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G. VAN DIJK

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SMOOTH AND ADMISSIBLE REPRESENTATIONS OF P-ADIC UNIPOTENT GROUPS

G. van Dijk

§1. Introduction

A representation π of a totally disconnected group G on a complex vector space V is said to be smooth if for each $v \in V$ the mapping

$$x \mapsto \pi(x)v \qquad (x \in G)$$

is locally constant. π is called admissible if in addition the following condition is satisfied: For any open subgroup K of G, the space of vectors $v \in V$ left fixed by $\pi(K)$ is finite-dimensional. An admissible representation is said to be pre-unitary if V carries a $\pi(G)$-invariant scalar product.

These representations play an important role in the harmonic analysis on reductive p-adic groups [6]. The aim of this paper is to emphasize their importance in harmonic analysis on unipotent p-adic groups. Let Ω be a p-adic field of characteristic zero. $\mathbf{G}$ will denote a connected unipotent algebraic group, defined over Ω and G its subgroup of Ω-rational points. Let $\mathbf{\mathfrak{G}}$ be the Lie algebra of $\mathbf{G}$ and $\mathfrak{G}$ its subalgebra of Ω-points. G is a totally disconnected group. We show:

(i) any irreducible smooth representation of G is admissible,
(ii) any irreducible admissible representation of G is pre-unitary.

Jacquet [7] has shown that (i) holds for reductive p-adic groups G. Actually, we make use of a remarkable lemma from [7]. The main tool for the proof of (i) and (ii) is the interference of so-called super-cuspidal representations, which are known to play a decisive role in the representation theory of reductive groups [6]. We apply some results of Casselman concerning these representations [3], which originally were only stated for $GL(2)$. For the proof, which is by induction on $\dim G$, one has to go to the three-dimensional p-adic Heisenberg group. A new version of von Neumann's theorem ([11], Ch. 2) is needed to complete the induction. All this is to be found in sections 2, 3, 4 and 5.

Section 6 is concerned with the Kirillov construction of irreducible unitary representations of G, which is standard now. In the next section we discuss the character formula, following Pukanszky [12]. As a byproduct we obtain a homogeneity property for the distribution, defined by a G-orbit O in $\mathscr{G}'$: if $\dim O = 2m$, then

$$\int_O \phi(tv)\,\mathrm{d}v = |t|^{-m} \int_O \phi(v)\,\mathrm{d}v \qquad (\phi \in C_c^\infty(\mathscr{G}'))$$

for all $t \in \Omega$, $t \neq 0$. Similar results are true for nilpotent orbits of reductive G in $\mathscr{G}$ [2]; there they form a substantial help in proving that the formal degrees of supercuspidal representations are integers, provided Haar measures are suitably normalized. Let Z denote the center of G.

Section 8 deals with square-integrable representations mod Z of G. Moore and Wolf [10] have discussed them for real unipotent groups. The main results still hold for p-adic groups.

Let π be an irreducible square-integrable representation mod Z of G. For any open compact subgroup K of G, let $m(\pi, 1)$ denote the multiplicity of the trivial representation of K in the restriction of π to K. Normalize Haar measures on G and Z in such a way that $\mathrm{vol}(K) = \mathrm{vol}(K \cap Z) = 1$. Choose Haar measure on G/Z accordingly. Then, according to a general theorem ([5], Theorem 2) one has:

$$m(\pi, 1) \leq \frac{1}{d(\pi)}, \quad \text{where } d(\pi) \text{ is the formal degree of } \pi.$$

Now assume in addition K to be a lattice subgroup of G: $L = \log K$ is a lattice in $\mathscr{G}$. Moreover, let $m(\pi, 1) > 0$. Then we have equality:

$$m(\pi, 1) = \frac{1}{d(\pi)}.$$

This is proved in section 9.

In section 10 we relate our results to earlier work of C.C. Moore [9] on these multiplicities, involving numbers of K-orbits. We conclude with an example in section 11.

§2. Smooth representations

We call a Hausdorff space X a totally disconnected (t.d.) space if it satisfies the following condition: Given a point $x \in X$ and a neighborhood U of x in X, there exists an open and compact subset ω of X such that $x \in \omega \subset U$. Clearly a t.d. space is locally compact.

Let X be a t.d. space and S a set. A mapping $f: X \to S$ is said to be smooth if it is locally constant. Let V be a complex vector space. We write $C^\infty(X, V)$ for the space of all smooth functions $f: X \to V$ and $C_c^\infty(X, V)$ for the subspace of those f which have compact support. If $V = \mathbb{C}$ we simply write $C^\infty(X)$ and $C_c^\infty(X)$ respectively. One can identify $C_c^\infty(X, V)$ with $C_c^\infty(X) \otimes V$ by means of the mapping $i: C_c^\infty(X) \otimes V \to C_c^\infty(X, V)$ defined as follows: If $f \in C_c^\infty(X)$ and $v \in V$, then $i(f \otimes v)$ is the function $x \mapsto f(x)v$ $(x \in X)$ from X to V.

Let G be a t.d. group, i.e. a topological group whose underlying space is a t.d. space. It is known that G has arbitrarily small open compact subgroups. By a representation of G on V, we mean a map $\pi: G \to \mathrm{End}(V)$ such that $\pi(1) = 1$ and $\pi(xy) = \pi(x)\pi(y)$ $(x, y \in G)$. A vector $v \in V$ is called π-smooth if the mapping $x \mapsto \pi(x)v$ of G into V is smooth.

