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THE STRUCTURE OF THE CUT LOCUS IN DIMENSION LESS THAN OR EQUAL TO SIX

Michael A. Buchner

0. Introduction

The main result is a structure theorem for the generic cut locus for a compact manifold M in dimensions 2, 3, 4, 5, and 6. We give a procedure which works for any stable cut locus in any dimension but the generic cut locus is stable only in dimensions ≤ 6. (See our earlier paper [1]). We actually carry out the computations in dimensions 2 and 3. The result is as follows.

THEOREM: *If* dim $M = 2$ *and* $p \in M$ *then the picture near a point* q *on a stable cut locus with respect to* p *is* (i) *a straight line through* q *or* (ii) *a straight line starting at* q *or* (iii) *three straight lines meeting at* q *any two of which have regular intersection (see section* 3 *for the definition of regular intersection). See figure* [1].

If dim $M = 3$ *and* $p \in M$ *then the picture near a point* q *on a stable cut locus with respect to* p *is* (i) *a plane through* q *or* (ii) *three planes meeting along a line through* q, *any two of the planes having regular intersection or* (iii) *the picture of* 6 *planes meeting along* 4 *lines all meeting at* q *obtained by viewing* q *as the barycenter of a tetrahedron and joining it to the* 4 *vertices or* (iv) *a half plane with* q *in the boundary or* (v) *a quarter plane glued onto a surface. See figure* [2]

As a curious byproduct of this theorem we obtain in Section 4 a decomposition theorem for compact 3 manifolds:

THEOREM: *Any compact differentiable* 3 *manifold has a compact subset which is locally homeomorphic to* (i), (ii), *or* (iii) *of figure* [2] *such that the complement of the compact subset is an open cell.*

This is related to a standard geometric topological procedure in Section 4.

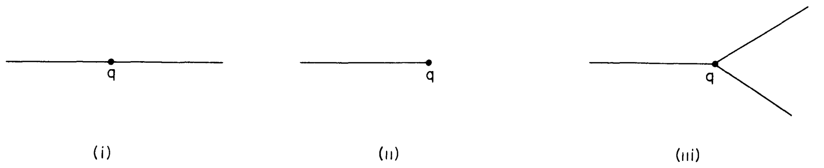

Figure 1.

Next we discuss the relationship of our calculations and results to those of other authors.

In his paper [12] Thom gives (without proof) the picture described above for a 2 manifold for a fixed metric and for generic point p on the manifold i.e., one allows the origin of the cut locus to be variable. (However the genericity appears to be contradicted by the result of Singer and Gluck [17] that there exists in $\mathbb{R}^3$ a strictly convex surface of revoltion having cut locus $C(p)$ nontriangulable for a nonempty open set of points p.)

D. Schaeffer in [11] obtains a similar local picture as in figures [1(i)], [1(ii)], and [1(iii)] for the shock waves for hyperbolic conservation law equations (for 1 space variable. There is no higher dimensional analogue). In a sense the role of stability in our work is played by an

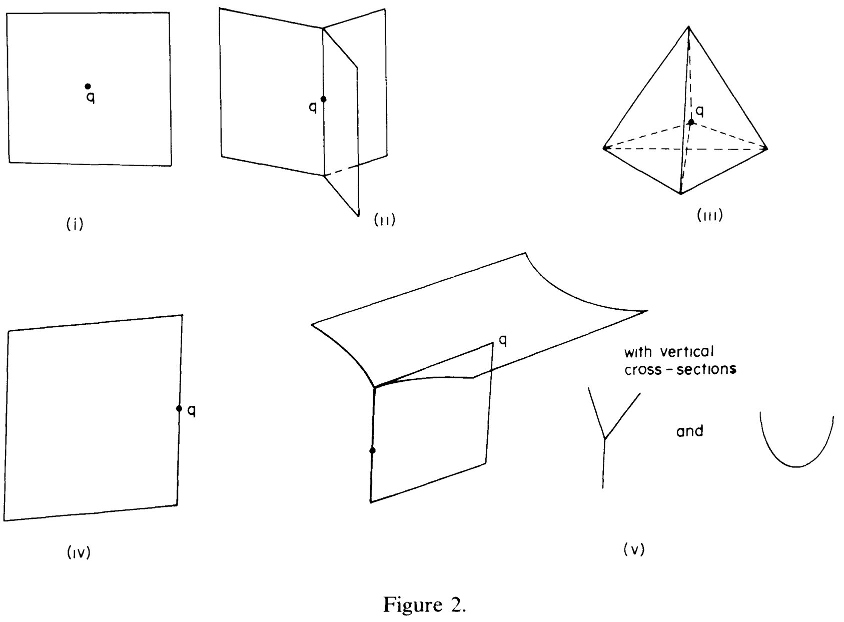

Figure 2.

assumption of uniform convexity together with a restriction to generic initial data in the work of Schaeffer.

In his review article [3] J.J. Duistermaat has given a global treatment of unfoldings and a multitransversality condition for globally stable unfoldings. Our multitransversality condition is a 'right-left' version of Duistermaat's 'right' version. (Right and left refer to allowing composition with diffeomorphisms on the right and on the left). We proved our multitransversality condition in an earlier paper [1] and the extra work involved beyond Duistermaat is not insignificant. We learnt right-left techniques from G. Wasserman [13].

