ANTONIO CASSA

A theorem on complete intersection curves and a consequence for the runge problem for analytic sets

Compositio Mathematica, tome 36, no 2 (1978), p. 189-202

<http://www.numdam.org/item?id=CM_1978__36_2_189_0>
A THEOREM ON COMPLETE INTERSECTION CURVES AND A CONSEQUENCE FOR THE RUNGE PROBLEM FOR ANALYTIC SETS

Antonio Cassa

Summary

The main goal of this article is to prove the following:

**Approximation Theorem:** Let $X$ be a Stein complex analytic manifold of dimension $n \geq 2$, $A$ a Runge and Stein open set of $X$ and $C$ a curve of $A$; there exists a sequence of curves $\{C_k\}_{k \geq 1}$ of $X$ such that:

$$C = \lim_{k \to \infty} (C_k|_A)$$

in the topological space $Z_1^+(A)$ of positive analytic 1-cycles of $A$.

The proof makes use essentially of the following:

**Complete Intersection Theorem:** For each relatively compact open set $B$ of $A$ there exist functions $g_1, \ldots, g_{n-1}$ holomorphic on $B$ such that the positive analytic 1-cycle defined by $g = (g_1, \ldots, g_{n-1})$ in $B$ is:

$$V_1(g) = C|_B + m_1 \cdot (D_1|_B) + \cdots + m_s \cdot (D_s|_B)$$

where $D_1, \ldots, D_s$ are curves of $X$ and $m_1, \ldots, m_s$ positive integers.

In fact if $\{g^{(k)}\}_{k \geq 1}$ is a sequence of maps $g^{(k)} : X \to \mathbb{C}^{n-1}$ holomorphic on $X$, having at least multiplicity $m_i$ on $D_i$ for each $i = 1, \ldots, s$ and converging to $g$, for $k$ big enough we have:

$$V_1(g^{(k)}) = C|_B + m_1 \cdot (D_1|_B) + \cdots + m_s \cdot (D_s|_B)$$
where the $C_k$ are curves of $X$; then in $Z^1(B)$:

$$C = \lim_{k \to \infty} (C_k|_B)$$

So every curve $C$ of $A$ can be approximated by curves of $X$ on every relatively compact open set $B$ of $A$, that is the restriction map:

$$Z_1(X) \longrightarrow Z_1(A)$$

has dense image in $Z_1(A)$.

Moreover if $C$ is irreducible in $A$ the curves $C_k$ can be chosen irreducible in $X$ and if $X$ is an open set of $C^n$ they can be taken algebraic.

This proves that the so-called Runge problem has always solution for analytic cycles of dimension one. This is no longer true in general for higher dimension; Cornalba and Griffiths show in [7] page 76 there exists a non trivial condition for the approximability of an analytic set.

Under that condition they state a general Runge problem for analytic sets that they solve in the case of codimension one.

In that article (as in [4]) the topology of $Z^d_+(X)$ is defined through the space of currents $\mathcal{D}^d(X)$; the properties of that topology are described in [11] and in a more geometric way in [3] or in [5].

I take the opportunity of thanking prof. A. Andreotti for all his help and mainly for his precious suggestions; likewise I wish to thank prof. M. Cornalba and prof. Ph. Griffiths for having communicated to me their ideas about the Runge problem.

**List of symbols**

- $\text{reg } V = \text{manifold of all the regular points of the analytic space } V.$
- $\text{sing } V = V - \text{reg } V = \text{subspace of the singular points of } V.$
- $T_x(V) = \text{Hom}_C(\mathcal{W}_x, \mathcal{W}_x^*; C) = \text{Zarinski tangent space at } x \in V.$
- $\text{dim } T_x(V) = \text{dim}_C T_x(V) = \text{embedding dimension} = \text{tangential dimension}.$
- $Z_d(W) = \text{topological group of the analytic } d\text{-cycles in the manifold } W.$
- $Z^+_d(W) = \text{cone in } Z_d(W) \text{ of the positive } d\text{-cycles of } W.$
- $V_d(f) = \text{positive } d\text{-cycle defined by the equation } f = 0, \text{ where } f: W \to \mathbb{C}' \text{ is an holomorphic map.}$
§1. The estimate of the rank of a sheaf using the Endromisbündel of Forster and Ramspott

Let $\mathcal{F}$ be a coherent sheaf on a complex analytic manifold $X$. For each point $x \in X$ the least number of generators of the stalk $\mathcal{F}_x$ is given by the dimension on $\mathbb{C}$ of the vector space $L_x(\mathcal{F}) = \mathcal{F}_x/(\pi_x \cdot \mathcal{F}_x)$. If this number is bounded in $X$ the sheaf $\mathcal{F}$ has finite rank, that is there exist sections $f_1, \ldots, f_r \in \Gamma(X, \mathcal{F})$ generating all the stalks $\mathcal{F}_x$ for every $x \in X$ (see [6]).

Taken a positive integer $s \leq r$, the existence of $s$ sections $g_1, \ldots, g_s \in \Gamma(X, \mathcal{F})$ with the same property is equivalent to the existence of an holomorphic section of a bundle $E(\mathcal{F}; f, r)$ on $X$ called Endromisbündel (see [8]).

The Endromisbündel is an open set of $X \times \mathbb{C}^s$ obtained subtracting analytic subspaces defined by the sections $f_1, \ldots, f_r$ and by the numbers $\{\dim \mathcal{F}_x; x \in X\}$.