Let V_∞ be the subspace of all π-smooth vectors. Then V_∞ is $\pi(G)$-stable. Let π_∞ denote the restriction of π on V_∞. π is said to be a *smooth representation* if $V = V_\infty$. Of course π_∞ is always smooth.

We call a smooth representation π on V irreducible if V has no non-trivial $\pi(G)$-invariant subspaces.

Let π be a representation of G on the complex vector space V. π is called *admissible* if

(i) π is smooth,

(ii) for any open subgroup K of G, the space of vectors $v \in V$ which are left fixed by $\pi(K)$, is finite-dimensional.

An admissible representation π of G on V is called *pre-unitary* if V carries a $\pi(G)$-invariant scalar product. Let $\mathcal{H}$ be the completion of V with respect to the norm, defined by the scalar product. Then π extends to a continuous unitary representation ρ of G on $\mathcal{H}$ such that $V = \mathcal{H}_\infty$ and $\pi = \rho_\infty$. It is well-known that π is irreducible if and only if ρ is topologically irreducible. Note that V is dense in $\mathcal{H}$.

Let π be a smooth representation of G on V and V' the (algebraic) dual of V. Then the dual representation π' of G on V' is given by

$$\langle v, \pi'(x)\lambda \rangle = \langle \pi(x^{-1})v, \lambda \rangle \qquad (x \in G, \lambda \in V', v \in V).$$

Put $\check{V} = (V')_\infty$ and $\check{\pi} = (\pi')_\infty$. Then $\check{\pi}$ is a smooth representation which is called contragredient to π. It is easily checked that π is admissible if and only if $\check{\pi}$ is.

Let H be a closed subgroup of G and σ a smooth representation of H on W. Then we define a smooth representation $\pi = \mathrm{ind}_{H\uparrow G}\,\sigma$ as follows: Let V denote the space of all smooth functions $f: G \to W$ such that

(1) $f(hx) = \sigma(h)f(x) \qquad (h \in H,\ x \in G)$,

(2) Supp f is compact mod H.

Then π is the representation of G on V given by

$$\pi(y)f(x) = f(xy) \qquad (x, y \in G, f \in V).$$

Let π_1, π_2 be two smooth representations of G on V_1 and V_2 respectively. We say that π_1 is equivalent to π_2 if there is a linear bijection $T: V_1 \to V_2$ such that $\pi_2(x)T = T\pi_1(x)$ for all $x \in G$.

§3. Smooth and admissible representations of the three-dimensional p-adic Heisenberg group

Let Ω be a p-adic field, i.e. a locally compact non-discrete field with a discrete valuation. There is an absolute value on Ω, denoted $|\cdot|$, which we assume to be normalized in the following way. Let dx be an additive Haar measure on Ω. Then $\mathrm{d}(ax) = |a|\,\mathrm{d}x$ ($a \in \Omega^*$). Let $\mathcal{O}$ be the ring of integers: $\mathcal{O} = \{x \in \Omega : |x| \le 1\}$; $\mathcal{O}$ is a local ring with unique maximal ideal P, given by $P = \{x \in \Omega : |x| < 1\}$. The residue-class field $\mathcal{O}/P$ has finitely many, say q, elements. P is a principal ideal with generator ϖ. So $P = \varpi\mathcal{O}$, $|\varpi| = q^{-1}$. Put $P^n = \varpi^n\mathcal{O}$ ($n \in \mathbb{Z}$).

Since P^n is a compact subgroup of the additive group of Ω and $\Omega = \bigcup_n P^n$, any additive character of Ω is unitary. Let $G = H_3$ be the 3-dimensional Heisenberg group over Ω:

$$G = \left\{ [x, y, z] = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix};\quad x, y, z \in \Omega \right\}.$$

G is a t.d. group. The group multiplication is given by:

$$[x, y, z][x', y', z'] = [x + x', y + y', z + z' + xy'].$$

THEOREM 1: (1) *Each irreducible smooth representation* π *of* H_3 *is admissible*; (2) *Each irreducible admissible representation* π *of* H_3 *is pre-unitary.*

We make use of the following result of Jacquet [7].

LEMMA 1: *Let* H *be a group and* ρ *an (algebraically) irreducible representation of* H *on a complex vector space* V *of at most denumerable dimensions. Then every operator* A *which commutes with* $\rho(H)$ *is a scalar.*

Let V be the space of π. Let $v \in V$, $v \neq 0$ and $K = \{g \in G : \pi(g)v = v\}$. Then K is open and G/K is denumerable. Since $V = \text{span}\{\pi(g)v : g \in G/K\}$, the lemma applies. $Z = \{[0,0,z] : z \in \Omega\}$ is the center of G. Therefore, there exists an additive (unitary) character ψ_π of Ω such that $\pi([0,0,z]) = \psi_\pi(z)I$ $(z \in \Omega)$, where I is the identity in $\text{End}(V)$. We have two cases:

(a) $\psi_\pi = 1$. Then π actually is a representation of $G/Z \simeq \Omega^2$ which is (again by the lemma) one-dimensional and, as observed above, unitary.

