delta}$. This chain will give rise to a chain of $U(g)$ modules, which is fundamental in the work [3].

Recall the two distinguished elements $t$ and $\tau$ of $W_k$ from §2. The highest weight of the special irreducible representation of $k$ which the $U(g)$ module $D_{\mu,\mu,\tau}(\mu)$ will contain is described in the corollary to the lemma below.

(3.6) Lemma: (i) $-\mu - \delta(H_\alpha)$ is a nonnegative integer for every $\alpha$ in $P \cap -P'$ and (ii) $\tau_\mu + \tau \delta - \tau \delta_k - \delta_k(H_\mu)$ is a nonnegative integer for every $\varphi$ in $P_k$.

Proof: By (2.4), (2.7) and (2.8), one sees that $-\mu(H_\alpha)$ is a positive integer for every $\alpha$ in $P \cap -P'$. The elements of $P \cap -P'$ are nonnegative integral linear combinations of elements of $\pi(q)$. Since $\delta(H_\alpha) = 1$ for every $\alpha$ in $\pi(q)$ it now follows that $-\mu - \delta(H_\alpha)$ is a nonnegative integer for every $\alpha$ in $P \cap -P'$.

To prove (ii) first suppose $\varphi$ lies in $P_k \cap P_k$. We will show that $\tau \mu - \delta_k(H_\mu)$ and $\tau \delta - \delta_k(H_\delta)$ are both nonnegative integers. For this it is enough to show that $\tau \mu(H_\mu)$ is a positive integer for every $\varphi$ in $P_k$ and that $\tau \delta(H_\delta)$ is a positive integer for every $\varphi$ in $\tau P_k$. By (2.6) there exists a finite dimensional representation of $g$ having a weight vector $v$ of weight $\mu$ with respect to the Cartan subalgebra $h$ and such that $v$ is annihilated by $[r'', r']$ (cf. (2.3)). Since $r'_k \subseteq r''$, $v$ is in particular annihilated by $[r'_k, r'_k]$. It is clear from this that $\mu(H_\mu)$ is a nonnegative integer for every $\varphi$ in $P_k$. In view of (2.7), $\mu(H_\mu)$ is then a positive integer for every $\varphi$ in $P_k$. Note that $\tau P_k = P_k$. Hence $\tau \mu(H_\mu)$ is a positive integer for every $\varphi$ in $P_k$. It remains to show that $\tau \delta(H_\delta)$ is a positive integer for every $\varphi$ in $\tau P_k$. For this consider the representation $\rho$ of $g$ having a weight vector $v$ of weight $\delta$ with respect to the Cartan subalgebra $h$ and such that $v$ is annihilated by $[r, r]$. Clearly then $v$ is annihilated by $[r_k, r_k]$, hence $\delta(H_\delta)$ is a nonnegative integer for every $\varphi$ in $P_k$. To show that $\delta(H_\delta)$ is nonzero we give the following reason: one can easily see that the stabilizer of $v$ in $g$ is exactly $r$. If $\delta(H_\delta)$ is zero for some $\varphi$ in $P_k$ then $X_\varphi$ would stabilize $v$. But $X_\varphi$ does not belong to $r$. Hence $\delta(H_\delta)$ is a positive integer for every $\varphi$ in $P_k$, so that $\tau \delta(H_\delta)$ is a positive integer for every $\varphi$ in $\tau P_k$.

Now suppose $\varphi$ lies in $P_k \cap -P_k$. Let $r(q)$ be the maximal reductive subalgebra of $q$ defined by $r(q) = h + \Sigma_{\alpha \in P \cap -P} (g^\alpha + g^{-\alpha})$. By (ii) $-\mu - \delta(H_\alpha)$ is a nonnegative integer for every $\alpha$ in $P \cap -P'$. Hence, if $n(r(q)) = \Sigma_{\alpha \in P \cap -P} g^\alpha$, there exists a finite dimensional representation of $r(q)$ and a weight vector for $h$ of weight $-\mu - \delta$ annihilated by all of
n_{r(q)}, hence in particular by \( k \cap n_{r(q)} \). Observe that \( P_k \cap -P'_k \) is precisely the set of roots in \( P_k \), whose corresponding root spaces span \( k \cap n_{r(q)} \). Thus there exists a finite dimensional representation of \( b + \Sigma_{\varphi \in P_k \cap -P'_k} (C \cdot X_\varphi + C \cdot X_{-\varphi}) \) with a weight vector for \( b \) of weight \( -\mu - \delta \) annihilated by \( X_\varphi \) for every \( \varphi \) in \( P_k \cap -P'_k \). Hence we conclude that \( -\mu - \delta(H^k_\varphi) \) is a nonnegative integer for every \( \varphi \) in \( P_k \cap -P'_k \). Since \( -\tau(P_k \cap -P'_k) = P_k \cap -P'_k \), \( \tau(\mu + \delta)(H^k_\varphi) \) is a nonnegative integer for every \( \varphi \) in \( P_k \cap -P'_k \). On the other hand \( \tau \delta_k = \delta_k^k = \text{half the sum of the roots in } P_k, \) while \( \delta_k + \delta_k^k(H^k_\varphi) = 0 \) for every \( \varphi \) in \( P_k \cap -P'_k \). Thus \( \tau \mu + \tau \delta - \tau \delta_k - \delta_k^k(H^k_\varphi) \) is a nonnegative integer for every \( \varphi \) in \( P_k \cap -P'_k \).