Let’s put for each integer $k \geq 0$:

$$Y_k(\mathcal{F}) = \{x \in X: \dim \mathcal{F}_x \geq k\}$$

the family $\{Y_k(\mathcal{F})\}_{k \geq 0}$ is a decreasing sequence of analytic subspaces of $X$ which are surely empty for $k \geq r + 1$.

On the analytic space $X_k(\mathcal{F}) = Y_k(\mathcal{F}) - Y_{k+1}(\mathcal{F})$ the Endromisbündel is a locally trivial holomorphic bundle whose fibre $F_{r, s, k}$ is homotopic to the manifold $W_{sk}$ of all the orthonormal $k$-frames of $\mathbb{C}^s$.

The main result of [8] (satzen 5 and 6) claims that if $X$ is holomorphically convex the existence of a holomorphic section of the Endromisbündel is equivalent to the existence of a continuous section.

Therefore the evaluation of the rank of $\mathcal{F}$ is a purely topological problem whose main ingredients are the spaces $Y_k(\mathcal{F})$ and the fibres $W_{sk}$.

The following proposition is a way to make sure the existence of a continuous section of $E(\mathcal{F}, f, s)$ supposing zero all the cohomology groups containing the obstructions.

**Proposition:** Let $X$ be a Stein manifold and $\mathcal{F}$ a coherent analytic sheaf having his rank bounded by an integer $s$.

If for each $k \geq 0$ and $q \geq 1$:

$$H^{q+1}(Y_k(\mathcal{F}), Y_{k+1}(\mathcal{F}); \pi_q(W_{sk})) = 0$$
then there exist $s$ sections $g_1, \ldots, g_s \in \Gamma(X, \mathcal{F})$ generating all the stalks of $\mathcal{F}$.

**Proof:** Let $f_1, \ldots, f_r$ be global sections of $\mathcal{F}$ generating all the stalks of $\mathcal{F}$; proceeding by induction on $h = r - k$ from 0 to $r$ we will prove there exists a continuous section of $E(\mathcal{F}, f, s)$ on $Y_{r-h}(\mathcal{F})$ for $h = 0, \ldots, r$.

If $h = 0$ since $Y_{r+1} = \emptyset$ the bundle $E(\mathcal{F}, f, s)$ is a locally trivial fibre bundle with fibre homotopic to $W_{sr}$; the condition $H^{q+1}(Y_r; \pi_q(W_{sr})) = 0$ is just the one we need to prove the existence of a continuous section on $Y_r$ (see [13] page 174).

Let's prove now we can extend a continuous section from $Y_{r-(h-1)}$ to $Y_{r-h}$; we can find a triangulation of $Y_{r-h}$ in such a way $Y_{r-(h-1)}$ is a subpolyhedron furnished of a neighborhood $U$ which is again a subpolyhedron of $Y_{r-h}$ and contractible on $Y_{r-(h-1)}$.

Since $E(\mathcal{F}, f, s)$ is an open set of $\mathbb{C}^n \times X$ choosing $U$ suitably small we can, first of all, extend our continuous section from $Y_{r-(h-1)}$ to $U$; then we can extend the section from $U - Y_{r-(h-1)}$ to $Y_{r-h} - Y_{r-(h-1)}$ because for each $q \geq 1$ we have:

$$H^{q+1}(Y_{r-h} - Y_{r-(h-1)}, U - Y_{r-(h-1)}; \pi_q(W_{s,r-h})) = 0$$

In fact:

$$H^{q+1}(Y_{r-h} - Y_{r-(h-1)}; U - Y_{r-(h-1)}) = H^{q+1}(Y_{r-h}, U)$$

$$= H^{q+1}(Y_{r-h}, Y_{r-(h-1)}) = H^{q+1}(Y_k, Y_{k+1}) = 0.$$

§2. Complete intersection curves

Let $C$ be a curve of an open set of $\mathbb{C}^n$ and $x_0$ a singular point of $C$, if $\dim t_{x_0}(C) = 2$ then the curve $C$ is complete intersection at $x_0$.

In fact there exist a manifold $M$ of dimension 2 in $\mathbb{C}^n$ and a neighborhood $U$ of $x_0$ such that $C \cap U \subset M \cap U$; restricting, in case, $U$ we can find a function $f_n$ holomorphic on $U$ such that $\mathcal{F}_{C \cap U \cap M} = f_n \cdot \mathcal{O}_{M \cap U}$ and functions $f_2, \ldots, f_{n-1}$ holomorphic on $U$ such that $\mathcal{F}_{M \cap U} = f_2 \cdot \mathcal{O}_U + \cdots + f_{n-1} \cdot \mathcal{O}_U$; therefore

$$\mathcal{F}_{C \cap U} = f_2 \cdot \mathcal{O}_U + \cdots + f_n \cdot \mathcal{O}_U.$$

The following two lemmas prove in most cases that if $t = \dim t_{x_0}(C)$ is bigger than 2, then adding to $C$ some lines $L_1, \ldots, L_{t-2}$ through $x_0$ the curve $C \cup (L_1 \cup \cdots \cup L_{t-2})$ is complete intersection at $x_0$. 


LEMMA 1: Let $C$ be a curve of an open set of $\mathbb{C}^n$ (with $n \geq 2$) and the origin $0$ a singular point of $C$.