(b) $\psi_\pi \neq 1$. Fix $w \in \check{V}$, $w \neq 0$. For any $v \in V$, put $c_v(g) = \langle \pi(g)v, w \rangle$ $(g \in G)$. The mapping $v \mapsto c_v$ is a linear injection of V into the space of smooth functions f on G, satisfying

$$f([x,y,z]) = \psi_\pi(z) f([x,y,0]).$$

Let K be a (small) open compact subgroup of G such that $\check{\pi}(k)w = w$ for all $k \in K$. Call $V_K = \{v \in V : \pi(k)v = v \text{ for all } k \in K\}$. Then $f = c_v$ satisfies

$$f(kg) = f(gk) = f(g) \qquad (g \in G;\ k \in K)$$

for all $v \in V_K$.

Write $g = [x,y,0]$, $k = [x',y',0]$. Then

$$f([x,y,0]) = f([x+x', y+y', xy']) = f([x+x', y+y', x'y]).$$

Hence

$$f([x,y,0]) = f([x+x', y, 0]) = f([x, y+y', 0]).$$

Therefore $f([x+x',\ y+y',\ 0]) = f([x,y,0])$ for all $x, y \in \Omega$ and x', y'

small (only depending on K, not on the particular choice of $v \in V_K$). Moreover:

$$f([x, y, 0]) = f([x, y, 0])\psi_\pi(xy') = f([x, y, 0])\psi_\pi(x'y)$$

for x', y' as above. Since $\psi_\pi \neq 1$, $f([x, y, 0]) = 0$ for x or y large enough (only depending on K, not on the particular choice of $v \in V_K$). Since $f([x, y, z]) = \psi_\pi(z)f([x, y, 0])$, f is completely determined by the values $f([x, y, 0])$, $(x, y \in \Omega)$. Consequently, $\dim V_K = \dim\{c_v : v \in V_K\} < \infty$. Part (1) of the theorem is now evident. To prove part (2) it suffices to take the following scalar product on V:

$$(v, v') = \int_\Omega \int_\Omega c_v([x, y, 0])\overline{c_{v'}([x, y, 0])}\, dx\, dy \qquad (v, v' \in V).$$

REMARK: It is clear that the same observations remain true for the higher dimensional p-adic Heisenberg groups.

§4. Supercuspidal representations

G is a t.d. group and π a smooth representation of G on V. By a matrix coefficient of π, we mean a function on G of the form

$$x \mapsto \langle \pi(x)v, \check{v} \rangle \qquad (x \in G)$$

where v and $\check{v}$ are fixed elements in V and $\check{V}$ respectively. Let Z denote the center of G. We call π a *supercuspidal* representation if each matrix coefficient of π has compact support modulo Z. The proof of Theorem 1 emphasizes the significance of this kind of representations. Actually, one has the following lemma.

LEMMA 2: *Let π be a smooth representation of H_3 such that $\pi([0, 0, z]) = \psi_\pi(z)I$ $(z \in \Omega)$ for some non-trivial additive character ψ_π of Ω. Then π is a supercuspidal representation.*

Assume, from now on, G to satisfy the second axiom of countability. Let π be an irreducible smooth representation of G on V. Then by Lemma 1, there is a character λ_π of Z such that $\pi(z) = \lambda_\pi(z)I$ $(z \in Z)$.

LEMMA 3: *Let π be an irreducible, admissible and supercuspidal representation of G on V. Assume λ_π unitary. Then π is pre-unitary* *and one has the following orthogonality relations: There exists a positive constant d_π (the formal degree of π), only depending on the choice of Haar measure* $\mathrm{d}\dot{g}$ *on G/Z such that*

$$\int_{G/Z} \langle \pi(g)u, \check{u}\rangle\langle \pi(g^{-1})v, \check{v}\rangle \,\mathrm{d}\dot{g} = d_\pi^{-1}\langle u, \check{v}\rangle\langle v, \check{u}\rangle$$

for all $u, v \in V$, $\check{u}$, $\check{v} \in \check{V}$.

To make π pre-unitary, choose any $w \in \check{V}$, $w \neq 0$ and define the following G-invariant scalar product on V:

$$(v, v') = \int_{G/Z} \langle \pi(g)v, w\rangle\overline{\langle \pi(g)v', w\rangle}\,\mathrm{d}\dot{g}.$$

π extends to an irreducible unitary representation on the completion $\mathcal{H}$ of V such that $\mathcal{H}_\infty = V$. The orthogonality relations now follow easily from those for irreducible unitary supercuspidal representations ([5], Theorem 1).

The following theorem is due to Casselman ([3], Theorem 1.6).