Historically, the cut locus was first introduced by Poincare [10] for compact simply connected surfaces of positive curvature. In 1935, Summer B. Meyers [7], [8], and J.H.C. Whitehead [15] both studied the cut locus. Whitehead showed that any compact n-dimensional Riemannian manifold decomposes into the cut locus and an open cell with the cut locus as the cell boundary. Meyers showed that for compact analytic surfaces the cut locus is a graph, for simply connected surfaces the graph is a tree, the end points of which are conjugate to the origin of the cut locus and are cusps of the locus of first conjugate points. This result of the cut locus for a compact analytic surface being a graph has been extended to arbitrary dimensions in a separate paper of ours [2] where the cut locus is proved to be a simplicial complex. In a recent paper V. Ozols [9] analyzed the local structure of the cut locus near cut points which are not conjugate to the origin of the cut locus. His local picture (for arbitrary metric) is that of hypersurfaces intersected with half spaces (of which our figures [1(i)] and [1(ii)] and [2(i)], [2(ii)], and [2(iii)] are examples). However, he does not obtain a finite classification and there is no general statement of regular intersection because there is no restriction on the metric. Of course to obtain a local picture near a cut point which is also a conjugate point without some restriction on the metric is not very likely.

After these calculations were carried out it was pointed out to us by D. Meyer and A. Bellaiche that the above pictures are amongst the pictures in Dubóis, Dufour and Stanek [16] in their study of catastrophes. Nevertheless that the above pictures in dimensions 2 and 3 (and others which can be derived similarly in dimensions 4, 5 and 6) are actually the generic cut locus pictures appears to be new. Moreover our methods are rather different.

Finally we wish to thank D. Meyer and A. Bellaiche for catching an error in the original version of this paper in which picture (v) was missing.

1. The Morse theory approach to the cut locus

Let M be a compact C^∞ manifold without boundary and let p be a point in M. The space of metrics will be viewed as the space of sections of the bundle of positive definite symmetric matrices over M and will have the C^∞ Whitney topology. If α is a metric on M then $C(p, \alpha)$ will denote the *cut locus* in M with respect to p and the metric α i.e. the set of those points q in M which are joined to p by a length minimizing geodesic which fails to minimize the length to points beyond q on the geodesic. The piecewise smooth paths on M, starting at p, will be denoted by $\Omega(M)$ (the parametrization is on the closed interval $[0, 1]$). For any choice of metric α the following defines a distance function in the space $\Omega(M)$

$$d(\omega, \omega') = \sup_{t \in [0,1]} d_\alpha(\omega(t), \omega'(t)) + \int_0^1 \left(\left\| \frac{d\omega}{dt} \right\|_\alpha - \left\| \frac{d\omega'}{dt} \right\|_\alpha \right)^2 dt$$

where $\| \ \|_\alpha$ is the norm associated to α and d_α the distance function in M associated to α. This topology will be called the α-topology. With respect to this topology the energy function

$$E(\alpha)(\omega) = \int_0^1 \|d\omega/dt\|^2 dt$$

is continuous.

In [1] we constructed a finite dimensional model of the path space which permits the study of $E(\alpha)$ for variable α (thus generalizing the well known construction in Morse theory for a fixed metric). To be precise, for a fixed metric α_0, we constructed

(a) an open subset U of the space of metrics containing α_0

(b) a manifold B_1 (finite dimensional) without boundary (an open subset of the product of M with itself a certain number of times)

(c) for each $\alpha \in U$ a topological embedding $j(\alpha): B_1 \to \Omega(M)$ ($\Omega(M)$ having the α-topology). The image $j(\alpha)(B_1)$ consists of certain piecewise geodesics (with respect to the metric α)

(d) a submanifold B of B_1 which is compact, has boundary and is zero codimensional

(e) a map $\mathrm{Pr}: B_1 \to M$ which is a submersion and such that $\mathrm{Pr}(B) = M$

The above constructed entities were shown to enjoy the following properties

(i) $H(\alpha) = E(\alpha) \circ j(\alpha)$ is smooth on B_1

(ii) the following diagram is commutative

$$\begin{array}{ccc} B & \xrightarrow{j(\alpha)} & \Omega(M) \\ & {\scriptstyle \mathrm{Pr}}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & M & \end{array}$$

where π is the endpoint projection map.

(iii) if $q \in M$ then the minima of $E(\alpha)$ along $\pi^{-1}(q)$ are in $j(\alpha)(B)$ and these minima correspond via $j(\alpha)$ to the minima of $H(\alpha)$ along $\mathrm{Pr}^{-1}(q)$.

Moreover, we proved two further facts

(1) $C(p, \alpha) = \{q \in M | H(\alpha) | \mathrm{Pr}^{-1}(q)$ has a degenerate minimum or at least two minima$\}$

(2) We can assume there is a closed interval $I \subset \mathbb{R}$ such that $H(\alpha)(B) \subset I$ for all $\alpha \in U$. Denote the projection $M \times \mathbb{R} \to M$ by π_1. Then if dim $M \leq 6$ the composition of maps

$$B \xrightarrow[(\mathrm{Pr}, H(\alpha))|B]{} M \times I \xrightarrow[\pi_1 | M \times I]{} M$$

is a stable composition for α in a residual subset of U. This means that there is a neighborhood Y of $H(\alpha) \mid B$ in $C^\infty(B, \mathbb{R})$, continuous maps

$$A_1 : Y \to \mathrm{Diff}^\infty(B_1)$$
$$A_2 : Y \to \mathrm{Diff}^\infty(M \times \mathbb{R})$$
$$A_3 : Y \to \mathrm{Diff}^\infty(M)$$

with $A_i(H(\alpha) \mid B) = \text{identity}$ such that for $\tilde{f} \in Y$ the following diagrams commute

$$\begin{array}{ccc} B_1 & \xleftarrow{A_1(\tilde{f})^{-1}} & B \\ {\scriptstyle (\mathrm{Pr}, H(\alpha))}\downarrow & & \downarrow{\scriptstyle (\mathrm{Pr}, \tilde{f})} \\ M \times \mathbb{R} & \xrightarrow{A_2(\tilde{f})} & M \times \mathbb{R} \end{array}$$

$$\begin{array}{ccc} M \times \mathbb{R} & \xleftarrow[A_2(\tilde{f})^{-1}]{} & M \times I \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow{A_3(\tilde{f})} & M \end{array}$$