This completes the proof of (3.6). (q.e.d.)

(3.7) **Corollary:** \(-t(\tau \mu + \tau \delta - \tau \delta_k - \delta_k^k)(H^k_\varphi)\) is a nonnegative integer for every \( \varphi \) in \( P_k \).

**Proof:** Clear since \(-tP_k = P_k\). (q.e.d.)

Let \( \pi_k \) be the set of simple roots of \( P_k \). For \( \varphi \) in \( P_k \), let \( s_\varphi \) be the reflection \( s_\varphi(\lambda) = \lambda - \lambda(H_\varphi) \varphi \) of \( b^X \). If \( \varphi \) lies in \( \pi_k, s_\varphi \) is called a simple reflection. For \( w \) in \( W_k \), the length \( N(w) \) of \( w \) is the smallest integer \( N \) such that \( w \) is a product of \( N \) simple reflections. A reduced word for \( w \) is an expression of \( w \) as a product of \( N(w) \) simple reflections. Choose any reduced word for the element \( \tau t \) of \( W_k \). Following the notation in [5, §4.15], we write it as

\[
\tau t = s_1 s_2 \ldots s_m
\]

where \( s_i = s_{\eta_i}, \eta_i = \varphi_{\eta_i}, \varphi_{\eta_i} \in \pi_k \). For \( \lambda \) in \( b^X \) and \( w \) in \( W_k \) write \( w'(\lambda) = w(\lambda + \delta_k) - \delta_k \). Having chosen the element \( \mu \) in \( F(P'': q, r) \), we now define elements \( \mu_i \) of \( b^X \) as follows:

\[
\mu_{m+1} = -t(\tau \mu + \tau \delta - \tau \delta_k - \delta_k) \quad \text{and}
\mu_i = (s_i s_{i+1} \ldots s_m)\mu_{m+1} \quad (i = 1, \ldots, m)
\]

(3.9) Note that \( \mu_1 = (\tau t)\mu_{m+1} = -\mu - \delta \) and that \( \mu_1 \) and \( \mu_{m+1} \) are independent of the reduced expression (3.8). We now define the positive integers \( e_i \) by

\[
e_i = \mu_{i+1} + \delta_k(H^k_{\mu_i}) \quad (i = 1, \ldots, m).
\]
With $\mu_i$ defined as above, the following inclusion relations between Verma modules are well known [2, 6]:

\[(3.11) \quad V_{k, P_k, \mu_1} \subseteq V_{k, P_k, \mu_2} \subseteq \cdots \subseteq V_{k, P_k, \mu_{m+1}}.\]

Define elements $v_1, v_2, \ldots, v_{m+1}$ of $V_{k, P_k, \mu_{m+1}}$ as follows: $v_{m+1}$ is the unique nonzero weight vector of $V_{k, P_k, \mu_{m+1}}$ of weight $\mu_{m+1}$. For $i = 1, 2, \ldots, m$, $v_i = X_{-\mu_i} \cdot v_i+1$. Then one knows that $v_i$ is of weight $\mu_i$ and that $V_{k, P_k, \mu_i} = U(k)v_i$. Associated to the reduced word (3.8) and $\mu$ in $F(P'' : q, r)$ is a fundamental chain of $U(g)$ modules: $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$. It will turn out that $W_1$ and $W_{m+1}$ are independent of the reduced expression (3.8). They are defined as follows: $W_1$ is defined to be $V_{P, -\mu - \delta}$ as in (3.3). Then $W_{m+1}$ is given by the following lemma.

\[(3.12) \text{Lemma: There exists a } U(g) \text{ module } W_{m+1} = U(g) \cdot v_{m+1} \text{ such that (a) } W_1 \text{ is a } U(g) \text{ submodule of } W_{m+1}, (b) v_1 \text{ belongs to } U(k)v_{m+1}, (c) v_{m+1} \text{ is a } P_k \text{ extreme weight vector (with respect to } b) \text{ of weight } \mu_{m+1}, (d) W_{m+1} \text{ is } X_\varphi \text{ free for all } \varphi \text{ in } P_k \text{ and (e) } W_{m+1} \text{ is a sum of } U(k) \text{ submodules of type } P_k.\]

**Proof:** Start with the inclusion of $V_{k, P_k, \mu_1}$ in $W_1$ given by Corollary 3.5 and the inclusion of $V_{k, P_k, \mu_1}$ in $V_{k, P_k, \mu_{m+1}}$ given by 3.11. By 3.5 we know that $W_1$ is $X_\varphi$ free for every $\varphi$ in $P_k$. Now [3, Lemma 4] gives us the module $W_{m+1}$ with the properties required in the lemma. (One easily sees that the results of [3, §2] do not depend on the assumption there that rank of $g = \text{rank of } k$). \(\text{q.e.d.}\)

\[(3.13) \text{Remark: If } V \text{ and } \tilde{V} \text{ are Verma modules for, say, } U(k) \text{ then the space of } U(k) \text{ homomorphisms of } V \text{ into } \tilde{V} \text{ has dimension equal to zero or one. Thus the inclusion of } V_{k, P_k, \mu_1} \text{ into } V_{k, P_k, \mu_{m+1}} \text{ given by (3.11) is independent of the reduced expression (3.8) for } \tau \text{. Hence also the } U(g) \text{ module } W_{m+1} \text{ and the inclusion of } W_1 \text{ in } W_{m+1} \text{ with the properties listed in Lemma 3.12 can be chosen to be independent of the reduced expression (3.8).}