Denoted by $L_1, \ldots, L_n$ the coordinate axes of $\mathbb{C}^n$ and written $L_0 = \{0\}$, if the following hypothesis is verified:

(i) the projection map $p : \mathbb{C}^n \to \mathbb{C}^2$ defined by $p(z_1, \ldots, z_n) = (z_{n-1}, z_n)$ is injective on $C$ in a neighborhood of $0$.

then a neighborhood $V$ of $0$, an integer $s = 0, \ldots, n-2$, a Stein neighborhood $U$ of $(L_0 \cup \cdots \cup L_s)$ and functions $f_1, \ldots, f_{n-1}$ holomorphic on $U$ exist such that:

(1) \{ $x \in U : f_i(x) = \cdots = f_{n-1}(x) = 0$ \} = $(C \cap V \cap U) \cup (L_0 \cup \cdots \cup L_s)$

(2) the germs $f_{1,x}, \ldots, f_{n-1,x}$ generate the stalk $\mathcal{F}_{C,x}$ for each $x \in C \cap V \cap U - \{0\}$.

PROOF: Let's proceed by induction on $n \geq 2$. For $n = 2$ the conclusion is well known. For $n \geq 3$ let's suppose we have already proved the lemma for all the curves $C'$ of $\mathbb{C}^n$ with $n' < n$ and let's prove it for the curves $C$ of $\mathbb{C}^n$.

Let's denote by $q : \mathbb{C}^n \to \mathbb{C}^{n-1}$ the projection along the axis $L_{n-2}$ defined by $q(z_1, \ldots, z_{n-2}, z_{n-1}, z_n) = (z_1, \ldots, 0, z_{n-1}, z_n)$ with values in $\mathbb{C}^{n-1} = \{z \in \mathbb{C}^n : z_{n-2} = 0\}$.

For the hypothesis (i) it is possible to find a neighborhood $V$ of $0$ where $q$ is injective on $C$. Rechoosing in case $V$ we can suppose the map $q : V \cap C \to q(V)$ proper; therefore $C' = q(C \cap V)$ is a curve of $V' = q(V)$ open neighborhood of $0$ in $\mathbb{C}^{n-1}$.

We can choose $V$ small enough to have also $\text{sing}(C) \cap V = \{0\} = \text{sing}(C')$.

The curve $C'$ of $\mathbb{C}^{n-1}$ in respect to the coordinates $z_1, \ldots, z_{n-2}, z_{n-1}, z_n$ verifies the hypothesis (i); for the induction there exist an integer $s' = 0, \ldots, n-3$, a Stein neighborhood $U'$ of $L_0 \cup \cdots \cup L_{s'}$ and functions $f_1', \ldots, f_{n-2}$ holomorphic on $U'$ verifying the theses (1) and (2).

Using (i) it can be verified that the restriction of $q$ gives a map $\hat{q} : (C \cap V) \cup (L_0 \cup \cdots \cup L_s) \to C' \cup (L_0 \cup \cdots \cup L_{s'})$ bijective and holomorphic, whose inverse is meromorphic, continuous and biholomorphic out of $0$. Likewise the function $m$ in $C' \cup (L_0 \cup \cdots \cup L_{s'})$ defined by $m(x') = z_{n-2}(\hat{q}^{-1}(x'))$ is meromorphic, continuous, holomorphic out of $0$ and vanishes on $(L_0 \cup \cdots \cup L_{s'})$. Therefore $m(x') = a'(x')/b'(x')$ everywhere $b'(x') \neq 0$ for two functions $a'$, $b'$ holomorphic on $C' \cup (L_0 \cup \cdots \cup L_{s'})$ with $b'$ not identically zero on any irreducible component and $a' = 0$ on $L_0 \cup \cdots \cup L_{s'}$.

Solving a $\mathcal{O}^*$-cohomological problem we can find two functions $a$ and $b$ holomorphic such that $b(x') \neq 0$ if $x' \neq 0$ and $m(x') = a(x')/b(x')$ for each $x' \neq 0$.  


Since $U'$ is Stein we can suppose $a$ and $b$ defined on $U'$; written $U = q^{-1}(U')$, $f_1 = f_1 \circ q$, \ldots, $f_{n-2} = f_{n-2} \circ q$, $f_{n-1} = (b \circ q) \cdot z_{n-2} - (a \circ q)$ the theses (1) and (2) are verified for the curve $C$ together with the lines $L_0, \ldots, L_{n-2}$ if $b(0) = 0$ or the lines $L_0, \ldots, L_r$ if $b(0) \neq 0$.

**Lemma 2:** Let $C$ be as in Lemma 1 and $n \geq 3$; there exists a coordinate system in $C^n$ verifying (i).

Moreover if $E$ is a measurable subset of $C^n \setminus \{0\}$ with Hausdorff measure $H_r(E) = 0$ for each $r > 2$, the coordinate system can be chosen in such a way to have:

$$(L_1 \cup \cdots \cup L_{n-2}) \cap E = \emptyset$$

**Proof:** For each $n$-uple of lines $L = (L_1, \ldots, L_n)$ in general position and for each $i = 1, \ldots, n-1$ let’s write $V_{L,i} = L_1 + \cdots + L_n$ and let’s denote $p_{L,i} : C^n \to V_{L,i}$ and $q_{L,i+1} : V_{L,i} \to V_{L,i+1}$ the natural projections.

