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THE DISCRIMINANT OF A REAL SIMPLE SINGULARITY

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Introduction

Let (X_0, x_0) be a germ of a hypersurface with an isolated singularity which is defined over $\mathbb{R}$. Such a singularity admits a semi-universal deformation $F: (X, x_0) \to (S, s_0)$, also defined over $\mathbb{R}$. We choose a nice representative $\tilde{F}: X \to S$ for F, which is invariant under complex conjugation. The image of the critical set of $\tilde{F}$, resp. of its real part $\tilde{F}_{\mathbf{R}}: X(\mathbf{R}) \to S(\mathbf{R})$, is called the complex, resp. real, discriminant and denoted by D, resp. $D_{\mathbf{R}}$. Various people, among whom Arnol'd, Pham and Thom conjecture that the complement of the complex discriminant has a contractible universal covering. Up to now, this has only been proved to be the case for the simple singularities. Thom also conjectures a sort of real analogue of this, namely that the connected components of the complement of the real discriminant are contractible. One of the aims of this paper is to show that this is indeed the case for the simple singularities. We should add here that Thom formulates his conjecture in a much stronger form than just given: it is claimed that the members of the canonical stratification of $S(\mathbf{R})$ (for this notion, see [6], Ch. I) are contractible. This too, can be proved for the simple singularities, but we shall not go into this matter in the present paper.

We briefly review the various sections. In section 1, our main results on real simple singularities are stated. The proof of these assertions requires an investigation of real discriminants of certain Coxeter groups, which is carried out in section 2. Since the proofs are valid for any finite Coxeter group, we have stated the results accordingly. As such, this section is completely independent of the previous one. In the last section we show how the period mapping enables us to reduce the assertions of section 1 to the results of section 2.

I am much indebted to a letter of E. Brieskorn. In this letter, Brieskorn communicated to me the result (2.2) of Tits (in its untwisted form) and pointed out how this might be used to describe the connected components in question in the simple case by means of the associated Coxeter group.

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1. Real forms of singularities

We begin with comparing the real and the complex discriminant. First a few general observations. Since D is the proper image of an analytic set, D is analytic in S by Remmert's theorem. Similarly, $D_{\mathbf{R}}$ is by the Tarski–Seidenberg theorem a closed semi-algebraic subset of $S(\mathbf{R})$. Furthermore D (resp. $D_{\mathbf{R}}$) is at each of its points of complex (resp. real) codimension one.

(1.1) PROPOSITION: *The real discriminant $D_{\mathbf{R}}$ is the closure of the real nonsingular part of D. The difference $D(\mathbb{R})-D_{\mathbf{R}}$ is a semi-algebraic subset of codimension ≥ 2 in $S(\mathbb{R})$.*

PROOF: If s is a real nonsingular point of D, then the fibre $X_s := \tilde{F}^{-1}(s)$ has precisely one singular point. Since the singular part of X_s is invariant under complex conjugation, this point must be real. Hence $s \in D_{\mathbf{R}}$. Conversely, let $s \in D_{\mathbf{R}}$ and suppose that there exists a neighbourhood of s in $D_{\mathbf{R}}$ which is entirely contained in the singular locus of D. Since such a neighbourhood contains a point where it is locally a real codimension-one submanifold of $S(\mathbf{R})$, the singular locus of D must be somewhere of complex codimension one in S. This is of course impossible and thus the proof of the first assertion is complete. To prove the second statement, we observe that the first one implies that $D(\mathbb{R})-D_{\mathbf{R}}$ is contained in the singular locus of D. As this singular locus is defined over $\mathbb{R}$ and of complex codimension ≥ 2 in S, it follows that its real part is of real codimension ≥ 2 in $S(\mathbb{R})$.