Having defined $W_1$ and $W_{m+1}$ as above, now for any given reduced word for $\tau$ such as (3.8), we define submodules $W_2, W_3, \ldots, W_m$ of $W_{m+1}$ by

\[(3.14) \quad W_i = U(g)v_i\]

where $v_i$ are the elements of $W_{m+1}$ defined after (3.11). We have
$W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ because $v_i$ belongs to $U(k)v_{i+1}$, $(i = 1, \ldots, m)$. The properties of this chain of $U(g)$ modules are summarized below from the work of [3, §3]:

(3.15) $W_i = V_{P_i - \mu_i - \delta}$ and each $W_i$ is the sum of its weight spaces with respect to $b$. Moreover as a $U(k)$ module $W_i$ is the sum of $U(k)$ submodules of type $P_k$.

(3.16) Each $W_i$ is a cyclic $U(g)$ module with a cyclic vector $v_i$, which is a $P_k$ extreme weight vector of weight $\mu_i$ with respect to $b$, $i = 1, \ldots, m + 1$.

(3.17) The $P_k$ extreme vectors of weight $\mu_i$ in $W_i$ are scalar multiples of $v_i$; for $i = 1, \ldots, m + 1$, the vector $v_i$ does not belong to $W_{i-1}$.

(3.18) Each $W_i$ is $X_\varphi$ free for every $\varphi$ in $P_k$ and $W_{i+1}/W_i$ is $m(\eta_i)$ finite ($i = 1, \ldots, m$).

(3.19) $v_i = X_{\gamma_i} v_{i+1}$ ($i = 1, \ldots, m$).

(3.20) Let $w$ be in $W_k$. Let $i = 1, \ldots, m$. Suppose $w'(\mu_{m+1})$ belongs to $J(W_i)$. Then $N(w)$ equals at least $m + 1 - i$.

We will not prove the properties (3.15) to (3.20) here since they are essentially proved in [3, Lemma 5]. Though (3.20) has the same form as [3, Lemma 5, vi] its proof is different in our case. It is important to first know the case $i = 1$ of (3.20) to carry over the inductive arguments of [3, §3] to our situation. To this end we prove the following lemma. Before that we make the following remark.

(3.21) Remark: Let $H'_q$ be the element of $h$ defined by $(H'_q, H) = \text{trace } (\text{ad } H|u)$, for every $H$ belonging to $h$, where $u$ is the unipotent radical of $q$. Since $q$ and $h$ are $\theta$ invariant $\theta(H'_q) = H'_q$; hence $H'_q$ belongs to $b$. One can easily prove the following: For every $\alpha$ in $P \cap -P'$, $\alpha(H'_q)$ equals zero; for every $\alpha$ in $P \cap P'$, $\alpha(H'_q)$ is a positive real number; and for every $\varphi$ in $P_k \cap -P'_k$, $\varphi(H'_q)$ equals zero while for every $\varphi$ in $P_k \cap P'_k$, $\varphi(H'_q)$ is a positive real number. (Observe that any $\varphi$ in $P_k \cap -P'_k$ is the restriction to $b$ of some $\alpha$ in $P \cap -P'$).

Now we come to the lemma which is basic to carry over the inductive arguments of [3, §3].
(3.22) **Lemma:** Let $w$ be in $W_k$. Suppose $w'(\mu_{m+1})$ belongs to $J(W)$. Then $N(w)$ is greater than or equal to $m$.

**Proof:** Since $w'(\mu_{m+1})$ belongs to $J(W)$ it is in particular a weight of $W_i$ of for $b$. Hence by (3.4), $w'(\mu_{m+1})$ is of the form $\mu_1 - \sum n_i \alpha_i | b$, where $n_i$ are nonnegative integers and $\alpha_i$ are in $P$. That is $w(\mu_{m+1} + \delta_k) - \delta_k = \mu_1 - \sum n_i \alpha_i | b = \tau(\mu_{m+1} + \delta_k) - \delta_k - \sum n_i \alpha_i | b$. Thus,

$$\tau(\mu_{m+1} + \delta_k) - w(\mu_{m+1} + \delta_k) = \sum n_i \alpha_i | b.$$