Since $p_{n-1} = (q_{n-1}) \circ \cdots \circ (q_2)$, if $p_{n-1}$ is not injective on $C$ in any neighborhood of $0$, then some of the projections $q_{i+1}$ (where $i = 1, \ldots, n-2$) is not injective on the set $p_i(C)$ in any neighborhood of $0$; therefore for each integer $j \geq 1$ there exist two points $z_j$ and $z'_j$ of $C$ such that $p_i(z_j)$ and $p_i(z'_j)$ are different, non zero, $|p_i(z_j)| < 1/j$, $|p_i(z'_j)| < 1/j$ and $(p_i(z_j) - p_i(z'_j)) \in L_i$.

Then the intersection $L_i \cap (p_i(C) - p_i(C))$ has interior part not empty in $L_i$, this set is in fact the image of the holomorphic map $d| : d^{-1}(L_i) \cap (C \times C) \to L_i$ where $d : C^n + C^n \to V_i$ is defined by $d(z', z'') = p_i(z') - p_i(z'')$ which is of rank one at least in some point containing in its image the sequence $\{(p_i(z_j) - p_i(z'_j))\}_{j \geq 1}$ infinite and converging to 0.

Written $G = \{g \in C^* : g = e^{a+bi}$ with $a, b \in \mathbb{Q}\}$, $S_i = p_i(C) - p_i(C)$, $S'_i = \cup_{g \in G} g \cdot S_1$ we have $L_i = \cup_{g \in G} g \cdot (L_i \cap S_i)$ and therefore $L_i \subset S'_i + (L_0 + \cdots + L_{i-1})$.

Let’s prove at this point that for each $i = 1, \ldots, n-2$ and for each $L' = (L_0, \ldots, L_{i-1}) \in \{L_0\} \times (\mathbb{P}^{n-1})^{i-1}$ (where $L_0 = \{0\}$) the set $R_{L'} = \{L \in \mathbb{P}^{n-1} : L \subset S'_i + L_0 + \cdots + L_{i-1}\}$ has measure zero in $\mathbb{P}^{n-1}$.

In fact written $T_{L'} = \cup_{L \in R_{L'}} L$, because $T_{L'} \subset S'_i + L_0 + \cdots + L_{i-1}$ and $H_r(S'_i) = 0$ if $r > 4$, it must be $H_r(T_{L'}) = 0$ for $r > 4 + 2(i - 1)$ and therefore $H_r(R_{L'}) = 0$ if $r > 2 + 2(i - 1) = 2i$ since $T_{L'} \cup \{0\} \approx R_{L'} \times C^*$, so we can conclude $\mu(R_{L'}) = H_{2n-2}(R_{L'}) = 0$ because $2n - 2 > 2i$.

We are able now to prove that for each $k = 1, \ldots, n-2$ there exist $k$ lines $L_1, \ldots, L_k$ in general position such that $(L_1 \cup \cdots \cup L_k) \cap E = \emptyset$ and for each $l = 1, \ldots, k$ written $L'_l = (L_0, \ldots, L_{l-1})$ we have $L'_l \notin R_{L'}$. 

A. Cassa [6]
For $k = n - 2$ the lemma will result proved.
For $k = 1$ we have to find $L_0 \in \mathbb{P}^{n-1}$ such that $L_1 \not\subseteq R_{L_0} \cup E'$ where $E'$ is the image of $E$ in $\mathbb{P}^{n-1}$ for the natural quotient map. It is possible to find $L_1$ since the set of directions to avoid has measure zero in $\mathbb{P}^{n-1}$.

Given the lines $L_1, \ldots, L_{k-1}$ with the properties listed above we have to find a line $L_k$ in general position in respect with the others and in such a way $L_k \not\subseteq R_{L'} \cup E'$.

Again it is possible to choose $L_k$ since the set of directions to avoid has measure zero.

**THEOREM 1:** Let $X$ be a Stein manifold of dimension $n \geq 3$, $A$ a Runge and Stein open set of $X$ and $C$ a curve of $A$.

For each relatively compact open set $B$ of $A$ there exist a curve $D$ of $X$ and functions $g_1, \ldots, g_{n-1}$ holomorphic on $B$ such that:

1. $\{x \in B : g_1(x) = \cdots = g_{n-1}(x) = 0\} = (C \cup D) \cap B$
2. The germs $g_{1,x}, \ldots, g_{n-1,x}$ generate the stalk $\mathcal{T}_{C,x}$ for each $x \in C \cap B - S$, where $S = \{x \in C \cap B : C$ is not complete intersection at $x\}$.

**PROOF:** Enlarging $B$ we can suppose it a Runge and Stein open set yet relatively compact in $A$.

The set $S$ contained in $\text{sing}(C) \cap B$ is finite; if $S = \emptyset$ the curve $C \cap B$ is locally complete intersection (ideal theoretically) and therefore it is complete intersection in $B$ (see [8] page 162, the Remark (b) to Corollary (2) of Theorem (9).

If $S \neq \emptyset$ let's write $S = \{x_1, \ldots, x_p\}$; we show that however fixed an integer $r = 1, \ldots, p$ for each $j = 1, \ldots, r$ there exist a curve $D_j$ of $X$, an open neighborhood $U_j$ of $D_j$ and functions $f_{j,1}, \ldots, f_{j,n-1}$ holomorphic on $U_j$ such that:

A. $\{x \in U_j \cap B : f_{j,1}(x) = \cdots = f_{j,n-1}(x) = 0\} = (C \cup D_j) \cap U_j \cap B$
B. The germs $f_{j,1,x}, \ldots, f_{j,n-1,x}$ generate $\mathcal{T}_{C,x}$ for each $x \in C \cap U_j - \{x_j\}$
C. $D_j \cap C \cap B = \{x_j\}$
D. $U_k \cap U_j \cap B = \emptyset$ for each $k < j \leq r$.