To save breath, we shall refer to a metric in the above mentioned residual set as a *cut-stable metric*. The reason for this is that as a consequence of the stability of the pair $((\mathrm{Pr}, H(\alpha)) \mid B, \pi_1 \mid M \times I)$ it is easy to see that $C(p, \alpha)$ is structurally stable i.e. if β is near α then there is a diffeomorphism $\phi: M \to M$ such that $\phi(C(p, \alpha)) = C(p, \beta)$ (in fact $\phi = A_3(H(\beta))$). Moreover ϕ depends continuously on β.

REMARK: It is not claimed that the converse is true i.e. it is not claimed that structural stability of $C(p, \alpha)$ implies α is a cut-stable metric. However if $C(p, \alpha)$ is structurally stable then $C(p, \alpha)$ is diffeomorphic (via an ambient diffeomorphism $\phi: M \to M$) to $C(p, \alpha')$ where α' is a cut-stable metric (in dimensions 2, 3, 4, 5 and 6). Consequently it is enough to compute the structure of those $C(p, \alpha)$ associated to cut-stable metrics.

2. The local and global structure of the energy function for cut-stable metrics

It will be assumed from now on that α is a given fixed cut-stable metric. Suppose now that $f: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is a C^∞ germ and $F: (\mathbb{R}^n \times \mathbb{R}^m, (0, 0)) \to (\mathbb{R}, 0)$ is a C^∞ germ such that $F(x, 0) = f(x)$. Then F is called an *unfolding* of f. If $G: (\mathbb{R}^n \times \mathbb{R}^m, (0, 0)) \to (\mathbb{R}, 0)$ is an unfolding of $g: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ then G is said to be equivalent to F if there is a germ $\Phi: (\mathbb{R}^n \times \mathbb{R}^m, (0, 0)) \to (\mathbb{R}^n \times \mathbb{R}^m, (0, 0))$ such that $\Phi(x, y) = (\phi(x, y), \psi(y))$, ψ a diffeomorphism $(\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$, $\phi(x, 0)$ a diffeomorphism $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, a germ $\lambda: (\mathbb{R}^m \times \mathbb{R}, (0, 0)) \to (\mathbb{R}, 0)$ with $\partial\lambda/\partial t(0, 0) > 0$, where t is the real variable, such that

$$F(x, y) = \lambda(y, G \circ \Phi(x, y))$$

If $K: U \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a C^∞ map and $(x_0, y_0) \in U$ then $K(x_0 + x, y_0 + y) - K(x_0, y_0)$ is an unfolding of $K(x_0 + x, y_0) - K(x_0, y_0)$. Denote the germ of this unfolding by $K_{(x_0, y_0)}$. The unfolding F of the germ f is said to be stable if there is a representative $\tilde{F}: V \to \mathbb{R}$ of F (V some neighborhood of $(0, 0)$) and a neighborhood $\mathscr{W}$ of $\tilde{F}$ in $C^\infty(V, \mathbb{R})$ such that given $\tilde{G}$ in $\mathscr{W}$ there is $(x_0, y_0) \in V$ such that F is equivalent to $\tilde{G}_{(x_0, y_0)}$.

The germ of $H(\alpha) - H(\alpha)(b)$ at any $b \in \mathrm{Int}\, B$ (the interior of B) is a stable unfolding. To be precise, let $b \in \mathrm{Int}\, B$ and observe that, since Pr is submersive, B has a local product structure at b with respect to which Pr is projection. The claim is that the germ of $H(\alpha) - H(\alpha)(b)$

at b is a stable unfolding with respect to this product structure. This can be established as follows: let D_r = open disc of radius r in $\mathbb{R}^q$ centered at $0 \in \mathbb{R}^q$. Let f = a C^∞ function defined on D_3 and let $J^k(U, \mathbb{R})$ denote the k jets of functions from U to $\mathbb{R}$. If $A \subset J^k(D_2, \mathbb{R})$ is an open set containing the k jets $j^k f(x)$ for all $x \in D_2$ and if $g \in C^\infty(D_2, \mathbb{R})$ then $g \mid D_1$ can be extended to $\tilde{g} \in C^\infty(D_3, \mathbb{R})$ such that $\tilde{g} \mid D_3 - D_2 = f \mid D_3 - D_2$ and such that $j^k \tilde{g}(D_2) \subset A$ provided g is sufficiently close to $f \mid D_2$. In fact let σ be a C^∞ function which is 1 on D_1 and 0 outside D_2 and let $\tilde{g} = f + \sigma(g - f)$. If this observation is combined with the following observation: for fixed $b \in B$ the point $A_1(h)(b)$ depends continuously on $h \in C^\infty(B, \mathbb{R})$ then the claim at the beginning of the paragraph follows easily. To be more specific: If g is a function defined on a neighborhood of b and close to $H(\alpha) - H(\alpha)(b)$ in this neighborhood then g can be extended to $\tilde{g}$ defined and close to $H(\alpha) - H(\alpha)(b)$ on B. Then $\tilde{g} + H(\alpha)(b)$ is close to $H(\alpha)$. So $\tilde{g} + H(\alpha)(b)$ at $A_1(\tilde{g} + H(\alpha)(b))(b)$ is equivalent to $H(\alpha)$ at b using the maps $A_1(\tilde{g} + H(\alpha)(b))$ and $A_2(\tilde{g} + H(\alpha)(b))$.