THEOREM 2: *Let ρ be an irreducible, admissible and supercuspidal representation of G on W such that $\rho(z) = \lambda(z)I$ ($z \in Z$), where λ is a unitary character of Z. Let π be any smooth representation of G on V such that $\pi(z) = \lambda(z)I$ ($z \in Z$). Given a G-morphism $f \neq 0$ from π to ρ, there exists a G-morphism splitting f.*

PROOF: Let $S_\lambda(G)$ denote the space of smooth functions h on G with compact support mod Z such that $h(xz) = h(x)\lambda(z^{-1})$ ($x \in G$, $z \in Z$). $S_\lambda(G)$ is a G-module, G acting by left translation. Fix $\check{w}_0 \in \check{W}$, $\check{w}_0 \neq 0$. The mapping F: $W \to S_\lambda(G)$, defined by

$$F(w)(x) = \langle \rho(x^{-1})w, \check{w}_0\rangle \qquad (w \in W,\ x \in G)$$

is a G-morphism. Choose $w_0 \in W$ and $v_0 \in V$ such that $\langle w_0, \check{w}_0\rangle = d_\rho$, $f(v_0) = w_0$. By P we denote the G-morphism from $S_\lambda(G)$ to V given by

$$P(h) = \int_{G/Z} h(x)\pi(x)v_0\,\mathrm{d}\dot{x} \qquad (h \in S_\lambda(G)).$$

Then $P \circ F$ is the G-morphism, splitting f:

$$\begin{aligned}\langle f\circ P\circ F(w),\check{w}\rangle &= \int_{G/Z}\langle\rho(x^{-1})w,\check{w}_0\rangle\langle f(\pi(x)v_0),\check{w}\rangle\,d\dot{x}\\ &= \int_{G/Z}\langle\rho(x^{-1})w,\check{w}_0\rangle\langle\rho(x)w_0,\check{w}\rangle\,d\dot{x}\\ &= d_\rho^{-1}\langle w_0,\check{w}_0\rangle\langle w,\check{w}\rangle \qquad \text{(by Lemma 3)}\\ &= \langle w,\check{w}\rangle \text{ for all } \check{w}\in\check{W}.\end{aligned}$$

Hence $f\circ P\circ F(w)=w$ for all $w\in W$.

Let us now turn back to H_3. The irreducible unitary representations of H_3 are well-known (cf. [11]). Their restrictions to the space of smooth vectors are admissible. Keeping in mind Theorem 1, we have therefore the following list of irreducible admissible representations of H_3. Let χ_0 denote any non-trivial additive character of Ω. Then:

(a) One-dimensional representations $\rho_{\mu,\nu}$ ($\mu,\nu\in\Omega$), trivial on Z; $\rho_{\mu,\nu}([x,y,z])=\chi_0(\mu x+\nu y)$.

(b) Supercuspidal representations ρ_λ ($\lambda\in\Omega^*$), non-trivial on Z, on the space $C_c^\infty(\Omega)$;

$$\rho_\lambda([x,y,z])f(t)=\chi_0(\lambda(z+ty))f(t+x)\qquad (f\in C_c^\infty(\Omega)).$$

We have the following analogue of the famous theorem of von Neumann for H_3 ([11], Ch. 2).

THEOREM 3: *Let π be a smooth representation of H_3 such that $\pi([0,0,z])=\chi_0(\lambda z)I$ ($z\in\Omega$) for some $\lambda\neq 0$. Then π is the (algebraic) direct sum of irreducible representations equivalent to ρ_λ.*

PROOF: Let V be the space of π. Due to Theorem 1, every irreducible subrepresentation of π is equivalent to ρ_λ. By Lemma 2, π is a supercuspidal representation. We shall prove the following: Given any G-invariant subspace W of V, $W\neq V$, there exists an irreducible subspace U of V such that $U\cap W=(0)$. An easy application of Zorn's Lemma then yields the theorem.

Let W be a proper G-invariant subspace of V. Put $\bar{V}=V/W$. $\bar{V}$ is a G-module; the action of G is a smooth and supercuspidal representation of G. Let $\bar{v}_0\in\bar{V}$, $\bar{v}_0\neq 0$. The G-module $\bar{V}_0$ generated by $\bar{v}_0$ contains a maximal proper G-module. Therefore $\bar{V}_0$ has an irreducible quotient, which is also supercuspidal, and admissible by Theorem 1. By Theorem 2, $\bar{V}_0$ and hence $\bar{V}$, even has an irreducible subspace, say $\bar{V}_1$, on which G acts as an admissible, supercuspidal representation. Let V_1+W be its pre-image in V. Then V_1+W is a G-invariant subspace of V and the canonical map from V to $\bar{V}$ induces a non-zero G-morphism from $V_1 + W$ to $\bar{V}_1$. Again Theorem 2 implies the existence of an irreducible subspace U of V such that $U \cap W = (0)$, $U + W = V_1 + W$. This concludes the proof of Theorem 3.

§5. Smooth and admissible representations of unipotent p-adic groups

Let Ω be a p-adic field of characteristic zero. By $\boldsymbol{G}$ we mean a connected algebraic group, defined over Ω, consisting of unipotent elements, with Lie algebra $\boldsymbol{\mathscr{G}}$. Let G, $\mathscr{G}$ be the sets of Ω-points of $\boldsymbol{G}$, $\boldsymbol{\mathscr{G}}$ respectively. We have the Ω-isomorphism of algebraic varieties exp: $\boldsymbol{\mathscr{G}} \to \boldsymbol{G}$, which map $\mathscr{G}$ onto G. Let 'log' denote its inverse. We shall call G a unipotent p-adic group and say that $\mathscr{G}$ is its Lie algebra.

Let Z be the center of G, its Lie algebra $\mathscr{Z}$. One has $\exp \mathscr{Z} = Z$. More generally: the exponential of a subalgebra of $\mathscr{G}$ is a unipotent p-adic subgroup of G, the exponential of an ideal in $\mathscr{G}$ is a normal subgroup of G.

