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A CYLINDER FLOW ARISING FROM IRREGULARITY OF DISTRIBUTION

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1. Introduction

In this paper we study skew product extensions of irrational rotations on the circle by the group of integers. Such extensions are determined by an integer valued Borel function on the circle. Even though 'most' such functions define ergodic extensions (cf. [6, Theorem 9.11]), it is in general difficult to check whether a particular function gives rise to an ergodic extension.

Here we consider the integer extension of an irrational rotation of the circle which is given by the function which takes values $+1$ and -1 on complementary halves of the circle. The corresponding skew product transformation is associated with the irregularity of distribution of the multiples of the irrational number (mod 1) defining the rotation. Throughout the paper we shall write Z for the integers, R for the real line, and $[a, b)$ for an interval which is closed at a and open at b.

2. A cylinder flow and its properties

Let $X = [0, 1) = R/Z$ denote the additive group of real numbers modulo 1 and let μ be the Lebesgue measure on X. By T we denote the transformation of X given by

$$Tx = x + \alpha_0 \,(\text{mod } 1) \tag{2.1}$$

for every $x \in X$, where

$$\alpha_0 = \frac{\sqrt{5} - 1}{4}.$$

Consider the function $f: X \to Z$ given by

$$f(x) = \begin{cases} +1 & \text{for } 0 \le x < 1/2, \\ -1 & \text{for } 1/2 \le x < 1. \end{cases} \tag{2.2}$$

We write λ for the Haar (= counting) measure on Z and put $\tilde{X} = X \times Z$ and $\rho = \mu \times \lambda$. The function f defines a Z-extension S of T by setting

$$S(x, n) = (Tx, n + f(x)), (x, n) \in \tilde{X}. \tag{2.3}$$

Clearly S is a measure preserving automorphism of the measure space (X, ρ). Our aim is to show that S is ergodic. Let, for every $n \ge 1$, $x \in X$,

$$a(n, x) = \sum_{k=0}^{n-1} f(T^k x). \tag{2.4}$$

If we put $a(0, x) = 0$ and $a(-n, x) = -a(n, T^{-n}x)$, $n \ge 1$, $x \in X$, we obtain a map $a: Z \times X \to Z$ which satisfies the cocycle equation

$$a(n, T^m x) - a(n + m, x) + a(m, x) = 0 \tag{2.5}$$

for every $m, n \in Z$, $x \in X$, and we also have

$$S^k(x, n) = (T^k x, n + a(k, x)) \tag{2.6}$$

for every $k \in Z$, $(x, n) \in \tilde{X}$.

PROPOSITION 2.1: *Let $E(a)$ be the set of all integers k which have the following property: For every Borel set $A \subset X$ of positive measure,*

$$\mu\Big(\bigcup_{n \in Z} (A \cap T^{-n}A \cap \{x : a(n, x) = k\}\Big) > 0. \tag{2.7}$$

Then the following is true:

(1) *$E(a)$ is a subgroup of Z,*

(2) *S is ergodic if and only if $E(a) = Z$.*

PROOF: See Theorem 3.9 and Corollary 5.4 in [6].

The first step in our proof of the ergodicity of S will be to show that

$$2 \in E(a). \tag{2.8}$$

For every $N \geq 0$ we consider the numbers $\{-n\alpha_0 + t/2 \pmod 1) : 0 \leq n < N, t = 0, 1\}$ and arrange them in increasing order: $\beta_0^{(N)} = 0 < \beta_1^{(N)} < \beta_2^{(N)} < \cdots < \beta_{2N+1}^{(N)} = 1$, say. It is easy to see that $\beta_k^{(N)} = \beta_{k+N+1}^{(N)} - 1/2$ for every $k \leq N$. For every $x \in X$ and every $N \geq 0$ we choose $k_N(x)$ such that

$$\beta_{k_N(x)}^{(N)} \leq x < \beta_{k_N(x)+1}^{(N)}$$

and put

$$I_N^1(x) = [\beta_{k_N(x)}^{(N)}, \beta_{k_N(x)+1}^{(N)}) \tag{2.9}$$

and

$$I_N^2(x) = [\beta_{k_N(x)-1}^{(N)}, \beta_{k_N(x)+2}^{(N)}) \tag{2.10}$$

whenever $1 \leq k_N(x) \leq 2N$. If $k_N(x) = 0$, we define $I_N^1(x)$ by (2.9) and replace (2.10) by

$$I_N^2(x) = [0, \beta_2^{(N)}) \cup [\beta_{2N}^{(N)}, 1). \tag{2.11}$$

For $k_N(x) = 2N + 1$, we define $I_N^1(x)$ by (2.9) and set

$$I_N^2(x) = [\beta_{2N}^{(N)}, 1) \cup [0, \beta_1^{(N)}). \tag{2.12}$$

Furthermore we define

$$I_N^3(x) = T^N(I_N^1(T^{-N}x)) \tag{2.13}$$

and

$$I_N^4(x) = T^N(I_N^2(T^{-N}x)). \tag{2.14}$$

LEMMA 2.2:

$$\lim_N \sup \sup_{x \in X} \frac{\mu(I_N^1(x))}{\mu(I_N^2(x))} < 6/7 \tag{2.15}$$