It is a consequence of unfolding theory that up to equivalence there are only three stable unfoldings $F : (\mathbb{R}^n \times \mathbb{R}^2, (0, 0)) \to (\mathbb{R}, 0)$ which have $d_x F(0, 0) = 0$, namely, if $y = (u, v)$ and $x = (x_1, \ldots, x_n)$ they are

$$\begin{cases} F(x_1, \ldots, x_n, u, v) = \sum_{i=1}^{n} \pm x_i^2 \\ F(x_1, \ldots, x_n, u, v) = x_1^3 + u x_1 + \sum_{i>1} \pm x_i^2 \\ F(x_1, \ldots, x_n, u, v) = \pm x_1^4 + u x_1^2 + v x_1 + \sum_{i>1} \pm x_i^2 \end{cases}$$

If it is required that 0 be a minimum for the germ $F \mid \mathbb{R}^n \times (0)$ then the only possibilities are

$$F(x_1, \ldots, x_n, u, v) = \sum_{i=1}^{n} x_i^2$$

$$F(x_1, \ldots, x_n, u, v) = x_1^4 + u x_1^2 + v x_1 + \sum_{i>1} + x_i^2$$

It is also a consequence of unfolding theory that up to equivalence there are only six stable unfoldings $F : (\mathbb{R}^n \times \mathbb{R}^3, (0, 0)) \to (\mathbb{R}, 0)$ which have $d_x F(0, 0) = 0$, namely if $y = (u, v, w)$ and $x = (x_1, x_2, \ldots, x_n)$ they are

$$\begin{cases} F(x_1, \ldots, x_n, u, v, w) = \sum_{i=1}^{n} \pm x_i^2 \\ F(x_1, \ldots, x_n, u, v, w) = x_1^5 + ux_1^3 + vx_1^2 + wx_1 + \sum_{i>1} \pm x_i^2 \\ F(x_1, \ldots, x_n, u, v, w) = x_1^3 + x_2^3 + ux_1x_2 + vx_1 + wx_2 + \sum_{i>2} \pm x_i^2 \\ F(x_1, \ldots, x_n, u, v, w) = x_1^3 - x_1x_2^2 + u(x_1^2 + x_2^2) + vx_1 + wx_2 + \sum_{i>2} \pm x_i^2 \end{cases}$$

and the suspension of the two others for $(\mathbb{R}^n \times \mathbb{R}^2, (0,0)) \to (\mathbb{R}, 0)$.

If, moreover, it is required that 0 be a minimum for the germ $F \mid \mathbb{R}^n \times \{0\}$ then there are only two possibilities; namely

$$F(x_1, \ldots, x_n, u, v, w) = \sum_{i=1}^{n} x_i^2$$

and

$$F(x_1, \ldots, x_n, u, v, w) = x_1^4 + ux_1^2 + vx_1 + \sum_{i>1} \pm x_i^2.$$

Thus far only the local structure of $H(\alpha)$ has been given. However, the fact that α is cut stable imposes global conditions as well. Recall some consequences of C^∞ stability theory in the following form, which will be convenient for the subsequent theory. Let N and P be C^∞ manifolds $K \subset N$ a closed submanifold of codimension 0, and $f: N \to P$ a C^∞ map such that $f \mid K$ is proper. The map $f \mid K$ is said to be stable if there exists a neighborhood $\mathcal{V}$ of $f \mid K$ in $C^\infty(K, P)$ and continuous maps $H_1: \mathcal{V} \to \mathrm{Diff}^\infty(N)$ $H_2: \mathcal{V} \to \mathrm{Diff}^\infty(P)$ such that $H_1(f \mid K) = \text{identity}$, $H_2(f \mid K) = \text{identity}$ and $g = H_2(g) \cdot f \cdot H_1(g) \mid K$ for all $g \in \mathcal{V}$. Then $f \mid K$ is stable if and only if for all finite subsets S of K such that $f(S) = \{y\}$ (a single point) the germ $f: (K, S) \to (P, y)$ is infinitesimally stable. But $f: (K, S) \to (P, y)$ is infinitesimally stable if and only if each germ $f: (K, s) \to (P, y)$ ($s \in S$) is infinitesimally stable *and* the vector spaces $df(s)(T_sK) = V_s$ are in general position i.e. if $S = \{s_1, s_2, \ldots, s_\ell\}$ then codim (in T_yP) of $V_{s_1} \cap V_{s_2} \cap \cdots \cap V_{s_\ell} =$ codim V_{s_1} + codim $V_{s_2} + \cdots +$ codim V_{s_ℓ}. (The reference for these ideas is Mather [4], [5] and [6]).

The map $(\mathrm{Pr}, H(\alpha)): B_1 \to M \times \mathbb{R}$ is stable when restricted to B and consequently must obey the tangent space codimension condition (taking $N = B_1$, $K = B$, $f = (\mathrm{Pr}, H(\alpha))$ and $P = M \times \mathbb{R}$).