Let G be a unipotent p-adic group.

THEOREM 4: *Each irreducible smooth representation π of G is admissible and pre-unitary.*

PROOF: We use induction on dim G. Lemma 1 is the main source to prove the theorem in case $\dim G = 1$. Assume $\dim G > 1$. Fix any non-trivial character χ_0 of Ω. By Lemma 1 there exists a (unitary) character λ_π of Z such that $\pi(z) = \lambda_\pi(z) I$ for all $z \in Z$. $\lambda_\pi \circ \exp$ is an additive character of $\mathscr{Z}$, hence $\lambda_\pi \circ \exp = \chi_0 \circ f$ for some $f \in \mathscr{Z}'$. Ker(f) is a subalgebra of $\mathscr{Z}$, $\exp(\operatorname{Ker} f) = \operatorname{Ker}(\lambda_\pi)$ therefore a unipotent p-adic subgroup of Z of codimension at most one. If $\dim Z > 1$ or $\dim Z = 1$ and $\lambda_\pi = 1$, π actually reduces to an irreducible representation π_0 of $G_0 = G/\operatorname{Ker} \lambda_\pi$. But $\dim G_0 < \dim G$. The theorem follows from the induction hypotheses.

It remains to consider the case: $\dim Z = 1$ and $\lambda_\pi \neq 1$. We will first show the existence of a unipotent p-adic subgroup G_1 of codimension one in G and an irreducible smooth representation π_1 of G_1 such that π is equivalent to $\operatorname{ind}_{G_1 \uparrow G} \pi_1$.

Let $Y_0 \in \mathscr{G}$ be such that, $[Y_0, \mathscr{G}] \subset \mathscr{Z}$, $Y_0 \notin \mathscr{Z}$. Put $\mathscr{G}_1 = \{U : [U, Y_0] = 0\}$. $\mathscr{G}_1$ is an ideal in $\mathscr{G}$ of codimension 1. Choose $X_0 \notin \mathscr{G}_1$ and define $Z_0 = [X_0, Y_0]$. Observe $Z_0 \in \mathscr{Z}$, $Z_0 \neq 0$. Then $\{X_0, Y_0, Z_0\}$ is a basis for a 3-dimensional subalgebra of $\mathscr{G}$ isomorphic to the Lie algebra of H_3. Let S denote the subgroup of G corresponding to this subalgebra and write, as usual,

$$[x, y, z] = \exp yY_0 \cdot \exp xX_0 \cdot \exp zZ_0 \qquad (x, y, z \in \Omega).$$

We can choose $\lambda \in \Omega$, $\lambda \neq 0$ with the following property:

$$\lambda_\pi([0,0,z]) = \chi_0(\lambda z) \qquad (z \in \Omega).$$

Let us assume, for the moment, that π is an irreducible smooth representation of G on V. By Theorem 3, the restriction of π to S is a direct sum of irreducible representations of S, all equivalent to the representation ρ_λ of S in $C_c^\infty(\Omega)$ given by

$$\rho_\lambda([x,y,z])f(t) = \chi_0(\lambda(z+ty))f(t+x) \qquad (f \in C_c^\infty(\Omega)).$$

So $V = \bigoplus_{i \in I} V_i^\lambda$ for some index-set I, each V_i^λ being isomorphic to $C_c^\infty(\Omega)$. We may regard I as a t.d. space in the obvious way. Then we have

$$V \simeq C_c^\infty(I, C_c^\infty(\Omega)) \simeq C_c^\infty(I) \otimes C_c^\infty(\Omega) \simeq C_c^\infty(\Omega, W),$$

where $W = C_c^\infty(I)$. Moreover, with these identifications,

$$\pi([x,y,z])f(t) = \chi_0(\lambda(z+ty))f(t+x) \qquad (f \in C_c^\infty(\Omega, W)).$$

Let G_1 denote the unipotent p-adic subgroup of G with Lie algebra $\mathcal{G}_1$. G_1 is a closed normal subgroup of G and $G = G_1.(\exp tX_0)_{t \in \Omega}$ (semi-direct product). Since Y_0 is in the center of $\mathcal{G}_1$, $\pi(G_1)$ and $\pi(\exp yY_0)$ $(y \in \Omega)$ commute. Recall

$$\pi(\exp yY_0)f(t) = \chi_0(\lambda ty)f(t) \qquad (y, t \in \Omega;\ f \in C_c^\infty(\Omega, W)).$$

Our aim now is to prove the following lemma.

LEMMA 4: *For each $t \in \Omega$, there exists a smooth representation $g_1 \mapsto \pi(g_1, t)$ of G_1 on W such that*

(a) $(\pi(g_1)f)(t) = \pi(g_1, t)\cdot f(t)$ *for all* $f \in C_c^\infty(\Omega, W)$, $g_1 \in G_1$ *and* $t \in \Omega$;

(b) $\pi(g_1, t+t_0) = \pi(\exp t_0X_0 \cdot g_1 \cdot \exp(-t_0X_0), t)$ *for all* $t, t_0 \in \Omega$, $g_1 \in G_1$.