PROOF: Let $(p_n/q_n, n \geq -2)$ be the sequence of best approximations of the irrational number $2\alpha_0 = (\sqrt{5}-1)/2$. It is given by $q_{-2} = p_{-1} = 0$, $p_{-2} = q_{-1} = 1$, and $q_n = q_{n-2} + q_{n-1}$, $p_n = p_{n-2} + p_{n-1}$ for $n \geq 0$. For any $\gamma \in R$ we write $\langle\langle \gamma \rangle\rangle$ for the distance of γ from the nearest integer. A classical result (see [3, p. 13]) states that

$$\lim_n \inf n\langle\langle 2n\alpha_0 \rangle\rangle = \lim_n q_n \langle\langle 2q_n\alpha_0 \rangle\rangle = 5^{-1/2}.$$

Consider now, for every n, the numbers $\{2k\alpha_0 \,(\mathrm{mod}\ 1): k = 1, \ldots, q_n\}$. We arrange them again in increasing order and denote this arrangement by $\gamma_1^{(n)} < \gamma_2^{(n)} < \cdots < \gamma_{q_n}^{(n)}$. Finally we put $\gamma_0^{(n)} = 0$ and $\gamma_{q_n+1}^{(n)} = 1$. It is easy to see that

$$r_n = \max_{i=0,\ldots,q_n} \gamma_{i+1}^{(n)} - \gamma_i^{(n)} \leq 2\langle\langle 2q_n\alpha_0 \rangle\rangle + 1/q_n.$$

Hence we have, for all sufficiently large n,

$$r_n \leq (1 + 4\sqrt{5})/q_n < 3/q_n.$$

In order to estimate $\mu(I_N^1(x))/\mu(I_N^2(x))$ we first note that, for every N,

$$\min_{0 \leq k < 2N+2} (\beta_{k+1}^{(N)} - \beta_k^{(N)}) = \tfrac{1}{2} \min_{0 \leq i \leq q_n} (\gamma_{i+1}^{(n)} - \gamma_i^{(n)}) = \frac{\langle\langle 2q_n\alpha_0 \rangle\rangle}{2},$$

and

$$\max_{0 \leq k \leq 2N+2} (\beta_{k+1}^{(N)} - \beta_k^{(N)}) \leq \tfrac{1}{2} \max_{0 \leq i \leq q_n} (\gamma_{i+1}^{(n)} - \gamma_i^{(n)}) < 3/2q_n,$$

where n is chosen so that $q_n < N \leq q_{n+1}$. For every $x \in X$ we thus have

$$\frac{\mu(I_N^1(x))}{\mu(I_N^2(x))} = \frac{\beta_{k_N(x)+1}^{(N)} - \beta_{k_N(x)}^{(N)}}{\mu(I_N^2(x))} \leq \frac{\beta_{k_N(x)+1}^{(N)} - \beta_{k_N(x)}^{(N)}}{\beta_{k_N(x)+1}^{(N)} - \beta_{k_N(x)}^{(N)} + \langle\langle 2q_n\alpha_0 \rangle\rangle}$$
$$< \frac{3/2q_n}{3/2q_n + 1/4q_n} = 6/7$$

for all sufficiently large N. The lemma is proved.

LEMMA 2.3:

$$\lim_N \sup \sup_{x \in X} \frac{\mu(I_N^3(x))}{\mu(I_N^4(x))} < 6/7. \tag{2.16}$$

PROOF: This is clear from Lemma 2.2.