Let's proceed by induction on $r$. Let $r = 1$; it is possible to find a holomorphic map $R : X \to C^n$ regular in $x_1$ and such that $R^{-1}(0) = \{x_1\}$ (see [8] page 161, Corollary 1 of Theorem 9). Replacing $R$ with another map (denoted again by $R$) near enough to $R$ we can have (see [9] page 168, Theorem 4):

1. for all the points $x$ of a neighborhood $W$ of $x_1 : R^{-1}(R(x)) \cap \overline{B} = \{x\}$
2. $R(x_1) = 0$
(3) \( \dim(R^{-1}(R(x))) = 0 \) for each \( x \in X \).

(4) \( R \) establishes a biholomorphism between \( W \) and \( W' = R(W) \) open set of \( C^n \).

Applying the Lemmas 1 and 2 to the curve \( C' = R(C \cap W) \) of the open set \( W' \) of \( C^n \) and to the set \( E = R(C) - \{0\} \), it is possible to find a coordinate system \( (z_1, \ldots, z_n) \) in \( C^n \) whose coordinate axes we denote by \( L_1, \ldots, L_n \) \((L_0 = \{0\})\), a neighborhood \( V' \) of 0 contained in \( W' \), an integer \( s = 0, \ldots, n - 2 \), a neighborhood \( U' \) of \( L_0 \cup \cdots \cup L_s \) and functions \( f_1, \ldots, f'_{n-1} \) holomorphic on \( U' \) such that:

1. \( \{z \in U' : f_1(x) = \cdots = f'_{n-1}(x) = 0\} = (C' \cap V' \cap U') \cup (L_0 \cup \cdots \cup L_s) \)

2. the germs \( f_{1,z}, \ldots, f'_{n-1,z} \) generate \( \mathcal{O}_{C^n,z} \) for each \( z \in C' \cap V' \cap U' - \{0\} \)

3. \( (L_0 \cup \cdots \cup L_s) \cap R(C) = \{0\} \).

Let's put \( D_1 = R^{-1}(L_0 \cup \cdots \cup L_s) \), since \( D_1 \cap C \cap \bar{B} = \{x_1\} \) we can find a neighborhood \( U_1 \) of \( D_1 \) contained in \( R^{-1}(U') \) such that \( C \cap (U_1 \cap \bar{B}) \subset C \cap W \); on \( U_1 \) let's define the functions \( f_{i,1} = f_{i,1} \circ R, \ldots, f_{i,n-1} = f_{i,n-1} \circ R \).

For these sets and functions the conditions (A) (B) (C) and (D) listed above are verified.

Let's suppose now \( r > 1 \) and we have found for each \( j = 1, \ldots, r - 1 \) a curve \( D_j \) of \( X \), a neighborhood \( U_j \) of \( D_j \) and functions \( f_{j,1}, \ldots, f_{j,n-1} \) satisfying the conditions (A) (B) (C) and (D) and let's show how to add a curve \( D_r \) a neighborhood \( U_r \) of \( D_r \) and functions \( f_{r,1}, \ldots, f_{r,n-1} \) in such a way the properties (A) (B) (C) and (D) are verified for each \( k < j \leq r \).

Again we consider an holomorphic map \( R_r : X \to C^n \) such that:

1. for all the points \( x \) of an open neighborhood \( W_r \) of \( x \), we have \( R_r^{-1}(R_r(x)) \cap \bar{B} = \{x\} \)

2. \( R_r(x_r) = 0 \)

3. \( \dim R_r^{-1}(R_r(x)) = 0 \) for each \( x \in X \).

As above we apply the Lemmas (1) and (2) to the curve \( C'_r = R_r(C \cap W_r) \) of the open set \( W'_r = R_r(W_r) \) of \( C^n \) and the set \( E_r = R_r(C \cup D_1 \cup \cdots \cup D_{r-1}) - \{0\} \); written \( D_r = R_r^{-1}(L_0 \cup \cdots \cup L_{s_r}) \), again we can find a neighborhood \( U_r \) of \( D_r \) such that \( C \cap U_r \cap B \subset C \cap W_r \) and define \( f_{r,1} = f_{i,1} \circ R_r, \ldots, f_{r,n-1} = f_{i,n-1} \circ R_r \); moreover since \( D_i \cap D_j \cap B = \emptyset \) if \( i \neq j \leq r \) we can choose \( U_i, \ldots, U_r \) in such a way to have \( U_i \cap U_j \cap B = \emptyset \) for each \( i \neq j \leq r \), and again these sets and functions satisfy the conditions (A) (B) (C) and (D).

Arrived with \( r \) to \( p \), let's put \( D = D_1 \cup \cdots \cup D_p \) and let's define for each \( i = 1, \ldots, p \) a coherent sheaf \( \mathcal{F}_i \) on \( U_i \cap B \) putting:

\[
\mathcal{F}_i = f_{i,1} \cdot \mathcal{O}_{|U_i \cap B} + \cdots + f_{i,n-1} \cdot \mathcal{O}_{|U_i \cap B}
\]
For each \( x \in U_i \cap B - D \) we have \( \mathcal{I}_{lx} = \mathcal{I}_{C,x} \).