(1.2) COROLLARY: *The connected components of $S(\mathbb{R})-D(\mathbb{R})$ and $S(\mathbb{R})-D_{\mathbf{R}}$ are in a canonical bijective correspondence.*

(1.3) It is almost never true that $D_{\mathbf{R}}$ and $D(\mathbb{R})$ coincide. Consider for instance the case when X_0 is defined by z^{k+1}. Then $\tilde{F}$ is the family of zeroes of the polynomials $z^{k+1}+s_{k-1}z^{k-1}+\cdots+s_1z+s_0$. If $s =$

$(s_0, \ldots, s_{k-1})$ is real, then the roots of such a polynomial can be split into complex conjugate pairs $z_1, \bar{z}_1, \ldots, z_j \bar{z}_j$ (with $\operatorname{Im} z_\alpha > 0$) and real roots $z_{2j+1} \leq \cdots \leq z_{k+1}$. Then $s \in D(\mathbb{R})$ iff two complex roots coincide and $s \in D_{\mathbf{R}}$ iff two real roots coincide. For what follows it is good to keep this example in mind (especially in section 2).

Let s be a point of $S(\mathbb{R}) - D(\mathbb{R})$. Complex conjugation induces an involution on X_s and hence an involution on $H_n(X_s; \mathbb{Z})$, where $n = \dim X_s$. We denote this involution by u_s. It is clear that u_s respects the intersection form and permutes the set of vanishing cycles. Since the monodromy group W_s (the image of the natural representation of $\pi_1(S - D, s)$ on $H_n(X_s; \mathbb{Z})$) is generated by Picard–Lefschetz transformations, u_s belongs to the normalizer N_s of W_s. The quotient group N_s/W_s is an entity which is independent of s. This allows us to drop the index s and to write N/W.

(1.4) Proposition: *If $n = \dim X_0$ is even, the image of u_s in N/W is independent of the choice of s in $S(\mathbb{R}) - D(\mathbb{R})$.*

Proof: This is clear if we let s vary in a fixed connected component of $S(\mathbb{R}) - D(\mathbb{R})$. It therefore suffices to show that if s and s' lie in adjacent components C and C' of $S(\mathbb{R}) - D(\mathbb{R})$, the corresponding involutions define the same element in N/W. (We say that two components are *adjacent* if their closures have a codimension-one submanifold in common.) For this purpose we choose an $\mathbb{R}$-immersion i of the complex unit disk Δ in S such that $i^{-1}D = \{0\}$, i is transverse to D, $i(\frac{1}{2}) = s$ and $i(-\frac{1}{2}) = s'$. The family $\{X_{i(t)}: t \in \Delta\}$ induced by i acquires a singularity over $t = 0$. This singularity is an ordinary double point, which is locally described over $\mathbf{R}$ by $t = -z_1^2 \cdots - z_\lambda^2 + z_{\lambda+1}^2 + \cdots + z_{n+1}^2$. This singularity determines (up to sign) a vanishing cycle δ in $H_n(X_s; \mathbb{Z})$. Over $X_{i(\epsilon)}$ ($\epsilon > 0$, small), it is represented by the n-sphere which is defined by taking $z_1, \ldots, z_\lambda$ purely imaginary and $z_{\lambda+1}, \ldots, z_{n+1}$ real in the above equation. Hence $u_s(\delta) = (-1)^\lambda \delta$. Similarly, there is a vanishing cycle $\delta' \in H_n(X_{s'}; \mathbb{Z})$ for which we have $u_{s'}(\delta') = (-1)^{n+1-\lambda}\delta'$. In order to compare u_s and $u_{s'}$ we fix a path which connects s with s' in Δ-$\{0\}$. This path identifies $H_n(X_{s'}; \mathbb{Z})$ with $H_n(X_s; \mathbb{Z})$ and δ' with $\pm\delta$. For elementary geometric reasons u_s and $u_{s'}$ coincide on the cycles not intersecting δ. By the above formulas we have $u_{s'}u_s^{-1}(\delta) = (-1)^{n+1}\delta$. Since $u_{s'}u_s^{-1}$ also respects the intersection form it follows that $u_{s'}u_s^{-1}$ is semi-simple (here we use that n is even). Since the Picard–Lefschetz reflection associated to δ is also semi-simple and has the same eigen value decomposition as $u_{s'}u_s^{-1}$, it must be equal to $u_{s'}u_s^{-1}$. In particular, the latter belongs to W_s.

(1.5) It follows from the previous proposition that to any real form of an even-dimensional isolated hypersurface singularity there is an associated element $\boldsymbol{u} \in N/W$ with $\boldsymbol{u}^2 = 1$, or, if we wish to view N/W as an abstract group, a conjugacy class of such elements.