Suppose $f_1, \ldots, f_\ell$ are C^∞ germs: $(\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \to (\mathbb{R}, 0)$ and

$\pi:(\mathbb{R}^n \times \mathbb{R}^m, (0,0)) \to (\mathbb{R}^m, 0)$ is projection on the second factor. Suppose $d_x f_i(0,0) = 0$ ($i = 1, 2, \ldots, \ell$) where x is the $\mathbb{R}^n$ variable and suppose

$$\sum_{i=1}^{\ell} \operatorname{codim} d(\pi, f_i)(0,0)(\mathbb{R}^n \times \mathbb{R}^m) = \operatorname{codim} \bigcap_{i=1}^{\ell} d(\pi, f_i)(0,0)(\mathbb{R}^n \times \mathbb{R}^m)$$

Let $y_1, \ldots, y_m$ be the $\mathbb{R}^m$ variables. Then the tangent space codimension restriction is equivalent to the assertion that the matrix

$$\begin{pmatrix} \dfrac{\partial(f_1 - f_2)}{\partial y_1}(0,0) & \cdots & \dfrac{\partial(f_1 - f_2)}{\partial y_m}(0,0) \\ \vdots & & \vdots \\ \dfrac{\partial(f_{\ell-1} - f_\ell)}{\partial y_1}(0,0) & \cdots & \dfrac{\partial(f_{\ell-1} - f_\ell)}{\partial y_m}(0,0) \end{pmatrix}$$

has rank $\ell - 1$. Note that this implies that $\ell \leq m + 1$. To put it another way: If $f = (\mathrm{Pr}, H(\alpha)) : B \to M \times \mathbb{R}$ and if $s_1, \ldots, s_\ell$ are paths in B from p to q each having minimum energy among such paths then $df(s_i)(T_{s_i}B)$ has dim m in $T_qM \times \mathbb{R}$. Since these hyperplanes are in general position we get $\ell \leq m + 1$.

As an immediate corollary one obtains the following:

PROPOSITION: *If $x \in C(p, \alpha)$ and α is cut stable then no more than* dim $M + 1$ *length minimizing geodesics join p to q.*

3. Computation of the local structure in dimensions 2 and 3

This section computes the local structure of $C(p, \alpha)$ for manifolds of dimension 2 and 3 where α is cut stable. The methods work in any dimensions but, of course, genericity is restricted to dimensions ≤ 6.

In order to compute what cases can and cannot occur for cut-stable metrics, recall the characterization of cut-stable metrics contained in the following theorem (proved in [1]):

THEOREM: *The function $H(\alpha)$ is cut stable if and only if it satisfies the $(2m + 2)$ nd. order r-multitransversality condition on B for $r \leq m + 2$.*

The meaning of this is the following: Let $J_0^k(n, 1)$ be the k jets of germs $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$. The vectorspace $\mathbb{R}^r\times[J_0^k(n,1)]^r$ is acted upon by the group $\mathbb{R}\times L^k(1)\times[L^k(n)]^r$ where $L^k(n)$ is k jets of diffeomorphisms $\phi:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$, the action of $\mathbb{R}$ on $\mathbb{R}^r$ is $\alpha(x_1,x_2,\ldots,x_n)=(x_1+\alpha,x_2+\alpha,\ldots,x_r+\alpha)$, and the actions of $L^k(n)$ and $L^k(1)$ on $J_0^k(n,1)$ are by composition on the right and left respectively. Now suppose $b_1,b_2,\ldots,b_r$ are points which project to q under Pr. Around each b_i we can put a local product neighborhood $U\times A_i$, $U\subset\mathbb{R}^m$ open, $A_i\subset\mathbb{R}^n$ open, with respect to which Pr is projection on the first factor. The function $H(\alpha)$ induces a map $j_1^kH(\alpha):U\times A_i\to\mathbb{R}\times J_0^k(n,1)$ by taking the k jet in the A_i space. This in turn induces a map ${}_rj_1^kH(\alpha):U\times A_1\times A_2\times\cdots\times A_r\to\mathbb{R}^r\times[J_0^k(n,1)]^r$. The multitransversality condition means that ${}_rj_1^kH(\alpha)$ is transversal at all points of $U\times A_1\times\cdots\times A_r$ to all orbits of $\mathbb{R}\times L^k(1)\times[L^k(n)]^r$ for $k=2m+2$ and $r\le m+2$ (this is independent of the choice of product neighborhoods).

This multitransversality condition yields an alternative proof of the tangent space codimension condition or equivalently the rank condition at the end of the last section. Let H_i be $H(\alpha)$ viewed on $U\times A_i$. Let $\zeta:\mathbb{R}^r\times[J_0^k(n,1)]^r\to\mathbb{R}^r$ be projection. The transversality of ${}_rj_1^kH(\alpha)$ at $(0,0,\ldots,0)$ to the orbit of ${}_rj_1^kH(\alpha)$ implies $\zeta(V)+$ (diagonal of $\mathbb{R}^r$) $=\mathbb{R}^r$ where V is the space spanned by the m partial derivatives of ${}_rj_1^kH(\alpha)$ at $(0,\ldots,0,0)$ with respect to the U variables $y_1,\ldots,y_m$. But $\zeta(V)$ is the space spanned by $\{(\partial H_1/\partial y_i(0,\ldots,0),\ldots,\partial H_r/\partial y_i(0,\ldots,0))\}_{i=1,\ldots,m}$. It follows that $\zeta(V)+$ (diagonal of $\mathbb{R}^r$) $=\mathbb{R}^r$ is equivalent to the condition that the matrix $(\partial(H_i-H_{i+1})/\partial y_j(0,\ldots,0))$ have rank $r-1$.