LEMMA 2.4: *Let $A \subset X$ be a Borel set with $\mu(A) > 0$. Then there exists an integer n such that*

$$\mu(A \cap T^{-n}A \cap \{x : |a(n,x)| = 2\}) > 0. \tag{2.17}$$

PROOF: For every $x \in A$, consider the sequences of intervals $I_N^k(x)$, $k = 1, \ldots, 4$, $N = 1, 2, \ldots$. By a well known theorem of Lebesgue we have, for every $k = 1, \ldots, 4$,

$$\lim_N \frac{\mu(I_N^k(x) \cap A)}{\mu(I_N^k(x))} = 1$$

for μ-*a.e.*$x \in A$. We can thus choose a subset $A_0 \subset A$ of positive measure and an integer $N_0 > 0$ with

$$\frac{\mu(I_N^k(x) \cap A)}{\mu(I_N^k(x))} > 29/30$$

for every $x \in A_0$, $k = 1, \ldots, 4$, $N \geq N_0$. Since $\int a(1,x)\, \mathrm{d}\mu(x) = 0$, there exists an integer $m \geq N_0$ such that

$$\mu(A_0 \cap T^{-m}A_0 \cap \{x : a(m,x) = 0\}) > 0$$

(cf. [1] or [6, Theorem 11.4]). Let $B = A_0 \cap T^{-m}A_0 \cap \{x : a(m,x) = 0\}$. Since $m \geq N_0$, we have, for every $x \in B$, $k = 1, 2$,

$$\frac{\mu(I_m^k(x) \cap A)}{\mu(I_m^k(x))} > 29/30, \tag{2.18}$$

and

$$\frac{\mu(T^m(I_m^k(x)) \cap A)}{\mu(T^m(I_m^k(x))} = \frac{\mu(I_m^{k+2}(T^m x) \cap A)}{\mu(I_m^{k+2}(T^m x))} > 29/30. \tag{2.19}$$

(2.15), (2.16), (2.18) and (2.19) together imply the following:

$$\begin{aligned}
\frac{\mu((I_m^2(x) \backslash I_m^1(x)) \cap A)}{\mu(I_m^2(x) \backslash I_m^1(x))} &> \frac{\mu(I_m^2(x)) \cdot 29/30 - \mu(I_m^1(x))}{\mu(I_m^2(x)) - \mu(I_m^1(x))} \\
&= \frac{(\mu(I_m^2(x)) - \mu(I_m^1(x))) \cdot 29/30 - \mu(I_m^1(x))/30}{\mu(I_m^2(x)) - \mu(I_m^1(x))} \\
&> \frac{(\mu(I_m^2(x)) - \mu(I_m^1(x))) \cdot 29/30 - (\mu(I_m^2(x)) - \mu(I_m^1(x)))/5}{\mu(I_m^2(x)) - \mu(I_m^1(x))} = 23/30,
\end{aligned}$$

provided that N_0 (and hence m) is sufficiently large. Similarly one proves that

$$\frac{\mu((I_m^2(x)\backslash I_m^1(x)) \cap T^{-m}A)}{\mu((I_m^2(x)\backslash I_m^1(x))} > 23/30.$$

This implies that, for every $x \in B$,

$$\mu((I_m^2(x)\backslash I_m^1(x)) \cap A \cap T^{-m}A) > 0. \tag{2.20}$$

Having established (2.20), let us take a look at the function $a(m, \cdot)$. Obviously $a(m, \cdot)$ is discontinuous exactly at the points $\beta_k^{(m)}$, $k = 0, \ldots, 2m+1$, and at each of these points we have

$$\lim_{x \to 0} |a(m, \beta_k^{(m)} + x) - a(m, \beta_k^{(m)} - x)| = 2.$$

Moreover, $C_m = \{x : a(m, x) = 0\}$ is a union of disjoint right half-open intervals of the form $[\beta_k^{(m)}, \beta_{k+1}^{(m)})$, and none of the points $\beta_k^{(m)}$ lies in the interior of C_m. It follows immediately that for every $x \in C_m$, and for every $y \in I_m^2(x)\backslash I_m^1(x)$, $|a(m, y)| = 2$. (2.20) implies now (2.17), and the proof is complete.

LEMMA 2.5: *For every Borel set $A \subset X$ with $\mu(A) > 0$ there exists an integer m such that*

$$\mu(A \cap T^{-m}A \cap \{x : a(m, x) = 2\}) > 0. \tag{2.21}$$

In other words, $2 \in E(a)$.

PROOF: Using (2.17) we can find an integer m_0 with $\mu(A \cap T^{-m_0}A \cap \{x : |a(m_0, x)| = 2\}) > 0$. If $\mu(A \cap T^{-m_0}A \cap \{x : a(m_0, x) = 2\}) > 0$, the lemma is proved. Otherwise we put $B = A \cap T^{-m_0}A \cap \{x : a(m_0, x) = -2\}$, and see that $\mu(B) > 0$. For every $y \in T^{m_0}B$ we have $T^{-m_0}y \in A$, and $a(-m_0, y) = -a(m_0, T^{-m_0}y) = 2$. This shows that $T^{m_0}B = A \cap T^{m_0}A \cap \{x : a(-m_0, x) = 2\}$ has positive measure and proves the lemma.