From now on we restrict our attention to the two-dimensional simple singularities. These are just the two-dimensional singularities for which W is finite. Since W is generated by reflections, it is a finite Coxeter group. The Coxeter groups which occur are A_l ($l = 1, 2, \ldots$), D_l ($l = 4, 5, \ldots$), E_6, E_7 and E_8, and the simple singularities are labeled accordingly. It is well-known that N/W is then naturally isomorphic to the group of automorphisms of the Coxeter diagram of W. The following table describes the real forms of the two-dimensional simple singularities. The fourth column indicates whether or not the corresponding element in N/W is trivial. In the nontrivial case it also specifies when this element is represented by minus the identity.

type	*equation*	N/W	$\boldsymbol{u}$
A_{2m}	$x^{2m+1} + y^2 + z^2$	$\mathbb{Z}/2$	trivial
A_{2m}	$x^{2m+1} + y^2 - z^2$	$\mathbb{Z}/2$	nontrivial (-1)
A_{2m+1}	$x^{2m+2} \pm (y^2 + z^2)$	$\mathbb{Z}/2$	trivial
A_{2m+1}	$x^{2m+2} + y^2 - z^2$	$\mathbb{Z}/2$	nontrivial (-1)
D_{2m}	$x^{2m-1} - xy^2 + z^2$	$\mathbb{Z}/2$ ($\mathscr{S}_3$ if $m = 2$)	trivial
D_{2m}	$x^{2m-1} + xy^2 + z^2$	$\mathbb{Z}/2$ ($\mathscr{S}_3$ if $m = 2$)	nontrivial
D_{2m+1}	$x^{2m} + xy^2 + z^2$	$\mathbb{Z}/2$	trivial
D_{2m+1}	$x^{2m} + xy^2 - z^2$	$\mathbb{Z}/2$	nontrivial (-1)
E_6	$x^4 + y^3 + z^2$	$\mathbb{Z}/2$	trivial
E_6	$x^4 + y^3 - z^2$	$\mathbb{Z}/2$	nontrivial (-1)
E_7	$x^3y + y^3 + z^2$	trivial	
E_8	$x^5 + y^3 + z^2$	trivial	

Note that in the case of D_4 the elements of order two in $N/W \simeq \mathscr{S}_3$ are all conjugate. We indicate how the above table is constructed. The relevant real forms have been classified by Arnol'd [2], while the groups N/W can be found in the tables of [3]. In order to determine $\boldsymbol{u}$ one proceeds as follows. According to A'Campo [1], the corresponding real singularity in the (x, y)-plane (forget the z-variable) admits a morsification with only two critical values. Using the fact that the suspension of a nonsingular complex fibre of the morsification is in a natural way homotopy equivalent to a nonsingular fibre of the original (two-dimensional) singularity one can easily determine the action of complex conjugation on its homology.

We now state our main results.

(1.6) THEOREM: *Recall that* $(X_0, 0)$ *is a two-dimensional simple singularity defined over* $\mathbf{R}$. *Let* $\boldsymbol{u} \in N/W$ *denote the element we associated to* $(X_0, 0)$ *in* (1.5). *Then the assignment* $s \in S(\mathbf{R}) - D(\mathbf{R}) \mapsto W$*-conjugacy class of* u_s *sets up a bijective correspondence between the connected components of* $S(\mathbf{R}) - D(\mathbf{R})$ *and* W*-conjugacy classes of involutions in the coset* $W\boldsymbol{u}$.

(1.7) THEOREM: *With the hypotheses of* (1.6), *let* s *be any point of* $S(\mathbf{R}) - D(\mathbf{R})$ *and let* K *denote the connected component of* $S(\mathbf{R}) - D_{\mathbf{R}}$ *containing it. Then*

(i) *K is homeomorphic to a real vector space (of the same dimension as $S(\mathbf{R})$, of course).*

(ii) *The image $W_{K,s}$ of the representation of $\pi_1(K - D(\mathbf{R}), s)$ on $H_2(X; \mathbf{Z})$ is as an abstract group in a unique way a Coxeter group. The representation of $W_{K,s}$ on $H_2(X_s; \mathbf{Z})$ is (up to a trivial factor) twice the ordinary representation.*

(iii) *The space $K - D(\mathbf{R})$ classifies the Artin group associated to $W_{K,s}$.*

(iv) The restriction $\tilde{F}_{\mathbf{R}}$: *$\tilde{F}_{\mathbf{R}}^{-1}K \to K$ is a trivial C^∞ fibre bundle. The euler characteristic of its fibre equals $1 + trace(u_s)$.*

For the term Artin group we refer to [4] and [5].