Now suppose $m=2$. Let $q\in C(p,\alpha)$. We have seen that $H(\alpha)\,|\,\mathrm{Pr}^{-1}(q)$ has at most three minima. If there is just one minimum this must be degenerate. The local picture of $C(p,\alpha)$ is entirely determined by the following set-up.

$$\begin{array}{ccc} (\mathbb{R}^n\times\mathbb{R}^2,0) & \xrightarrow{H} & \mathbb{R} \\ \Big\downarrow{\scriptstyle \pi} & & \\ (\mathbb{R}^2,0) & & \end{array}$$

where $H(x_1,\ldots,x_n,u,v)=x_1^4+ux_1^2+vx_1+\Sigma_{i>1}x_i^2$ and π is projection on the second factor. Then the germ of $C(p,\alpha)$ at q is equivalent to the germ of $\{(u,v)|H|\pi^{-1}(u,v)$ has a degenerate minimum or at least two minima$\}$ via a diffeomorphism $(M,q)\to(\mathbb{R}^2,0)$. The term $\Sigma_{i>1}x_i^2$ is irrelevant to the calculation. What is being sought is the set of (u, v) such that there exists $x \neq \bar{x}$ satisfying

$$\begin{cases} 4x^3 + 2ux + v = 0 \text{------------------------------} & (1) \\ 4\bar{x}^3 + 2u\bar{x} + v = 0 \text{------------------------------} & (2) \\ x^4 + ux^2 + vx = \bar{x}^4 + u\bar{x}^2 + v\bar{x} \text{-----------------------} & (3) \end{cases}$$

Equation (3) yields $x^3 + x^2\bar{x} + x\bar{x}^3 + \bar{x}^3 + u(x + \bar{x}) + v = 0$ and if we subtract $\frac{1}{2}((1) + (2))$ we get

$$-x^3 - \bar{x}^3 + x^2\bar{x} + x\bar{x}^3 = 0$$

or

$$-(x + \bar{x})(x - \bar{x})^2 = 0.$$

Consequently $x = -\bar{x}$. It follows that $v = 0$ and $u < 0$. Conversely if $v = 0$ and $u < 0$ then (1) and (2) and (3) are easily seen to be satisfied.

REMARK: It is interesting to compare this and subsequent calculations with those of D. Schaeffer in his paper [11] on the quasi-linear conservation law. A completely different problem in partial differential equations leads to calculations which have similarity to ours. But there is an important difference: the generic structure of the cut locus can be computed in dimensions 2, 3, 4, 5 and 6 while the Lax function in [11] is defined on a space fibered over space-time where space is one-dimensional.

The calculations above may be interpreted as follows: if $q \in C(p, \alpha)$ and there is a single degenerate geodesic joining p to q then (a) $C(p, \alpha)$ near q is equivalent to a straight line starting at q (b) points on $C(p, \alpha)$ near q have exactly two nondegenerate length minimizing geodesics joining them to p.

The next possibility is that there are exactly two minima b_1 and b_2 of $H(\alpha) | \mathrm{Pr}^{-1}(q)$ both of which are nondegenerate. Locally this reduces to the following set-up:

$$\begin{array}{ccccccc} \mathbb{R} & \xleftarrow{H_1} & (\mathbb{R}^n \times \mathbb{R}^2, 0) & & (\mathbb{R}^n \times \mathbb{R}^2, 0) & \xrightarrow{H_2} & \mathbb{R} \\ & & \downarrow \pi & & \downarrow \pi & & \\ & & (\mathbb{R}^2, 0) & \xrightarrow{\simeq} & (\mathbb{R}^2, 0) & & \end{array}$$

where

$$H_1(x_1, \ldots, x_n, u, v) = \lambda\left(\sum_{i=1}^{n} x_i^2, u, v\right)$$

$$H_2(x_1, \ldots, x_n, u, v) = \mu\left(\sum_{i=1}^{n} x_i^2, \phi(u, v)\right),$$

ϕ is a germ of a diffeomorphism and $\partial\lambda/\partial z(0,0,0) > 0$ (z is the first variable) and $\partial\mu/\partial z(0,0,0) > 0$. By redefining μ we can take H_2 to be $\mu(\Sigma_{i=1}^{n} x_i^2, u, v)$. Then $C(p, \alpha)$ near q is equivalent to the set of (u, v) satisfying the equation

$$\lambda(0, u, v) = \mu(0, u, v).$$

The codimension restriction is $(\partial(\lambda - \mu)/\partial u(0,0,0), \partial(\lambda - \mu)/\partial v(0,0,0)) \neq (0,0)$. It follows that $C(p, \alpha)$ near q is a 1-dimensional submanifold every point of which is joined to p by exactly two nondegenerate length minimizing geodesics.

Consider the germ ${}_2j_1^6 H(\alpha) : (\mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^n, (0,0,0)) \to \mathbb{R}^2 \times J_0^6(n, 1) \times J_0^6(n, 1)$ determined by $H(\alpha)$ at the two minima b_1 and b_2. Suppose that b_1 is a degenerate minimum for $H(\alpha) \mid \mathrm{Pr}^{-1}(q)$. Then the codimension of the orbit through ${}_2j_1^6 H(\alpha)(0,0,0)$ is $\geq 1 + n + n + 2$. It is not possible for ${}_2j_1^6 H(\alpha)$ to be transversal to this orbit for dimension reasons. Consequently the only possibility for two minima is the one already discussed. In the case of three minima multitransversality gives a germ ${}_3j_1^6 H(\alpha) : (\mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, (0,0,0,0)) \to \mathbb{R}^3 \times [J_0^6(n, 1)]^3$. If one of the minima is degenerate then the codimension of the orbit through ${}_3j_1^6 H(\alpha)(0,0,0)$ is $\geq 1 + n + n + n + 2$ and again transversality is impossible. This implies that the only possibility remaining is three nondegenerate minima. In this case the germ of $C(p, \alpha)$ is determined by three functions $\lambda(z, u, v)$, $\mu(z, u, v)$ and $\zeta(z, u, v)$ where $(z, u, v) \in \mathbb{R}^3$ and $\partial\lambda/\partial z(0,0,0) > 0$, $\partial\mu/\partial z(0,0,0) > 0$ and $\partial\zeta/\partial z(0,0,0) > 0$. To be precise, $C(p, \alpha)$ is equivalent near q to the set of (u, v) satisfying one of the following four conditions:

(1) $\zeta(0, u, v) = \mu(0, u, v)$ and $\lambda(0, u, v) > \zeta(0, u, v)$

(2) $\zeta(0, u, v) = \lambda(0, u, v)$ and $\mu(0, u, v) > \zeta(0, u, v)$

(3) $\lambda(0, u, v) = \mu(0, u, v)$ and $\zeta(0, u, v) > \lambda(0, u, v)$

(4) $\lambda(0, u, v) = \mu(0, u, v) = \zeta(0, u, v)$

In addition the codimension restriction is that the matrix

$$\begin{pmatrix} \dfrac{\partial(\lambda-\mu)}{\partial u}(0,0,0) & \dfrac{\partial(\lambda-\mu)}{\partial v}(0,0,0) \\ \dfrac{\partial(\mu-\zeta)}{\partial u}(0,0,0) & \dfrac{\partial(\mu-\zeta)}{\partial v}(0,0,0) \end{pmatrix}$$

have rank 2. Consequently the local normal form is 3 lines (embedded half closed intervals) meeting at a point with any pair of their tangents at their common boundary point having regular intersection (Definition: If $E_1, E_2, \ldots, E_r$ are vector subspaces of the vector-space E then regular intersection for the E_i means $\operatorname{codim} \bigcap_i E_i = \Sigma_i \operatorname{codim} E_i$). The geometric interpretation is that there are 3 nondegenerate length-minimizing geodesics joining p to q and exactly 2 joining p to points on $C(p, \alpha)$ near q.

A simple (but amusing) observation is that for the standard sphere in $\mathbb{R}^3$ and $p =$ North Pole the fact that the South Pole is unstable can be seen in two ways (a) the ellipsoid with unequal axes has as cut locus a closed line segment (b) infinitely many length minimizing geodesics join the north pole to the south pole.

In dimension 3 the possibilities are the following: There are either 2, 3, or 4 nondegenerate minima for $H(\alpha) \mid \mathrm{Pr}^{-1}(\mathrm{q})$. Then there could be a single degenerate minimum. There cannot, however, be two degenerate minima for $H(\alpha) \mid \mathrm{Pr}^{-1}(q)$. For suppose this is possible and consider the map

$${}_2j_1^8H(\alpha): U \times A_1 \times A_2 \to \mathbb{R}^2 \times J_0^8(n, 1) \times J_0^8(n, 1).$$

(where the jet is taken in the fiber direction at the two degenerate minima). The codimension of the orbit through ${}_2j_1^8H(\alpha)(0,0,0)$ is $> 1 + n + n + 2$ for at the degenerate minimum $H(\alpha) \mid \mathrm{Pr}^{-1}(q)$ has the form $x_1^4 + \Sigma_{i>1} x_i^2$).

In the case of one degenerate minimum and one nondegenerate minimum the argument is considerably more involved. In a neighborhood of q the picture is the following: The point q may be taken to be $0 \in \mathbb{R}^3$. There are two functions given $H_1, H_2: \mathbb{R}^n \times \mathbb{R}^3 \to \mathbb{R}$ with $H_1(x_1, \ldots, x_n, u, v, w) = \lambda(x_1^4 + ux_1^2 + vx_1 + \Sigma_{i>1} x_i^2, u, v, w)$ and $H_2(x_1, \ldots, x_n, u, v, w) = \rho(x_1^2 + \Sigma_{i>1} x_i^2, u, v, w)$. In addition $\partial\lambda/\partial z(0,0,0,0) > 0$ and $\partial\rho/\partial z(0,0,0,0) > 0$ where z is the first variable.

Now let $\gamma: \mathbb{R}^2 \times J_0^8(n, 1) \times J_0^8(n, 1) \to \mathbb{R}^2 \times \mathbb{R}_{x_1} \times \mathbb{R}_{x_1^2}$ be projection on the constants and the coefficients of x_1 and x_1^2 in the first factor of $J_0^8(n,1)$. Transversality of $d({}_2j_1^8H(\alpha))(0,0,0)(\mathbb{R}^3\times\mathbb{R}^n\times\mathbb{R}^n)$ to the tangent space of the orbit of ${}_2j_1^8H(\alpha)(0,0,0)$ is equivalent to $\gamma[d({}_2j_1^8H(\alpha))(0,0,0)(\mathbb{R}^3\times(0)\times(0))]+(\text{diagonal in } \mathbb{R}^2)\times(0)\times(0)=\mathbb{R}^2\times\mathbb{R}_{x_1}\times\mathbb{R}_{x_1^2}$. The first summand is the space spanned by

$$\left(\frac{\partial\lambda}{\partial u}(0),\frac{\partial\rho}{\partial u}(0),0,\frac{\partial\lambda}{\partial z}(0)\right)$$

$$\left(\frac{\partial\lambda}{\partial v}(0),\frac{\partial\rho}{\partial v}(0),\frac{\partial\lambda}{\partial z}(0),0\right)$$

and

$$\left(\frac{\partial\lambda}{\partial w}(0),\frac{\partial\rho}{\partial w}(0),0,0\right)$$

The transversality condition becomes simply $\partial\lambda/\partial w(0)\neq\partial\rho/\partial w(0)$.

Consider now the function $f(u,v,w)$ defined to be equal to

$$\rho(0,u,v,w)-\lambda([x_1(u,v)]^4+u[x_1(u,v)]^2+v[x_1(u,v)],u,v,w)$$

where $x_1(u,v)$ is the unique minimum for $x_1^4+ux_1^2+vx_1$ for (u,v) outside the set $A=\{(u,v)\mid u<0, v=0\}$. On account of $\partial\lambda/\partial w(0)\neq\partial\rho/\partial w(0)$ and the behaviour of $x_1(u,v)$ it is fairly straightforward that $f(u,v,w)=0$ defines w (uniquely) as a function of u and v for (u,v) in the complement of A.