We can now define a sheaf \( \mathcal{I} \) on \( B \) writing:

\[
\mathcal{I}_x = \begin{cases} 
\mathcal{I}_{lx} & \text{if } x \in B \cap U_i \\
\mathcal{I}_{C,x} & \text{if } x \in B - D
\end{cases}
\]

The sheaf \( \mathcal{I} \) is well defined and coherent; moreover

\[
\dim L_x(\mathcal{I}) = \begin{cases} 
1 & \text{for each } x \in B - (C \cup D) \\
n - 1 & \text{for each } x \in (C \cup D) \cap B
\end{cases}
\]

therefore the sheaf \( \mathcal{I} \) has limited rank on \( B \).

To complete the theorem’s proof we have to check that the rank of \( \mathcal{I} \) is just \( n - 1 \).

For what has been reported in §1 since we have:

\( Y_0(\mathcal{I}) = Y_1(\mathcal{I}) = B, \quad Y_2(\mathcal{I}) = \cdots = Y_{n-1}(\mathcal{I}) = (C \cup D) \cap B \) and

\( Y_r(\mathcal{I}) = \emptyset \) for each \( r \geq n \), we have to prove that for each \( q \geq 1 \):

\[
H^{q+1}((C \cup D) \cap B; \pi_q(W_{n-1,1})) = 0
\]

and

\[
H^{q+1}(B, (C \cup D) \cap B; \pi_q(W_{n-1,1})) = 0
\]

The first cohomology groups vanish because \( (C \cup D) \cap B \) is a Stein curve; for the second we have \( W_{n-1,1} \approx S^{2n-3} \), therefore \( \pi_q(W_{n-1,1}) = 0 \) for each \( 1 \leq q \leq 2n - 4 \).

For \( q \geq 2n - 3 \geq n \geq 3 \) from the exact sequence:

\[
\cdots \longrightarrow H^q((C \cup D) \cap B; G) \longrightarrow H^{q+1}(B, (C \cup D) \cap B; G) \longrightarrow H^{q+1}(B; G) \longrightarrow H^{q+1}((C \cup D) \cap B; G) \longrightarrow \cdots
\]

where \( G = \pi_q(W_{n-1,1}) \), it follows:

\[
H^{q+1}(B, (C \cup D) \cap B; G) \cong H^{q+1}(B; G) \cong 0
\]

because \( H^{q+1}((C \cup D) \cap B; G) = 0 = H^{q+1}(B; G) \) for each \( q \geq n \geq 3 \) (see [2] and [12]).

When \( X \) is an open set of \( \mathbb{C}^n \) we can prove something more precise:

**Theorem 1**: Let \( X \) be a Stein open set of \( \mathbb{C}^n \) \( (n \geq 3) \), \( A \) a Runge and Stein open set of \( X \) and \( C \) a curve of \( A \).
If the set:
\[ S = \{ x \in C : C \text{ is not complete intersection at } x \} \]
is finite, then for each \( x \in S \) there exists a finite family of lines \( L_{x,0}, \ldots, L_{x,s_x} \) through 0 such that the curve:
\[ ((C \cup \bigcup_{i=1}^{s_x} L_{x,i})) \cap A \]
is a set-theoretically complete intersection in \( A \).

More precisely there exist functions \( g_1, \ldots, g_{n-1} \) holomorphic on \( A \) such that:

1. \( \{ x \in A : g_1(x) = \cdots = g_{n-1}(x) = 0 \} = (C \cup \bigcup_{i=1}^{s_x} L_{x,i}) \cap A \)

2. the germs \( g_{1,x}, \ldots, g_{n-1,x} \) generate the stalk \( \mathcal{T}_{C,x} \) for each \( x \in C - S \).

**PROOF:** As in the theorem 1 forgetting about \( B \) or \( \bar{B} \) and using as maps \( R_x : X \to C^n \) the translations sending the points \( x_r \) in 0.

**THEOREM 2:** Let \( X \) be a Stein manifold of dimension \( n \geq 2 \), \( A \) a Runge and Stein open set of \( X \) and \( C \) a curve of \( A \).

For each relatively compact open set \( B \) of \( A \) there exist a holomorphic map \( g : B \to C^{n-1} \) and a positive 1-cycle \( D \in Z_1^+(X) \) such that:

\[ V_1(g) = C|_B + D|_B. \]

**PROOF:** Let's prove first the theorem when \( n \geq 3 \); enlarging \( B \) we can suppose it Runge and Stein in \( A \). For the Theorem 1 there exist a map \( g : B \to C^{n-1} \) and a curve \( D \) of \( X \) such that:

1. \( \{ x \in B : G(x) = 0 \} = (C \cup D) \cap B \)

2. \( g_{1,x}, \ldots, g_{n-1,x} \) generate \( \mathcal{T}_{C,x} \) for each \( x \in \text{reg}(C) \cap B \).

Let's denote by \( D \) the sum of the components of the cycle \( V_1(g) \) not contained in \( C \); \( D \) is a cycle of \( X \) and we have:

\[ V_1(g) = m_1 \cdot (C|_B) + \cdots + m_r \cdot (C|_B) + D|_B \]

where \( C_1, \ldots, C_r \) are curves contained in \( C \cap B \) decomposing it in its irreducible components, and \( m_1, \ldots, m_r \) are positive integers.