The reader will find slightly more precise results in the last section. In particular, we there give a description of $W_{K,s}$ in terms of u_s.

(1.8) THEOREM: *In case the group N/W is generated by minus the identity the real forms of the corresponding singularity have isomorphic real discriminants (this occurs for A_*, D_{odd}, E_6, E_7* and *E_8).*

2. Real discriminants of Coxeter groups

(2.1) Throughout this section we fix a real finite dimensional vector space V and a finite Coxeter group $W \subset GL(V)$. Our basic reference for such groups is Bourbaki [3, Ch. 4,5].

We begin with briefly recalling the invariant theory for this situation. Via the contragredient representation W acts on the dual V^* of

V and hence on the symmetric algebra $S(V^*)$ of V^*. On the other hand, W also acts on the complexification $V_\mathbf{C}$ of V. The orbit space for this action is in a natural way an affine algebraic variety whose coordinate ring is given by the W-fixed part of $S(V^*)$, tensorized with $\mathbb{C}$. We denote this orbit space by S_W and let $q: V_\mathbf{C} \to S_W$ be the natural map. It is well-known that $S(V^*)^W$ is a polynomial algebra in dim V unknowns. This implies that S_W is as an algebraic variety isomorphic to $\mathbb{C}^{\dim V}$.

The critical set of q is of course the union of irregular orbits. In this case it is just the union of the complex fixed-point hyperplanes of the reflections in W. Its image in S_W is a hypersurface which is called the discriminant. We denote it by D_W.

Since $S(V^*)^W$ is defined over $\mathbf{R}$, so is S_W. But sometimes there also exist a few "twisted" real forms for S_W which we proceed to describe. Let N_W denote the normalizer of W. By definition, W is a normal subgroup of N_W and the quotient group N_W/W can be identified with the group of automorphisms of the Coxeter diagram of W. This quotient also acts on the algebra $S(V^*)^W$ (on the left) and hence on the orbit space S_W (on the right). The real form for S_W we are going to define will depend on an element $\boldsymbol{u} \in N_W/W$ with $\boldsymbol{u}^2 = 1$. Such a $\boldsymbol{u}$ will be fixed for the remainder of this section. Now, a real form for S_W is (by definition, actually) equivalent to endowing its coordinate ring $S(V^*)^W \otimes \mathbb{C}$ with an anti-involution. As anti-involution we take here the tensor product of $\boldsymbol{u}$ and complex conjugation. Then $z \in V_\mathbf{C}$ is mapped to $S_W(\mathbf{R})$ if and only if for all $\phi \in S(V^*)^W$, we have $\phi(\bar{z}) = (\phi \cdot \boldsymbol{u})(z)$. We shall give a more explicit description of $q^{-1}S_W(\mathbf{R})$. For any involution u of V we let $V = V_u^+ \oplus V_u^-$ be its eigen space decomposition and we pose $V_u = V_u^+ \oplus iV_u^- \subset V_\mathbf{C}$. Note that V_u is a real form of $V_\mathbf{C}$ in the sense that $V_\mathbf{C} = V_u \oplus_\mathbf{R} \mathbb{C}$.

(2.2) Proposition (Tits): *The union of real orbits, $q^{-1}S_W(\mathbf{R})$, is the union of the V_u's, where u runs over all involutions in the coset $\boldsymbol{u}W$.*

Proof: If u is an involution of V, then u acts as complex conjugation on V_u. So if u is an involution in $\boldsymbol{u}W$ and $z \in V_u$, then for each $\phi \in S(V^*)^W$ we have $\phi(\bar{z}) = (\phi \cdot u)(z)$. This proves that $q(z) \in S_W(\mathbb{R})$.