Having proved (2.8) we now proceed to show that $E(a)$ also contains 1. To do this we consider the group Z_2 of integers mod 2 and the map $\tilde{a}$: $Z \times X \to Z_2$ given by $\tilde{a}(n, x) = a(n, x)$ (mod 2). As before we define an extension $\tilde{S}$ of T which now acts on $Y = X \times Z_2$:

$$\tilde{S}(x, \tilde{k}) = (Tx, \tilde{k} + \tilde{a}(1, x)) = (Tx, k + a(1, x) \pmod 2) \tag{2.22}$$

for every $(x, \tilde{k}) \in Y$. If $\tilde{\lambda}$ denotes the measure on Z_2 which assigns to each point mass 1/2, we see that $\tilde{S}$ is an isomorphism of Y which preserves the measure $\tilde{\rho} = \mu \times \tilde{\lambda}$. It is very easy to check that $\tilde{S}$ is ergodic. This implies

THEOREM 2.6: *Let $X = [0, 1) = R/Z$ be the group of real numbers modulo 1, and let μ be the Lebesgue measure on X. If λ denotes the counting measure on Z, consider the measure space $(X \times Z, \mu \times \lambda)$ and the measure preserving automorphism S of $(X \times Z, \mu \times \lambda)$ given by*

$$S(x, n) = (x + \alpha_0 \pmod 1), n + f(x)),$$

where

$$\alpha_0 = \frac{\sqrt{5} - 1}{4}$$

and

$$f(x) = \begin{cases} +1 & \text{for } 0 \leq x < 1/2 \\ -1 & \text{for } 1/2 \leq x < 1. \end{cases}$$

Then S is ergodic.

PROOF: We have already shown that $E(a)$ contains 2 (cf. Lemma 2.6). If $E(a) \neq Z$, it must be equal to $2Z = \{2n : n \in Z\}$. Lemma 3.10 in [6] (or a simple direct proof) shows that this would force the automorphism $\tilde{S}$ in (2.22) to be nonergodic, which is absurd. Hence $E(a) = Z$, and S is ergodic by Proposition 2.1. The theorem is proved.

3. Concluding remarks

One of the main problems in the theory of uniform distribution is the question how well a sequence of the form $(n\alpha \pmod 1), n = 1, 2, \ldots)$, α irrational, is distributed in the unit interval X. To measure the regularity of distribution one defines the discrepancy D_N of the sequence by

$$D_N = \sup_{0 \leq a < b < 1} \frac{1}{N} \left| \sum_{k=1}^{N} (\chi_{(a,b)}(k\alpha \pmod 1)) - (b - a) \right|$$

for every $N \geq 1$, where $\chi_{[a,b)}$ denotes the characteristic function of the interval $[a, b)$. This immediately leads to an investigation of the functions $c^\alpha_\beta : Z \times X \to R$ given by

$$c^\alpha_\beta(n, x) = \begin{cases} \sum_{k=0}^{n-1} (\chi_{[0,\beta)}(x + k\alpha \ (\mathrm{mod}\ 1)) - \beta & \text{for } n \geq 1, \\ 0 & \text{for } n = 0, \\ -c^\alpha_\beta(-n, x + n\alpha \ (\mathrm{mod}\ 1)) & \text{for } n < -1. \end{cases}$$

Such functions satisfy the cocycle equation (2.5) for the irrational rotation $Tx = x + \alpha$ (mod 1). From [5] it follows that the sequence $(c^\alpha_\beta(n, x),\ n = 1, 2, \ldots)$ is unbounded for every irrational α, every $x \in X$, and for every β which is not of the form $\beta = m\alpha$ (mod 1) for some m. Theorem 2.6 in this note gives a more precise result concerning the irregularity with which the numbers $n\alpha_0$, $n = 1, 2, \ldots$, fall into the two halves $[0, 1/2)$ and $[1/2, 1)$ of the unit interval. An equivalent formulation of Theorem 2.6 is the following statement:

For every pair of sets A, B of positive measure in the unit interval X and for every integer k we can find an integer n and a point $x \in A$ such that

(1) $x + n\alpha_0 \,(\mathrm{mod}\ 1) \in B$,
(2) $c^{\alpha_0}_{1/2}(n, x) = k/2$.

The property of α_0 which was used in the proof of Theorem 2.6 was that $\lim_n \inf n\langle\langle n\alpha_0 \rangle\rangle > 0$, and the proof will work for any irrational α with this property. Since this paper was written, the result of Theorem 2.6 has been extended to all irrational numbers α (cf. [4] and [6, Theorem 12.8]). Further results in this direction have been achieved in [2].

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