Write $\mu'_{m+1} = - t \mu_{m+1}$. Hence

$$-\tau(\mu'_{m+1} + \delta_k) + wt(\mu'_{m+1} + \delta_k) = \sum n_i \alpha_i | b \tag{3.23}$$

where $n_i$ are nonnegative integers and $\alpha_i$ are in $P$. The left side of the equality in (3.23) is the sum of $wt(\mu'_{m+1} + \delta_k) - (\mu'_{m+1} + \delta_k)$ and $(\mu'_{m+1} + \delta_k) - \tau(\mu'_{m+1} + \delta_k)$. We claim that (3.23) implies

$$P_k \cap - wt P_k \text{ is contained in } P_k \cap - \tau P_k. \tag{3.24}$$

To see this enumerate the elements of $P_k \cap - wt P_k$ in a sequence $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ such that $\epsilon_1$ is a simple root of $P_k$ and $\epsilon_{i+1}$ is a simple root of $s_{\epsilon_1} s_{\epsilon_2} \ldots s_{\epsilon_i} P_k$ $(i = 1, \ldots, k - 1)$. Then $wt = s_{\epsilon_1} \ldots s_{\epsilon_i}$ (cf. (5.4.15.10) and [7, 8.9.13]). By induction on $i$ one can show that $(\mu'_{m+1} + \delta_k) - s_{\epsilon_1} \ldots s_{\epsilon_i} (\mu'_{m+1} + \delta_k)$ can be written as $\sum d_{ij} \epsilon_j$ where $d_{ij}$ are positive integers. Thus $(\mu'_{m+1} + \delta_k) - wt(\mu'_{m+1} + \delta_k)$ can be written as $d_1 \epsilon_1 + d_2 \epsilon_2 + \cdots + d_k \epsilon_k$ where $d_1$ are positive integers. Similarly $(\mu'_{m+1} + \delta_k) - \tau(\mu'_{m+1} + \delta_k)$ can be written as $d'_1 \epsilon'_1 + d'_2 \epsilon'_2 + \cdots + d'_k \epsilon'_k$ where $d'_i$ are positive integers and $(\epsilon'_1, \ldots, \epsilon'_k)$ is an enumeration of $P_k \cap - \tau P_k$. With these observations we can write

$$-\tau(\mu'_{m+1} + \delta_k) + wt(\mu'_{m+1} + \delta_k) \tag{3.25}$$

where $d_1', \ldots, d_k'$, $d_1, \ldots, d_k$ are positive integers. Let $H'_q$ be the element of $h$ defined by $(H'_q, H) = \text{trace}(ad H|u)$, where $u$ is the unipotent radical of $q$. Then $H'_q$ belongs to $b$. We can apply remark (3.21) to (3.25) and conclude that $[-\tau(\mu'_{m+1} + \delta_k) + wt(\mu'_{m+1} + \delta_k)](H'_q)$ is a strictly negative real number unless (3.24) holds. But by looking at the right hand side of (3.23) and applying remark (3.21), we see that $[-\tau(\mu'_{m+1} + \delta_k) + wt(\mu'_{m+1} + \delta_k)](H'_q)$ is a nonnegative real number.
Thus we have proved the validity of (3.24). Now (3.24) implies that $N(wt)$ is less than or equal to $N(\tau)$. But note that $N(wt) = N(t) - N(w)$, while $N(\tau) = N(t) - N(\tau t) = N(t) - m$. Hence $N(w)$ is greater than or equal to $m$.

(q.e.d.)

(3.22) enables us to carry over the inductive arguments in [3, §3] without any further change and obtain the properties (3.15) to (3.20).

§4. The $k$-finite quotient $U(g)$ module of $W_{m+1}$

The difference between the special situation in [3] and our more general situation becomes more apparent in this section which parallels [3, §4].

Start with an arbitrary reduced word (3.8) for $\tau t$ and let $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ be a fundamental chain of $U(g)$ modules satisfying (3.15) through (3.20). Recall $W_1 = V_{P,-\mu-\delta}$. Recall the subset $\pi(q) \subseteq \pi$ corresponding to the parabolic subalgebra $q$. For $\alpha$ in $\pi$ and $\lambda$ in $h^X$ define $s^X_\alpha(\lambda) = s_\alpha(\lambda + \delta) - \delta$. By Lemma 3.6, $-\mu - \delta(H_\alpha)$ is a nonnegative integer for every $\alpha$ in $P \cap -P'$, hence in particular for every $\alpha$ in $\pi(q)$. Thus one has the inclusion of the Verma modules $V_{P,s^X\alpha(-\mu-\delta)} \subseteq V_{P,-\mu-\delta}$ for every $\alpha$ in $\pi(q)$. We now define a $U(g)$ submodule

$$W_0 = \sum_{\alpha \in \pi(q)} V_{P,s^X\alpha(-\mu-\delta)} \text{ of } W_1.$$  

As is well known the Verma modules have unique proper maximal submodules. Let $I$ be the proper maximal $U(g)$ submodule of $V_{P,-\mu-\delta}$. Then each $V_{P,s^X\alpha(-\mu-\delta)} (\alpha \in \pi(q))$ is contained in $I$. Hence

$$v_1 \text{ does not belong to } W_0.$$  

Now fix some $i$, $(i = 1, \ldots, m)$. Define a $U(g)$ submodule (relative to some reduced word (3.8) for $\tau t$) $\bar{W}_i$ of $W_{m+1}$ as follows: Let $W_{i,0}$ be the $U(g)$ submodule of all vectors in $W_{m+1}$ that are $m(\eta_i)$ finite mod $W_{i-1}$; once $W_{i,0}, \ldots, W_{i,p-1}$ are defined, $W_{i,p}$ is the $U(g)$ submodule of all vectors in $W_{m+1}$ that are $m(\eta_{i+p})$ finite mod $W_{i,p-1}$, $p = 1, 2, \ldots, m - i$. We have $W_{i,0} \subseteq \cdots \subseteq W_{i,m-i}$. We then define $\bar{W}_i = W_{i,m-i}$. Define

$$\bar{W} = W_m + \bar{W}_1 + \bar{W}_2 + \cdots + \bar{W}_m.$$
Thus for each reduced expression (3.8) for $\tau$, we have defined a $U(g)$ submodule $\tilde{W}$ of $W_{m+1}$.