Obviously, this lemma implies $\pi \simeq \mathrm{ind}_{G_1 \uparrow G}\, \pi_1$ where π_1 is given by $\pi_1(g_1) = \pi(g_1, 0)$ $(g_1 \in G_1)$. The irreducibility of π yields the irreducibility of π_1.

To prove the lemma, we start with a linear map $A: C_c^\infty(\Omega, W) \to$

$C_c^\infty(\Omega, W)$, commuting with all operators $\pi(\exp yY_0)$ $(y \in \Omega)$. Thus:

$$\{A(\chi_0(y\,\cdot)f(\cdot)\}(t) = \chi_0(ty)(Af)(t)$$

for all $t, y \in \Omega$ and $f \in C_c^\infty(\Omega, W)$.

Since $C_c^\infty(\Omega)$ is closed under Fourier transformation, we can easily establish the following: Given $\phi \in C^\infty(\Omega)$ and an open compact subset K of Ω, there exists an integer $m > 0$, $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $y_1, \ldots, y_m \in \Omega$ such that

$$\phi(t) = \sum_{i=1}^{m} \lambda_i \chi_0(y_i t) \qquad (t \in K).$$

For $\phi \in C^\infty(\Omega)$ let L_ϕ denote the linear map $C_c^\infty(\Omega, W) \to C_c^\infty(\Omega, W)$ given by $L_\phi f(t) = \phi(t)f(t)$ $(f \in C_c^\infty(\Omega, W))$. Then, putting $K = \operatorname{Supp} f \cup \operatorname{Supp} Af$, we obtain:

$$\{A(L_\phi f)\}(t) = A\left(\sum_{i=1}^{m} \lambda_i \chi_0(y_i\,\cdot)f(\cdot)\right)(t)$$

$$= \sum_{i=1}^{m} \lambda_i \chi_0(y_i t) Af(t) = \{L_\phi(Af)\}(t)$$

$(t \in \Omega,\ f \in C_c^\infty(\Omega, W))$. Hence $AL_\phi = L_\phi A$ for every $\phi \in C^\infty(\Omega)$. In particular we have: $\pi(g_1)L_\phi = L_\phi \pi(g_1)$ for all $g_1 \in G_1$, $\phi \in C^\infty(\Omega)$. Let ψ_n denote the characteristic function of P^n. In addition, put $L_t\phi(s) = \phi(s-t)$ $(s, t \in \Omega$, ϕ any function on $\Omega)$. Define:

$$\pi(g_1, t)w = \pi(g_1)(L_t\psi_n \otimes w)(t) \qquad (g_1 \in G_1, t \in \Omega, w \in W).$$

Here, as usual, $L_t\psi_n \otimes w$ is identified with the function $s \mapsto L_t\psi_n(s) \cdot w$ $(s \in \Omega)$. $\pi(g_1, t)$ is well-defined: assuming $n' \le n$, we obtain

$$\pi(g_1)(L_t\psi_n \otimes w)(t) = \pi(g_1)(L_t\psi_{n'} \cdot L_t\psi_n \otimes w)(t).$$

But this equals, by the above result,

$$L_t\psi_n(t)\pi(g_1)(L_t\psi_{n'} \otimes w)(t) = \pi(g_1)(L_t\psi_{n'} \otimes w)(t).$$

Let us show now that $\pi(g_1, t)$ satisfies condition (a) of Lemma 4. Fix $f \in C_c^\infty(\Omega, W)$ and determine integers $m, n > 0$, $t_1, \ldots, t_m \in \Omega$ and

$w_1, \ldots, w_m \in W$ such that

$$f = \sum_{i=1}^{m} L_{t_i}\psi_n \otimes w_i.$$

Then

$$\begin{aligned}
\pi(g_1)f(t) &= \pi(g_1)\Big(\sum_{i=1}^{m} L_{t_i}\psi_n \otimes w_i\Big)(t) \\
&= \sum_{i=1}^{m} \pi(g_1)(L_{t_i}\psi_n \otimes w_i)(t) \\
&= \sum_{i=1}^{m} \{L_t\psi_n \cdot \pi(g_1)(L_{t_i}\psi_n \otimes w_i)\}(t) \\
&= \sum_{i=1}^{m} \pi(g_1)(L_t\psi_n \cdot L_{t_i}\psi_n \otimes w_i)(t) \\
&= \sum_{i=1}^{m} \{L_{t_i}\psi_n \cdot \pi(g_1)(L_t\psi_n \otimes w_i)\}(t) \\
&= \sum_{i=1}^{m} L_{t_i}\psi_n(t) \cdot \pi(g_1, t)w_i \\
&= \pi(g_1, t) \cdot f(t) \qquad (t \in \Omega,\, g_1 \in G_1).
\end{aligned}$$

Condition (b) is also fulfilled. Indeed,

$$\begin{aligned}
&\pi(\exp t_0X_0 \cdot g_1 \cdot \exp - t_0X_0, t)w \\
&= \pi(\exp t_0X_0)\pi(g_1)\pi(\exp - t_0X_0)(L_t\psi_n \otimes w)(t) \\
&= \pi(g_1)\pi(\exp - t_0X_0)(L_t\psi_n \otimes w)(t + t_0).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\pi(\exp - t_0X_0)(L_t\psi_n \otimes w)(u) &= L_t\psi_n \otimes w(u - t_0) \\
&= L_{t+t_0}\psi_n \otimes w(u) \qquad (u \in \Omega).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\pi(\exp t_0X \cdot g_1 \cdot \exp - t_0X_0, t)w \\
&= \pi(g_1)(L_{t+t_0}\psi_n \otimes w)(t + t_0) = \pi(g_1, t + t_0)w.
\end{aligned}$$

Finally, it is easily checked, that condition (a) forces $g_1 \mapsto \pi(g_1, t)$ $(g_1 \in G_1)$ to be a smooth representation of G_1 for each $t \in \Omega$. This concludes the proof of Lemma 4.