Conversely, let $z \in V_\mathbf{R}$ be such that for all $\phi \in S(V^*)^W$, $\phi(\bar{z}) = (\phi \cdot \boldsymbol{u})(z)$. Choose any representative $u_0 \in N_W$ for $\boldsymbol{u}$. Since $(\phi \cdot \boldsymbol{u})(z) = \phi(u_0 z)$, it follows that there exists a $w \in W$ such that $\bar{z} = w u_0 z$. The stabilizer W_z of z is a Coxeter group and is clearly also the stabilizer of $\bar{z}$. This implies that wu_0 permutes the chambers of W_z. Since W_z acts transitively on its chambers, we can find a $w' \in W_z$ such that $w'wu_0$ leaves a chamber of W_z invariant. Put $u = w'wu_0$. The proof will be complete if we show that $u^2 = 1$ and $u(z) = \bar{z}$. The latter is clear, for $uz = w'\bar{z} = \bar{z}$. This implies that $u^2 \in W_z$. On the other hand, u^2 leaves a chamber of W_z invariant and hence $u^2 = 1$.

Let $D_W(\mathbf{R})$ denote the real part of D_W, i.e. the intersection of D_W and $S_W(\mathbf{R})$. Proposition (1.1) suggests to define the real discriminant $D_{W,\mathbf{R}}$ as the closure of the real nonsingular part of D_W. Then $D_{W,\mathbf{R}}$ is a closed semi-algebraic subset of $D_W(\mathbf{R})$, which at each point is of real codimension one in $S_W(\mathbf{R})$. Writing H_s for the fixed-point hyperplane of a reflection s, it follows from (2.2) and the observations in (2.1) that $q^{-1}D_W(\mathbf{R})$ is the union of intersections $V_u \cap H_{s,\mathbf{C}}$, where u runs over the involutions in $\mathbf{Wu}$ and s over the reflections in W. We give a similar description for $q^{-1}D_{W,\mathbf{R}}$.

(2.3) Lemma: *The inverse image $q^{-1}D_{W,\mathbf{R}}$ is the union of intersections $V_u \cap H_{s,\mathbf{C}}$, where u runs over the involutions in* $\mathbf{Wu}$ *and s over the reflections in W which commute with u.*

Proof: An element $z \in V_u$ maps to a simple point of D_W if and only if there exists a *unique* reflection s, W which fixes z. Since z is also fixed by u, it is fixed by the reflection usu^{-1}. Hence $usu^{-1} = s$. This proves the lemma, since for a generic point of $V_u \cap H_{s,\mathbf{C}}$, there are no other reflection hyperplanes than $H_{s,\mathbf{C}}$ containing it.

(2.4) We remark that although *id* and $-id$ may represent different elements in N_W/W, the map $z \in V_\mathbf{C} \mapsto iz \in V_\mathbf{C}$ induces an isomorphism between the associated real orbit spaces and their real discriminants.

If u is an involution of V and $s \in W$ a reflection commuting with u, then s leaves the subspaces V_u^+ and V_u^- invariant and s acts as a reflection on one of them and trivially on the other. We let $W_u^\pm$ denote the Coxeter subgroup of W generated by the reflections in W which leave $V_u^\mp$ pointwise fixed. It is clear that $W_u^+ \times W_u^-$ is a Coxeter subgroup of W which acts as such on V_u. A chamber in V_u for this action will be called a *cell for u* and is usually denoted by $\mathscr{C}$. Observe that a cell for u determines u without ambiguity.

(2.5) Lemma: *Let u be an involution in the coset* $\mathbf{u}W$. *Any cell for u maps to a connected component of $S_W(\mathbf{R}) - D_{W,\mathbf{R}}$ and the assignment $u \mapsto q$ (cell for u) sets up a bijective correspondence between W-con-*

jugacy classes of involutions in $W\boldsymbol{u}$ *and connected components of* $S_W(\mathbb{R}) - D_{W,\mathbb{R}}$.

PROOF: That the image of a cell is a connected component of the complement of the real discriminant follows from (2.3) and the obvious fact that a cell is connected. If two cells $\mathscr{C}$, $\mathscr{C}'$ for involutions u, u' respectively have the same image in the orbit space, then $\mathscr{C}' = w\mathscr{C}$ for some $w \in W$. This implies $u' = w^{-1}uw$ and hence the correspondence is injective. Surjectivity is clear.