(4.4) **Proposition:** For any reduced word (3.8) for $\tau$, define the $U(g)$ submodule $\tilde{W}$ of $W_{m+1}$ as above. Then $v_{m+1}$ does not belong to $\tilde{W}$. If $\lambda \in h^X$ is such that $W_{m+1}$ has a nonzero $P_\lambda$ extreme weight vector (with respect to $b$) of weight $\lambda$ which is nonzero mod $\tilde{W}$, then $(\tau^t)\lambda$ is a $P_\lambda$ extreme weight of $W_1/W_0$.

**Proof:** We refer to the proof of [3, Lemma 9].

Since we do not have a full chain of $U(g)$ modules corresponding to a reduced word for $t$ as in [3] but only a shorter chain corresponding to a reduced word for $\tau$, we have to work more to obtain a $k$-finite quotient $U(g)$ module of $W_{m+1}$. We now define

(4.5) $W_\chi = \sum \tilde{W}$, the summation being over all reduced expressions (3.8) for $\tau$.

(4.6) **Lemma:** $v_{m+1}$ does not belong to $W_\chi$. Let $\lambda \in h^X$ be such that there is a $P_\lambda$ extreme vector in $W_{m+1}$ of weight $\lambda$ which is nonzero mod $W_\chi$. Then $(\tau^t)\lambda(H_\phi^k)$ is a nonnegative integer for every $\varphi$ in $P_\lambda \cap -P^t_\lambda$.

**Proof:** $v_{m+1}$ is a $P_\lambda$ extreme weight vector in $W_{m+1}$ of weight $\mu_{m+1}$. From (3.7) and the definition of $\mu_{m+1}$, we know that $\mu_{m+1}(H_\phi^k)$ is a nonnegative integer for every $\varphi$ in $P_\lambda$. Now suppose $v_{m+1}$ belongs to $W_\chi$. Since $W_\chi = \sum \tilde{W}$, $W_\chi$ is a quotient of the abstract direct sum $\bigoplus \tilde{W}$, the summation being over all reduced words (3.8) for $\tau$. We can then apply [3, Lemma 7] and conclude that for some reduced word (3.8) for $\tau$, the corresponding $\tilde{W}$ has a nonzero $P_\lambda$ extreme vector of weight $\mu_{m+1}$. This vector has to be a nonzero scalar multiple of $v_{m+1}$ in view of (3.17). Hence $v_{m+1}$ belongs to that $\tilde{W}$. But this contradicts (4.4). This proves the first assertion in (4.6).

Next let $\lambda$ be as in the lemma. Let $c$ be the reductive component of $q$ defined by $c = h + \Sigma_{a \in P \cap -P}(g^a + g^{-a})$. We claim that $W_1/W_0$ is $c$-finite. For this it is enough to show that the image $\bar{v}_1$ in $W_1/W_0$ of $v_1$ is $c$-finite. For any $\alpha$ in $\pi(q)$ the submodule $V_{\alpha,P_\lambda \cap -P_\lambda}$ of $W_1$ coincides with $U(g)X_\alpha^{(H_\phi^k)+1} \cdot v_1$ (cf. [2, 7.1.15]). Thus we have $W_0 = \Sigma_{\alpha \in \pi(q)} U(g)X_\alpha^{(H_\phi^k)+1} \cdot v_1$. Hence the annihilator in $U(g)$ of $\bar{v}_1$ contains $U(g)X_\alpha^{(H_\phi^k)+1}$ for every $\alpha$ in $\pi(q)$. This suffices in view of [2, 7.2.5] to conclude that $\bar{v}_1$ is $c$-finite. Thus $W_1/W_0$ is $c$-finite.
Let $c_k = c \cap k$. Then in particular $W_1/W_0$ is $c_k$-finite. But note that $c_k = b + \Sigma_{\varphi \in P_k \cap -P_k} (C \cdot X_{\varphi} + C \cdot X_{-\varphi})$.

Now choose some reduced word (3.8) for $\tau \varphi$ and relative to it define $\tilde{W}$ as in (4.3). Note that $\tilde{W} \subseteq W_X$. For $\lambda$ as in the lemma, choose a $P_k$ extreme weight vector $v$ in $W_{m+1}$ which is nonzero mod $W_X$ and is of weight $\lambda$. Then $v$ is in particular nonzero mod $\tilde{W}$. Hence from (4.4), $(\tau \varphi)\lambda$ is a $P_k$ extreme weight of $W_1/W_0$. Since $W_1/W_0$ is $c_k$-finite, it now follows that $(\tau \varphi)\lambda(H^k_\varphi)$ is a nonnegative integer for every $\varphi$ in $P_k \cap -P_k$.

(q.e.d.)

For our proof of the $k$-finiteness of $W_{m+1}/W_k$, we need one more lemma.