This surface together with the part of the half-plane $u\leq 0$, $v=0$ given by

$$\rho(0,u,0,w)\geq\lambda\left(-\frac{u^2}{4},u,o,w\right)$$

gives the diagram [2] (v).

Consider now the case of four nondegenerate minima for $H(\alpha)\mid \mathrm{Pr}^{-1}(q)$. Near q the cut locus $C(p,\alpha)$ is determined by four functions $\{\mu_i(z,u,v,w)\}_{i=1}^4$ with $\partial\mu_i/\partial z(0)>0$ such that

$$\begin{pmatrix}\frac{\partial(\mu_1-\mu_2)}{\partial u}(0,0,0,0) & \frac{\partial(\mu_1-\mu_2)}{\partial v}(0,0,0,0) & \frac{\partial(\mu_1-\mu_2)}{\partial w}(0,0,0,0)\\ \frac{\partial(\mu_2-\mu_3)}{\partial u}(0,0,0,0) & \frac{\partial(\mu_2-\mu_3)}{\partial v}(0,0,0,0) & \frac{\partial(\mu_2-\mu_3)}{\partial w}(0,0,0,0)\\ \frac{\partial(\mu_3-\mu_4)}{\partial u}(0,0,0,0) & \frac{\partial(\mu_3-\mu_4)}{\partial v}(0,0,0,0) & \frac{\partial(\mu_3-\mu_4)}{\partial w}(0,0,0,0)\end{pmatrix}$$

has rank 3.

For instance there are six conditions of type $\mu_1(0, u, v, w) = \mu_2(0, u, v, w)$ and $\mu_3(0, u, v, w) > \mu_1(0, u, v, w)$ and $\mu_4(0, u, v, w) > \mu_1(0, u, v, w)$. There are four of the type $\mu_1(0, u, v, w) = \mu_2(0, u, v, w) = \mu_3(0, u, v, w)$ and $\mu_4(0, u, v, w) > \mu_1(0, u, v, w)$. Then there is $\mu_1(0, u, v, w) = \mu_2(0, u, v, w) = \mu_3(0, u, v, w) = \mu_4(0, u, v, w)$. This gives the following picture. Imagine q as the barycenter of a tetrahedron. Then the four lines to the vertices determine six surfaces. This picture is obtained because any triple of surfaces meet at one point (the same point for any triple) and there their tangents have regular intersection. See diagram 2[iii].

The case of 3 nondegenerate minima gives as local picture of $C(p, \alpha)$ near q three surfaces meeting in a line on which q lies, the line being the boundary of the 3 surfaces and tangent spaces at q of the three surfaces having pairwise regular intersection. Finally in the case of 2 nondegenerate minima the local picture is a surface passing through q. (These are diagrams 2[ii] and 2[i] respectively).

Then there is the geometric interpretation: in the barycenter–tetrahedron picture 4 length minimizing nondegenerate geodesics join p to the barycenter, 3 join p to any point on the 4 lines and 2 join p to any point on the 6 surfaces. The geometric interpretation for the other cases is clear.

By now the reader can verify routinely that the case of one degenerate minimum yields diagrams 2[iv] i.e. a half plane with q on the boundary and in fact with every point on the boundary being conjugate to p.

4. A connection with geometric topology

This section points out a curious connection with the study of 3-dimensional manifolds. We are not sure of its ultimate significance (if any).

Let M be any compact 3-dimensional differentiable manifold. Recall that a theorem of A. Weinstein [14] (in contradiction to a conjecture of H. Rauch) has shown that on any compact differentiable manifold not homeomorphic to S^2 there exists a metric α for which the cut locus contains no points conjugate to p. It follows by elementary continuity considerations that there is a neighborhood of α in the space of metrics for which this is true. In view of the density of cut-stable metrics in dimension 3 there must be a cut-stable metric whose associated cut locus is free of conjugate points. Consequently we have the following conclusion.

THEOREM: *Any compact* 3-*dimensional manfiold can be decomposed into the disjoint union of an open cell and a compact subset whose local picture is one of three types namely figures* [2(i)], [2(ii)] *or* [2(iii)].

Now this is not entirely surprising to geometric topologists. For it is standard procedure to do the following: First triangulate the manifold. Then choose a maximal tree in the triangulation. Next take all the two and three cells (in the first barycentric subdivision of the triangulation) dual to the one and zero cells of the maximal tree. Together they form an open 3 cell. Its complement is a 2-dimensional subcomplex. What is not clear (at least to me) is what the local picture of this two dimensional subcomplex looks like. No doubt there are only finitely many types. But if so can one eliminate all but the ones given in the above theorem. I would hazard (not very educated) guess that on a first try it will not be possible to reduce to the above three pictures. Very likely at least the figure [2(iv)] and [2(v)] will be amongst those present. This raises the question of whether on the geometric topological level there is a way of eliminating all but the above three pictures say by choosing the triangulation and the maximal tree with greater care. If there is such a procedure does it correspond in some way to the differential geometric procedure outlined above?

It remains to be seen whether or not this is a worthwhile endeavour.

REMARK: As Alan Weinstein points out, if only the above decomposition theorem is desired, the route can be considerably shortened. For one may work entirely within the set $\mathscr{A}$ of metrics whose cut loci have no conjugate points. In particular, our multitransversality condition will be very simple (i.e. transversality to the diagonal in $\mathbb{R}^r$) and will hold for an open dense set in $\mathscr{A}$ without dimension restriction. Consequently there is a decomposition theorem in any dimension analogous to the above.

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