Our next aim is to analyse which points in a cell are identified when we map it to the orbit space. For this purpose we fix an involution u in $\boldsymbol{u}W$ and a cell $\mathscr{C} = C^+ \times iC^-$ for u. This enables us to simplify the notation a little: we write $W^\pm$ and $V^\pm$ instead of $W_u^\pm$ and $V_u^\pm$.

(2.6) LEMMA: *Let* s *be a reflection in* W *which leaves a point of* $\mathscr{C}$ *invariant. Then* s *and* usu *commute.*

PROOF: As u permutes the irreducible components of W, there is no loss of generality in assuming that W is irreducible. We do so and we put $s' = usu$. Since s has a fixed point in $\mathscr{C}$ we have $s \notin W^+ \times W^-$ and hence $s' \neq s$. So the subgroup W' of W generated by s and s' is of rank two. Suppose ss' is of order ≥ 3. We distinguish three cases.

(a) rank $W = 2$. Since s fixes points in C^+ and C^-, their union is contained in H_s. But this contradicts the fact that $C^+ \cup C^-$ generates V.

(b) rank $W \geq 3$ and $(ss')^4 \neq 1$. According to the classification of Coxeter groups, this forces ss' to be of order 3 or 5. Then the reflection $ss's$ (resp. $ss'ss's$) equals its u-conjugate and is therefore contained in $W^+ \cup W^-$. This contradicts the fact that this reflection has a fixed point in $\mathscr{C}$.

(c) rank $W \geq 3$ and $(ss')^4 = 1$. Then the classification shows that s and s' are not conjugate in W (for W then extends to a root system of which the roots corresponding to s and s' have different length). As observed earlier, u induces an automorphism of the Coxeter diagram of W. Since the Coxeter diagram is a tree, at least one vertex is left fixed by u. So u preserves at least one conjugacy class of reflections in W. As there are at most two such classes, s and s' must be conjugate in W. Contradiction.

The above proof uses the classification in a mild way. Indeed, by employing more subtle (and elaborate) arguments, its use can be avoided altogether. The preceding lemma has an interesting consequence for the stabilizer of a point in $\mathscr{C}$.

(2.7) LEMMA: *Let $z \in \mathscr{C}$. Then its stabilizer W_z splits into a direct product $W'_z \times W''_z$ whose factors are interchanged by conjugation with u: $W''_z = uW'_zu$.*

PROOF: Suppose not. Then W_z contains an irreducible Coxeter subgroup W' which is normalized by u. We choose a N_W-invariant riemannian inner product $\langle\,,\rangle$ on V and we let R denote the set of vectors $r \in V$ with $\langle r, r\rangle = 1$ and $sr = -r$ for some reflection s in W'. Clearly, two such vectors are mutually perpendicular iff the corresponding reflections in W' commute. Note that W' leaves R invariant. For any $r \in R$ there exists a chain $S_r = \{r_0 = r, \ldots, r_k = ur_0\}$ with $\langle r_i, r_{i+1}\rangle \neq 0$ for $i = 0, \ldots, k-1$. We choose r and S_r such that S_r has minimal cardinality. This minimality condition implies that $\langle r_i, r_j\rangle = 0$ for $|i-j| > 1$ and $\langle r_i, ur_j\rangle = 0$ for $\{i, j\} \neq \{0, k-1\}$. Now, the element $r' := s_{r_{k-1}} \ldots s_{r_1} r_0$ belongs to R and is of the form $\Sigma_{i=0}^{k-1} c_i r_i$ with $c_0 = 1$ and $c_{k-1} \neq 0$. It follows that $\langle r', ur'\rangle = \langle r_0 + c_{k-1}ur_0, ur_0 + c_{k-1}r_0\rangle = 2c_{k-1}\langle r_0, ur_0\rangle \neq 0$. This implies that $s_{r'}$ and $us_{r'}u$ don't commute, thus contradicting (2.6).