(4.7) **Lemma:** Let $\eta$ be in $b^X$. Suppose $\eta(H^k_\varphi)$ is nonnegative for every $\varphi$ in $P_k$. Let $s$ be in $W_k$. Suppose $(\tau ts \varphi)\eta(H^k_\varphi)$ is nonnegative for every $\varphi$ in $P_k \cap -P_k$. Then $N(\tau \varphi) = N(\tau ts \varphi) + N(s^{-1})$.

**Proof:** $(\tau ts \varphi)\eta = \tau ts(\eta + \delta_k) - \delta_k$. Since $\eta(H^k_\varphi)$ is nonnegative for every $\varphi$ in $P_k$, $\tau ts(\eta + \delta_k)(H^k_\varphi)$ is negative for every $\varphi$ in $-\tau ts P_k$. Also $-\delta_k(H^k_\varphi)$ is negative for every $\varphi$ in $P_k$. Hence $(\tau ts \varphi)\eta(H^k_\varphi)$ is negative for every $\varphi$ in $(-\tau ts P_k) \cap P_k$. Hence the assumption implies

(4.8) \[ P_k \cap -\tau ts P_k \subseteq \text{complement of } P_k \cap -P_k \text{ in } P_k. \]

Note that $tP_k = -P_k$ and $\tau P_k = P_k$. So, $-P_k = \tau t P_k$. So, the complement of $P_k \cap -P_k$ in $P_k$ is $P_k \cap -\tau t P_k$. Hence from (4.8) we have

(4.9) \[ P_k \cap (-\tau ts P_k) \subseteq P_k \cap (-\tau t P_k). \]

Let $(\epsilon_1, \epsilon_2, \ldots, \epsilon_j, \epsilon_{j+1}, \ldots, \epsilon_m)$ be an enumeration of the elements of $(-\tau t P_k) \cap P_k$ such that $\epsilon_1$ is a simple root of $P_k$, $\epsilon_2$ is a simple root of $s_{\epsilon_1} P_k$, $\epsilon_{i+1}$ is a simple root of $s_{\epsilon_i}s_{\epsilon_{i-1}} \ldots s_{\epsilon_1} P_k$ ($i = 1, \ldots, m - 1$). Because of (4.9) we can further assume $(\epsilon_1, \ldots, \epsilon_j)$ is an enumeration of $(-\tau ts P_k) \cap P_k$. Let

\[ \varphi_i = s_{\epsilon_1} \ldots s_{\epsilon_{i-1}}(\epsilon_i) \quad (i = 1, \ldots, m) \quad (\varphi_1 = \epsilon_1). \]

Then $\varphi_i$ belongs to $\pi_k$. One can show that $\tau t = s_{\varphi_m} \ldots s_{\varphi_1}$ and a reduced word for $\tau t$ is

(4.10) \[ \tau t = s_{\varphi_1}s_{\varphi_2} \ldots s_{\varphi_m} \]
(cf. [5, 4.15.10] and [7, 8.9.13]). Similarly $\tau ts = s_{\epsilon_1} \ldots s_{\epsilon_i}$ and a reduced word for $\tau ts$ is

$$\tau ts = s_{\varphi_1} \ldots s_{\varphi_i}.$$  

Note that $N(\tau t) = m$ and $N(\tau ts) = \ell$. Now from (4.10) and (4.11) it is clear that $s^{-1} = s_{\varphi_{i+1}} \ldots s_{\varphi_{n}}$ is a reduced word for $s^{-1}$. These observations substantially prove the lemma. 

(q.e.d.)

(4.12) REMARK: With the data assumed in Lemma 4.7 we have actually proved more than what is asserted in (4.7): There exists a reduced word $\tau t = s_{\varphi_1} \ldots s_{\varphi_i} s_{\varphi_{i+1}} \ldots s_{\varphi_n}$ for $\tau t$ such that $s^{-1} = s_{\varphi_{i+1}} \ldots s_{\varphi_n}$.

The following proposition gives the $k$-finite $U(g)$ module quotient of $W_{m+1}$.

(4.13) PROPOSITION: The $U(g)$ module $W_{m+1}/W_\chi$ is $k$-finite.

PROOF: Let $\bar{v}_{m+1}$ be the image of $v_{m+1}$ in $W_{m+1}/W_\chi$. Since $U(g) \bar{v}_{m+1} = W_{m+1}/W_\chi$, it suffices to prove that $U(k) \cdot \bar{v}_{m+1}$ has finite dimension over $C$. For this again, by well known facts [2, 7.2.5] it suffices to prove that the annihilator of $\bar{v}_{m+1}$ in $U(k)$ contains $X^e_\varphi$ for every $\varphi$ in $\pi_k$, where $e(\varphi) = \mu_{m+1}(H^k_\varphi) + 1$ (observe that in view of (3.7), $\mu_{m+1}(H^k_\varphi)$ is a nonnegative integer for every $\varphi$ in $\pi_k$). Thus it suffices to show that for every $\varphi$ in $\pi_k$.