We have just to prove that \( m_1 = \cdots = m_r = 1 \); let \( i = 1, \ldots, r \) and \( x_i \in \text{reg}(C_i) \cap B \), at \( x_i \) we can find a coordinate system \( (z_1, \ldots, z_n) \) such
that \( g_1 = z_1, \ldots, g_{n-1} = z_{n-1}; \) in this coordinate system \( V_1(g) \) is the \( n \)th axis counted only once.

If \( n = 2 \), enlarging in case the open set \( B \) we can suppose it Runge and Stein in \( X \) and with smooth boundary. Therefore (see [2]) we have \( H_3(X, B; \mathbb{Z}) = 0 \) and the group \( H_2(X, B; \mathbb{Z}) \) is free of finite rank; then the restriction map:

\[
r : H^2(X; \mathbb{Z}) \longrightarrow H^2(B; \mathbb{Z})
\]

is surjective.

Therefore there exists a positive divisor \( D \) of \( X \) such that \( r(c(D)) = -c(C_{|B}) \), that is \( c(D_{|B} + C_{|B}) = 0 \).

Since the divisor has Chern class zero, there exist a holomorphic map \( g : B \to \mathbb{C} \) such that: \( V_1(g) = C_{|B} + D_{|B} \).

§3. Approximation of curves

**Theorem 3:** Let \( X \) be a Stein manifold of dimension \( n \geq 2 \), \( A \) a Runge and Stein open set of \( X \) and \( C \) an irreducible curve of \( A \).

There exists a sequence of irreducible curves \( \{C_k\}_{k \geq 1} \) such that:

\[
\lim_{k \to \infty} (C_k \cap A) = C
\]

in the space of positive 1-cycles \( Z_1^+(A) \).

**Proof:** Let \( \{B_i\}_{i \geq 1} \) be a sequence of relatively compact open sets of \( A \) which are Runge and Stein and invade \( A \).

For each \( i \geq 1 \) for the Theorem 3 we can find irreducible curves \( D_{i1}, \ldots, D_{is_i} \) of \( X \) and a map \( g : B_i \to \mathbb{C}^{n-1} \) such that:

\[
V_1(g_i) = (C \cap B_i) + m_{i1} \cdot (D_{i1} \cap B_i) + \cdots + m_{is_i} \cdot (D_{is_i} \cap B_i).
\]

Let’s write \( T_i = (\mathcal{T}_{D_{i1}})^{m_{i1}} \cap \cdots \cap (\mathcal{T}_{D_{is_i}})^{m_{is_i}} \), since \( g_i \in [\Gamma(B_i, \mathcal{T}_{\mathcal{T}_{B_i}})]^{n-1} \) for theorem 11 at page 241 of [9] there exists a sequence of maps \( \{g^{(k)}_i\}_{k \geq 1} \subset [\Gamma(X, \mathcal{T}_i)]^{n-1} \) converging to \( g_i \) on \( B_i \); therefore for the prop. 7 of [5] we have:

\[
V_1(g_i) = \lim_{k \to \infty} (V_1(g_i^{(k)}))_{|B_i}.
\]

Let’s denote by \( T_{ik} \) the sum of the terms of \( V_1(g_i^{(k)}) \) whose support
is not in $D_i = D_{i_1} \cup \cdots \cup D_{i_l}$; we can write:

$$V_i(g_i^{(k)}) = T_{ik} + m_{i_1}^{(k)} \cdot D_{i_1} + \cdots + m_{i_l} \cdot D_{i_l}$$

where $m_{ij}^{(k)} \geq m_{ij}$ for each $k \geq 0$ and $j = 1, \ldots, s_r$.

Let's fix a point $x_{ij} \in \text{reg}(D_{ij}) \cap B_{ij}$ and choose in a neighborhood of $x_{ij}$ a coordinate system where $D_{ij}$ is the first coordinate axis; let's call $R$ and $L$ respectively a cube of center $x_{ij}$ and $L$ the normal hyperplane to $D_{ij}$ in $x_{ij}$; for the Bochner–Martinelli formula (see [10]) we have:

$$m_{ij} = \int_{L \cap 0R} \frac{\lambda(g_i)}{|g_i|^{4n+2}} \quad \text{and} \quad m_{ij}^{(k)} \leq \int_{L \cap 0R} \frac{\lambda(g_i^{(k)})}{|g_i^{(k)}|^{4n+2}}$$

for $k$ big enough, where $\lambda(g)$ is a form whose coefficients are polynomials in $g$ and its derivatives.

For the integral continuity for $k$ big enough we have $m_{ij} \geq m_{ij}^{(k)}$.

Therefore:

$$V_i(g_i^{(k)}) = T_{ik} + m_{i_1} \cdot D_{i_1} + \cdots + m_{i_l} \cdot D_{i_l}$$

and then subtracting the common terms between $V_i(g_i^{(k)})$ and $V_i(g_i)$:

$$C \cap B_I = \lim_{k \to \infty} (T_{ik}|_{B_I}).$$

For the convergence is a local property (see [5]) we have:

$$C = \lim_{i \to \infty} (T_{ii}).$$

To complete the proof we need only to prove the following:

**Lemma:** Let $X$ be a manifold of dimension $n \geq 2$, $A$ an open set of $X$ and $C$ an irreducible curve of $A$.

If there exists a sequence of 1-cycles $\{T_k\}_{k \geq 1} \subset Z_1'(X)$ such that:

$$C = \lim_{k \to \infty} (T_{k|A})$$

then there exists a sequence of irreducible curves $\{C_k\}_{k \geq 1}$ of $X$ such that:

$$C = \lim (C_k \cap A).$$
LEMMA’S PROOF: It’s enough to prove the lemma for each relatively compact open set $B$ of $A$.