(2.8) COROLLARY: *Let $z \in \mathscr{C}$ and $w \in W$ be such that $wz \in \mathscr{C}$. Then there exists a $w_0 \in W_z$ with the property that ww_0 leaves $\mathscr{C}$ invariant.*

PROOF: We first note that $uw^{-1}uw(z) = uw^{-1}\overline{w(z)} = uww^{-1}(\bar{z}) = z$. Write $uw^{-1}uw$ as a product $w'w''$ with $w' \in W'_z$ and $w'' \in W''_z$. Since the inverse of $uw^{-1}uw$ equals its u-conjugate we must have $w''^{-1}w'^{-1} = uw'u^{-1} \cdot uw''u^{-1}$. It follows from the preceding lemma that then $w'^{-1} = uw''u^{-1}$. Hence $uw^{-1}uw = uw''^{-1}uw''$, in other words ww''^{-1} commutes with u. We claim that $w_0 := w''^{-1}$ is as desired. Since ww_0 commutes with u, it permutes the cells for u. Moreover $ww_0\mathscr{C} \cap \mathscr{C} \neq \phi$. Hence $ww_0\mathscr{C} = \mathscr{C}$.

(2.9) Let $\rho \in \mathscr{C}$ be any point on the barycentric halfline of $\mathscr{C}$. Any element of N_W which leaves $\mathscr{C}$ invariant fixes ρ. We set $W_{\mathscr{C}} := \{w'uw'u^{-1} : w' \in W'_\rho\}$. This notation will be justified in a moment. Note that (2.7) implies that $W_{\mathscr{C}}$ is a subgroup of W and as a group is isomorphic to W'_z. Moreover u commutes with the elements of $W_{\mathscr{C}}$. Hence $W_{\mathscr{C}}$ leaves V_u invariant and permutes the cells for u. As $W_{\mathscr{C}}$ stabilizes ρ it follows that $W_{\mathscr{C}}$ leaves $\mathscr{C}$ invariant. Observe that the representation of $W_{\mathscr{C}}$ on V_u is up to a trivial factor the complexification of the ordinary representation of $W_{\mathscr{C}} \cong W'_\rho$ as a Coxeter group.

(2.10) PROPOSITION: *The $W_{\mathscr{C}}$-orbit space of $\mathscr{C}$ maps bijectively onto a connected component of $S_W(\mathbf{R}) - D_{W,\mathbf{R}}$. This orbit space is homeomorphic to a vector space and its regular part is a classifying space for the Artin group of $W_{\mathscr{C}}$.*

PROOF: To prove the first assertion, it suffices by (2.5) to show that the induced map $W_{\mathscr{C}}\backslash\mathscr{C} \to q(\mathscr{C})$ is injective. Suppose $z, z' \in \mathscr{C}$ lie on the same W-orbit. Then by (2.8) there exists a $w \in W$ with $z' = wz$ and $w\mathscr{C} = \mathscr{C}$. Hence $w\rho = \rho$. We write $w = w'w''$ with $w' \in W'_\rho$ and $w'' \in W''_\rho$. Since w leaves $\mathscr{C}$ invariant, w commutes with u and hence $w'' = uw'u^{-1}$. This implies $w \in W_{\mathscr{C}}$ and the required injectivity follows. Now let F denote the fixed-point set of $W_{\mathscr{C}}$ in V_u and $F^\perp$ its orthogonal complement in V_u (relative an N_W-invariant riemannian metric). Then $F^\perp$ is $W_{\mathscr{C}}$-invariant and the action of $W_{\mathscr{C}}$ on $F^\perp$ is the complexification of the Coxeter representation. Using the fact that each intersection $\mathscr{C} \cap (z + F^\perp)$, $z \in F$ is starlike, it is easy to construct a $W_{\mathscr{C}}$-equivariant homeomorphism of $\mathscr{C}$ onto $(\mathscr{C} \cap F) + F^\perp$. The last two assertions now follow from known properties of the $W_{\mathscr{C}}$-action on $F^\perp$: the $W_{\mathscr{C}}$-orbit space of $F^\perp$ is homeomorphic to a vector space (see the remarks made in (2.1)) and according to a theorem of Deligne the regular $W_{\mathscr{C}}$-orbit space of $F^\perp$ classifies the Artin group associated to $W_{\mathscr{C}}$.