$$X^e_\varphi \cdot v_{m+1} \text{ belongs to } W_\chi.$$  

Suppose (4.14) is not true. Choose a $\varphi$ in $\pi_k$, such that $X^e_\varphi v_{m+1}$ does not belong to $W_\chi$. Then $X^e_\varphi v_{m+1}$ is a $P_k$ extreme vector of weight $s_\varphi(\mu_{m+1})$ in $W_{m+1}$ which is nonzero mod $W_\chi$. Hence by (4.6), $(\tau ts_\varphi) \mu_{m+1}(H^k_\varphi)$ is a nonnegative integer for every $\varphi$ in $P_k \cap -P'_k$. We can now apply (4.7) and (4.12) and conclude that there exists a reduced word

$$\tau t = s_{\varphi_1} s_{\varphi_2} \ldots s_{\varphi_{i-1}} s_{\varphi_i} (\varphi' \in \pi_k)$$

for $\tau t$ such that

$$\varphi'_m = \varphi.$$
Take the reduced word (4.15) for \( \tau t \) in (3.8) and consider the corresponding modules \( W_m \) and \( \tilde{W} \). By definition \( W_m \subseteq \tilde{W} \). But in the fundamental chain \( W_1 \subseteq \cdots \subseteq W_m \subseteq W_{m+1} \) associated to the reduced word (4.15) for \( \tau t \), the module \( W_m \) is simply \( U(g) \cdot X_{r q}^{e(q)} v_{m+1} \). This is clear from the definitions (cf. (3.14) and the definition of \( v_i \) after (3.11)) and (4.16). Thus it follows that \( X_{r q}^{e(q)} v_{m+1} \in \tilde{W} \subseteq W_\lambda \). But this is a contradiction to the hypothesis. Thus (4.14) is true and proved and with that also the \( k \)-finiteness of \( W_{m+1} \big/ W_\lambda \).

(q.e.d.)

§5

Let \( b \) be a Cartan subalgebra of \( k \) and \( h \) its centralizer in \( g \), so that \( h \) is a \( \theta \) stable Cartan subalgebra of \( g \). Let \( P \) be a system of positive roots for \( (g, h) \) such that \( \theta(P) = P \). Let

\[
n^+ = \sum_{\alpha \in P} g^\alpha
\]

and

\[
n^- = \sum_{\alpha \in P} g^{-\alpha}.
\]

The following fact is standard if \( b = h \), but it remains true in our general case.

(5.1) **Lemma:** Let \( U^b \) be the centralizer of \( b \) in \( U(g) \). If the set \( P \) of positive roots satisfies \( \theta P = P \), we have a unique homomorphism

\[
(5.2) \quad \beta_P : U^b \to U(h)
\]

such that for any \( y \) in \( U^b \)

\[
(5.3) \quad y = \beta_P(y)(\text{mod } U(g)n^+).
\]

**Proof:** We have

\[
(5.4) \quad U(g) = U(n^- + h) \bigoplus U(g)n^+
\]

and this decomposition is stable under \( \text{ad} \, H \) for every \( H \) in \( h \), i.e. \( \text{ad} \, H(U(n^- + h)) \subseteq U(n^- + h) \) and \( \text{ad} \, H(U(g)n^+) \subseteq U(g)n^+ \). For \( y \) in \( U^b \), let \( y = y_0 + y_1 \) be its decomposition with respect to (5.4). Define
\( \beta_p(y) = y_0. \) We claim \( \beta_p(y) \) belongs to the subalgebra \( U(h) \) of \( U(n^- + h) \). Since \( y \) is in \( U^b \), \( y_0 \) and \( y_1 \) are also in \( U^b \). Let \( S(n^- + h) \) and \( S(h) \) denote the symmetric algebras and \( \lambda \) the symmetrizer map of \( S(n^- + h) \) onto \( U(n^- + h) \). Then for \( H \) in \( b \), \( \lambda^{-1}(y_0) \) is annihilated by \( ad H \) (extended as a derivation to \( S(n^- + h) \)). It is enough to show that \( \lambda^{-1}(y_0) \) belongs to \( S(h) \). Using \( (1.14) \), one can show that there exists an element \( X_p \) in \( b \) such that \( \alpha(X_P) \) is a nonzero real number for every \( \alpha \) in \( \Delta \) (= the roots of \((g,h)) \) and such that \( P \) consists of precisely those \( \alpha \) in \( \Delta \) such that \( \alpha(X_P) \) is positive. It is then clear that in \( S(n^- + h) \), the null space for \( ad X_P \) is just \( S(h) \). Since \( ad X(\lambda^{-1}(y_0)) = 0 \) for every \( X \) in \( b \), in particular \( ad X_P(\lambda^{-1}(y_0)) = 0 \). Hence \( \lambda^{-1}(y_0) \) belongs to \( S(h) \), so that \( \beta_p(y) \) belongs to \( U(h) \).

Now suppose \( y \) and \( y' \) are in \( U^b \). Let \( y = y_0 + y_1 \) and \( y' = y_0 + y_1' \) be their decomposition with respect to \( (5.4) \), so that \( \beta_p(y') = y_0 \) and \( \beta_p(y) = y_0 \). Then \( yy' = y_0y_0' + y_0y_1' + y_1y_0' + y_1y_1' \). Clearly \( y_0y_0' \) belongs to \( U(h) \) and \( y_0y_1' + y_1y_0' \) belongs to \( U(g)\). Also \( y_1y_0' \in U(g)U(h) \subseteq U(g)U(h)U(h) \). Thus \( y_0y_0' \) is the component of \( yy' \) in \( U(n^- + h) \) with respect to \( (5.4) \). We already know that this component is in \( U(h) \). Thus \( \beta_p \) is a homomorphism of algebras. (q.e.d.)