COROLLARY: *Each irreducible smooth representation of G is monomial.*

Let us continue the proof of Theorem 4. By induction we assume that π_1 is admissible and pre-unitary. Hence $\pi = \mathrm{ind}_{G_1 \uparrow G}\, \pi_1$ is pre-unitary. Let K be an open subgroup of G and let V_K denote the space of all $f \in C_c^\infty(\Omega)$ such that $\pi(g)f = f$ for all $g \in K$. Let $f \in V_K$. Since

$$\pi(\exp xX_0)f(t) = f(x+t) \qquad (x, t \in \Omega),$$

there exists an integer $n > 0$, only depending on K, such that f is constant on cosets of P^n.

The relation

$$\pi(\exp yY_0)f(t) = \chi_0(\lambda yt)f(t) \qquad (y, t \in \Omega)$$

implies that $\mathrm{Supp}\, f \subset P^m$ for some integer $m > 0$, only depending on K. Assume $m < n$. Then $P^m = \bigcup_{i=1}^k (t_i + P^n)$ for some $t_1, \ldots, t_k \in \Omega$. Now consider the mapping

$$f \mapsto (f(t_1), \ldots, f(t_k))$$

of V_K into W^k. This mapping is linear and injective. Since

$$(\pi(g_1)f(t) = \pi_1(\exp tX_0 \cdot g_1 \cdot \exp - tX_0)f(t) \qquad (g_1 \in G_1, t \in \Omega)$$

we obtain that $f(t_i)$ is fixed by $\exp t_iX_0 \cdot (K \cap G_1) \exp(-t_iX_0)$, being an open subgroup of G_1 ($i = 1, 2, \ldots, k$). Therefore, each $f(t_i)$ stays in a finite-dimensional subspace of W. Consequently $\dim V_K < \infty$.

We have shown that π is admissible. This concludes the proof of Theorem 4.

REMARK: Similar to the proof of Theorem 4 one can easily show that the restriction of an irreducible unitary representation of G to its subspace of smooth vectors is an admissible representation of G.

§6. Kirillov's theory

Let G be as in §5. What remains is to describe the irreducible unitary representations of G. This is done by Kirillov [8] for the real groups G and, as observed by Moore [9], the whole machinery works in the p-adic case as well. For completeness and for later purposes, we give the result.

Given $f \in \mathscr{G}'$, put $B_f(X, Y) = f([X, Y])$ $(X, Y \in \mathscr{G})$. B_f is an alternating bilinear form on $\mathscr{G}$. A subalgebra $\mathfrak{H}$ of $\mathscr{G}$ which is at the same time a maximal totally isotropic subspace for B_f is called a *polarization* at f. Polarizations at f exist ([4], 1.12.10). They coincide with the subalgebra's $\mathfrak{H} \subset \mathscr{G}$ which are maximal with respect to the property that $\mathfrak{H}$ is a totally isotropic subspace for B_f (cf. [8], Lemma 5.2, which carries over to the p-adic case with absolutely no change). Let $\mathfrak{H}$ be any subalgebra of $\mathscr{G}$ which is a totally isotropic subspace for $B_f : f[\mathfrak{H}, \mathfrak{H}] = 0$. Put $H = \exp \mathfrak{H}$. We may define a character χ_f of H by the formula:

$$\chi_f(\exp X) = \chi_0(f(X))^{1} \qquad (X \in \mathfrak{H}).$$

Let $\rho(f, \mathfrak{H}, G)$ denote the unitary representation of G induced by χ_f.

THEOREM 5 ([8], [9]):
(i) *$\rho(f, \mathfrak{H}, G)$ is irreducible if and only if $\mathfrak{H}$ is a polarization at f,*
(ii) *each irreducible unitary representation of G is of the form $\rho(f, \mathfrak{H}, G)$,*
(iii) *$\rho(f_1, \mathfrak{H}_1, G)$ and $\rho(f_2, \mathfrak{H}_2, G)$ are unitarily equivalent if and only if f_1 and f_2 are in the same G-orbit in $\mathscr{G}'$.*

§7. The character formula

The main reference for this section is [12]. G acts on $\mathscr{G}$ by Ad and hence on $\mathscr{G}'$ by the contragredient representation. It is well-known (and can be proved similar to the real case) that all G-orbits in $\mathscr{G}'$ are closed.

Let us fix a non-trivial (unitary) character χ_0 of the additive group of Ω.

We shall choose a Haar measure dg on G and a translation invariant measure dX on $\mathscr{G}$ such that $dg = \exp(dX)$.