3. The period mapping over the reals

Our main tool for the proof of the theorems (1.6, 7, 8) will be the period mapping, introduced in [8]. We briefly recall its construction. Let $(X_0, 0)$ be a two-dimensional simple singularity defined over R. Then X_0 can be taken weighted homogeneous and for its semi-universal deformation we may choose a weighted homogeneous representative $\tilde{F}: X = \mathbf{C}^{2+l} \to \mathbf{C}^l = S$, which is also defined over $\mathbf{R}$. On X (resp. S) we choose a nowhere vanishing holomorphic form ω (resp. α) of maximal dimension and weighted homogeneous in an obvious sense. Both ω and α are unique up to a scalar factor. Then on each nonsingular fibre X_s of F there exists a unique holomorphic 2-form $\omega(s)$ with the property that $\omega(s) \wedge F^*(\alpha) = \omega$ along X_s. Now fix a base point $s \in S - D$ and let s' be any other point of $S - D$. Then a continuous path from s' to s enables us to displace $\omega(s)$ along this path to a 2-dimensional cohomology class on X_s. The ambiguity caused by the choice of such a path is eliminated if we assign to s' the W_s-orbit of this class, rather than the class itself. We thus get a map from $S-D$ to the W_s-orbit space of $H^2(X_s;\mathbf{C})$, which is called the period mapping. We denote this mapping by P and, in accordance to the notation of the previous section, the orbit space by S_W. The following was shown in [8].

(3.1) The period mapping is holomorphic and extends to an isomorphism from S onto S_W which maps D onto D_W.

Now we assume that ω and α are defined over $\mathbf{R}$ and that our base point s is in $S(\mathbf{R})-D(\mathbf{R})$. We let K denote the connected component of $S(\mathbf{R})-D_{\mathbf{R}}$ which contains s. If $s'\in S(\mathbf{R})$, we write $V_{s'}$ for $H^2(X_{s'},\mathbf{R})$ and we let $V_{s'}^+\oplus V_{s'}^-$ be the eigen space decomposition of $u_{s'}^*$.

(3.2) LEMMA: *For any $s'\in S(\mathbf{R})$, the cohomology class $[\omega(s)]\in V_{s',\mathbf{C}}$ lies in $V_{s'}^+\oplus iV_{s'}^-$.*

PROOF: Let $\delta\in H_2(X_{s'};\mathbf{R})$ be arbitrary. Since $\omega(s')$ has real coefficients, we have

$$\int_{u_{s'}(\delta)}\omega(s)=\int_{\delta}\overline{\omega(s)}=\overline{\int_{\delta}\omega(s)}.$$

This clearly implies the lemma.

We endow S_W with the real structure defined by the image $\boldsymbol{u}$ of u_s in N/W.

(3.3) COROLLARY: *The (extended) period mapping P maps $S(\mathbf{R})$ onto $S_W(\mathbf{R})$.*

PROOF: The preceding lemma and (1.4) show that P maps $S(\mathbf{R})$ into the W_s-orbit space of

$$\cup\{V_u^+\oplus iV_u^-: u\in \boldsymbol{u}W \text{ and } u^2=1\}.$$

By (2.2) the latter equals $S_W(\mathbf{R})$. It then follows from (3.1) that $P(S(\mathbf{R}))$ is a submanifold of $S_W(\mathbf{R})$. As it is open and closed in $S_W(\mathbf{R})$, it must be equal to $S_W(\mathbf{R})$.

(3.4) Now we can prove the assertions made in (1.6, 7, 8). Since P maps D onto D_W and $S(\mathbf{R})$ onto $S_W(\mathbf{R})$, it follows that P maps $D(\mathbf{R})$ onto $D_W(\mathbf{R})$ and $D_{\mathbf{R}}$ onto $D_{W,\mathbf{R}}$. Then (1.6) is immediate from (2.5) and the first three statements of (1.7) follow from (2.10) and the preceding discussion (2.9). As for (1.7)-iv we observe that $\tilde{F}_{\mathbf{R}}^{-1}(K) \to K$ is a trivial bundle because K is contractible. The assertion concerning the euler characteristic of X_s is a consequence of the Lefschetz fixed point formula. Finally, (1.8) is implied by (2.4).

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