The centralizer \( U^k \) of \( k \) in \( U(g) \) is contained in \( U^b \). As usual interpret elements of \( S(h) \) as polynomials on \( h^X \). For any \( \varphi \) in \( h^X \), define a homomorphism \( \chi_{P,\varphi} \) of \( U^k \) into \( C \) as follows:

\[
(5.5) \quad \chi_{P,\varphi}(y) = \beta_p(y)(\varphi) \quad (y \in U^b).
\]

The main results of the previous sections can now be formulated.

Let \( b_0 \) be a Cartan subalgebra of \( k_0 \) and \( b \) its complexification. Let \( q \) be a \( \theta \) stable parabolic subalgebra of \( g \) containing \( b \). The centralizer \( h \) of \( b \) in \( g \) is a Cartan subalgebra of \( g \) and \( q \) contains \( h \). Let \( r \) be a \( \theta \) stable Borel subalgebra of \( g \) contained in \( q \) (cf. \( (1.15) \) and \( (1.2) \)). Let \( P \) be the set of positive roots for \( (g,h) \) corresponding to \( r \). Define the \( \theta \) stable Borel subalgebra \( r' \subseteq q \) by \( (2.1) \). Choose a \( \theta \) stable positive system \( P'' \) of roots of \( (g,h) \) having properties \( (2.3) \) and \( (2.4) \). Denote by \( F(P'', q, r) \) the set of all elements \( \mu \) in \( h^X \) having properties \( (2.6) \) and \( (2.7) \). Now choose a \( \mu \) in \( F(P'', q, r) \) and recall the objects associated to it in §§3, 4.

We can now state

\[
(5.6) \quad \text{Theorem: Let } q \text{ be a } \theta \text{ stable parabolic subalgebra. Let } \mu \in F(P'', q, r). \text{ Let } W_{P', q, r} = W_{m+1}/W_X \text{ (cf. (3.12) and (4.5)). Then } W_{P', q, r}(\mu) \text{ is a } k \text{ finite } U(g) \text{ module having the following properties:}
\]
(i) $W_{P^{r},q,r}(\mu) = U(g)\tilde{v}_{m+1}$, where $\tilde{v}_{m+1}$ is the image of the vector $v_{m+1}$ of $W_{m+1}$. The irreducible finite dimensional representation of $k$ with highest weight $-t(\tau\mu + \tau\delta - \tau\delta_k - \delta_k)$ occurs with multiplicity one in $W_{P^{r},q,r}(\mu)$. The corresponding isotypical $U(k)$ submodule of $W_{P^{r},q,r}$ is $U(k)\tilde{v}_{m+1}$; on this space elements of $U^k$ act by scalars given by the homomorphism $\chi_{P^{r},-\mu-\delta}$.

(ii) If $\tau_\lambda$ is an irreducible finite dimensional representation of $k$ with highest weight $\lambda$ with respect to $P_k$, then the multiplicity of $\tau_\lambda$ in $W_{P^{r},q,r}(\mu)$ is finite; it is zero if $\lambda$ is not of the form $(\tau)\lambda(-\mu - \delta - \sum_{\varphi \in P_m \varphi})|b$ where $m_\varphi$ are nonnegative integers.

PROOF: By (4.13), we know that $W_{P^{r},q,r}(\mu)$ is nonzero and $k$-finite. By (4.6) the vector $v_{m+1}$ of $W_{m+1}$ does not belong to $W_k$. The image of $v_{m+1}$ in $W_{P^{r},q,r}(\mu)$ is $P_k$ extreme of weight $-(\tau\mu + \tau\delta - \tau\delta_k - \delta_k)$ (which is dominant by (3.7)) and this image generates an irreducible $k$-module with highest weight $-t(\tau\mu + \tau\delta + \tau\delta_k - \delta_k)$ with respect to $P_k$.

Based on the preceding sections one can complete the proof of the theorem in the same way as [3, Theorem 1].

It is easy to conclude from (5.6) that $W_{P^{r},q,r}(\mu)$ has a unique proper maximal $U(g)$ submodule and hence $W_{P^{r},q,r}(\mu)$ has a unique nonzero quotient $U(g)$ module which is irreducible. We denote this $U(g)$ module by $D_{P^{r},q,r}(\mu)$. The following theorem is now immediate from (5.6).

(5.7) Theorem: Let $\mu \in F(P^{r},q,r)$. Up to equivalence there exists a unique $k$-finite irreducible $U(g)$ module $D_{P^{r},q,r}(\mu)$ having the following property: The finite dimensional irreducible $U(k)$ module with highest weight $-t(\tau\mu + \tau\delta - \tau\delta_k - \delta_k)$ (with respect to $P_k$) occurs with multiplicity one in $D_{P^{r},q,r}(\mu)$ and the action of $U^k$ on the corresponding isotypical $U(k)$ submodule is given by the homomorphism $\chi_{P^{r},-\mu-\delta}$.

The uniqueness follows from the well known theorem of Harish Chandra [4]: An irreducible $k$-finite $U(g)$ module $M$ is completely determined by a nonzero isotypical $U(k)$ submodule of $M$ and the action of $U^k$ on it.
REFERENCES


School of Math.
Bombay, 400 005 India