Let $x$ be a regular point of $C$, we can find a coordinate system in a neighborhood of $x$ making $C$ a line; let $P_x$ be a polycylinder with center $x$ in this coordinate system. For $k$ big enough the analytic set $(\text{supp}(T_k)) \cap P_x$ is regular because each normal plane to $C$ meets, in $P_x$, the space $\text{supp}(T_k)$ in a simple point for the Bochner–Martinelli formula; moreover $(\text{supp}(T_k)) \cap P_x$ is a connected manifold and there exists an irreducible curve $C_{k_x}$ of $X$ such that $T_{k|P_x} = C_{k_x|P_x}$ for each $k$ bigger than a suitable $k_x$.

Let’s fix in $\text{reg}(C)$ a sequence of connected compact sets invading $\text{reg}(C)$ (such a sequence can be constructed using a triangulation of the connected smooth manifold $\text{reg}(C)$); let’s call $U$ a compact neighborhood of $\text{sing}(C) \cap B$ small enough to be contained in a Stein open set of $B$.

Since the set $(B - U) \cap \text{reg}(C)$ is relatively compact in $\text{reg}(C)$ there exists a connected compact set $K$ of $\text{reg}(C)$ containing the set $(B - U) \cap \text{reg}(C)$ and it is possible to find a finite number of points $x_1, \ldots, x_m$ of $K$ and polycylinders $P_{x_1}, \ldots, P_{x_m}$ centered in those points such that $P = \bigcup_{i=1}^{m} P_{x_i} \supset K$; therefore we have $C \cap B \subset P \cup U$.

Moreover whenever $P_{x_i} \cap P_{x_j} \neq \emptyset$ we can find a point $x_{ij} \in P_{x_i} \cap P_{x_j}$, a polycylinder $P_{ij}$ centered in $x_{ij}$ contained in $P_{x_i} \cap P_{x_j}$ and an integer big enough $k_{ij}$ such that $(\text{supp}(T_k)) \cap P_{ij}$ is non-empty and irreducible for each $k \geq k_{ij}$.

Since $P$ is connected for $k \geq k = \max\{k_{x_i}, k_{ij}\}$ the irreducible curve representing $T_k$ in each $P_{x_i}$ must be the same, that is there exists an irreducible curve $C_k$ of $X$ for each $k \geq k$ such that: $T_{k|P} = C_k|P$.

Moreover for $k$ big enough we have $(\text{supp}(T_k)) \cap B \subset (P \cap U) \cap B$ (see the Remark 5 of [5]); then $T_{k|B \cap U} = C_{k|B \cap U}$, that is $T_{k|B - U} = C_{k|B - U}$ and at last $T_{k|B} = C_{k|B}$.

THEOREM 4: Let $X$ be an holomorphically convex open set of $\mathbb{C}^n$ ($n \geq 2$), $A$ a Runge and holomorphically convex open set of $X$ and $C$ an analytic irreducible curve of $A$. There exists a sequence of algebraic curves $\{C_k\}_{k \geq 1}$ of $\mathbb{C}^n$ irreducible in $X$ such that:

$$\lim_{k \to \infty} (C_k \cap A) = C$$

in the space of positive analytic 1-cycles $Z_1^+(A)$. 

}[13] A theorem on complete intersection curves 201
PROOF: Trivial for $n = 2$.

For $n \geq 3$ following Theorem 3 let’s observe that, being $X$ an open set of $\mathbb{C}^n$, we can take as curves $D_{i_1}, \ldots, D_{i_k}$ some lines of $\mathbb{C}^n$ as in Lemma 1 and therefore the section of the sheaf $\mathcal{F}_j = (\mathcal{F}_{D_{i_1}})^{\mu_{i_1}} \cap \cdots \cap (\mathcal{F}_{D_{i_k}})^{\mu_{i_k}}$ are generated by some polynomials $p_{i_1}, \ldots, p_{i_k}$ of $\mathbb{C}^n$; that is for each $j = 1, \ldots, n - 1$ it holds:

$$ (g_j) = \sum_{i=1, \ldots, n} h_{ijl} \cdot p_{il} $$

for some functions $h_{ijl}$ holomorphic on $B_i$.

Moreover we can choose the open sets $B_i$ to be Runge in $\mathbb{C}^n$ and then find sequences of polynomials $\{q_{ijl}^{(k)}\}_{k \geq 1}$ of $\mathbb{C}^n$ converging to $h_{ijl}$ on $B_i$.

Denoting $(g_{ijl})_k = \sum_{i=1, \ldots, n} q_{ijl}^{(k)} \cdot p_{il}$, the positive 1-cycles $\{T_{ik}\}$ are algebraic and even more so the curves $\{C_k\}_{k \geq 1}$.

REFERENCES


(Obblato 2–XII–1976) Università degli Studi
Istituto Matematico U. Dini
Vialé Morgagni 571A
I-50 134 Firenze, Italia.