Let $f \in \mathscr{G}'$, $\mathfrak{H}$ a polarization at f and O the orbit of f in $\mathscr{G}'$. Put $\pi = \rho(f, \mathfrak{H}, G)$. Given $\psi \in C_c^\infty(G)$, we know that $\pi(\psi)$ is an operator of finite rank (§5, Remark). Put $\psi_1(X) = \psi(\exp X)$ $(X \in \mathscr{G})$. Then $\psi_1 \in C_c^\infty(\mathscr{G})$. The Fourier transform of ψ_1 is defined by:

[1] Here χ_0 is (as usual) a fixed non-trivial additive character of Ω.

$$\hat{\psi}_1(X') = \int_{\mathscr{G}} \psi_1(X)\chi_0(\langle X, X'\rangle)\, \mathrm{d}X \qquad (X' \in \mathscr{G}').$$

Observe that $\hat{\psi}_1 \in C_c^\infty(\mathscr{G}')$.

THEOREM 6: *There exists a unique positive G-invariant measure* $\mathrm{d}v$ *on O such that for all* $\psi \in C_c^\infty(G)$:

$$\operatorname{tr} \pi(\psi) = \int_O \hat{\psi}_1(v)\, \mathrm{d}v.$$

Note that the right-hand side is finite, because $\mathrm{d}v$ is also a measure on $\mathscr{G}'$, since O is closed in $\mathscr{G}'$.

Pukanszky's proof of ([12], Lemma 2),[2] goes over to our situation with no substantial change. Observe that each $\psi \in C_c^\infty(G)$ is a linear combination of functions of the form $\phi * \tilde{\phi}$ $(\phi \in C_c^\infty(G))$ where $\tilde{\phi}$ is given by $\tilde{\phi}(g) = \overline{\phi(g^{-1})}$ $(g \in G)$. The algorithm to determine $\mathrm{d}v$ (given $\mathrm{d}g$ and $\mathrm{d}X$ such that $\mathrm{d}g = \exp(\mathrm{d}X)$) is similar to that given by Pukanszky:

(i) Put $K = \exp \mathfrak{H}$, $\Gamma = K\backslash G$. Choose invariant measures $\mathrm{d}k$ and $\mathrm{d}\gamma$ on K and Γ respectively such that $\mathrm{d}g = \mathrm{d}k\, \mathrm{d}\gamma$.
(ii) Choose a translation invariant measure $\mathrm{d}H$ on $\mathfrak{H}$ such that $\mathrm{d}k = \exp(\mathrm{d}H)$.
(iii) Let $\mathrm{d}X'$ and $\mathrm{d}H'$ denote the dual measures of $\mathrm{d}X$ and $\mathrm{d}H$ respectively.
(iv) Let $\mathfrak{H}^\perp = \{X' \in \mathscr{G}' : \langle \mathfrak{H}, X'\rangle = 0\}$. Take $\mathrm{d}H^\perp$ on $\mathfrak{H}^\perp$ such that $\mathrm{d}X' = \mathrm{d}H'\, \mathrm{d}H^\perp$.
(v) Let S be the stabilizer of f in G. Then $S \subset K$. Choose $\mathrm{d}\lambda$ on $S\backslash K$ such that $\mathrm{d}\lambda$ is the inverse-image of $\mathrm{d}H^\perp$ under the bijection

$$Sk \mapsto k^{-1} \cdot f \qquad (k \in K)$$

of $S\backslash K$ onto $f + H^\perp$.
(vi) Finally, put $\mathrm{d}v$ = image of $\mathrm{d}\lambda\, \mathrm{d}\gamma$ under the bijective mapping $Sg \mapsto g^{-1} \cdot f$ $(g \in G)$ of $S\backslash G$ onto O.

The invariant measure $\mathrm{d}v$ depends on the choice of the character χ_0. Taking instead of χ_0 the character $x \mapsto \chi_0(tx)$ for some $t \in \Omega$, $t \neq 0$, we obtain, by applying the above algorithm, the following homogeneity

[2] Part (d) of his proof has to be omitted here.

property for $\mathrm{d}v$:

COROLLARY: *Let O be a G-orbit in $\mathscr{G}'$ of dimension $2m$. Then*

$$\int_O \phi(tv)\,\mathrm{d}v = |t|^{-m} \int_O \phi(v)\,\mathrm{d}v$$

for all $\phi \in C_c^\infty(\mathscr{G}')$ and all $t \in \Omega$, $t \neq 0$.

Observe that we may choose in the corollary $\mathrm{d}v$ to be any G-invariant positive measure on O.

Let O be as above. O carries a *canonical measure* μ, which is constructed as follows. For any $p \in O$, define $\alpha_p : G \to O$ by $\alpha_p(a) = a \cdot p$ $(a \in G)$. The kernel of the differential β_p of α_p, $\beta_p : \mathscr{G} \to T_p$ (T_p = tangent space to O in p) coincides with the radical of the alternating bilinear form B_p on $\mathscr{G}$. Let $\mathrm{Stab}_G(p)$ be the stabilizer of p in G. Then also, $\mathrm{Ker}\,\beta_p$ = Lie algebra of $\mathrm{Stab}_G(p)$. Hence B_p induces a non-degenerate alternating bilinear form ω_p on T_p. In this way a 2-form ω is defined on O. One easily checks that ω is G-invariant (cf. [12] for the real case). Let $d = 2m$ be the dimension of O. Assume $d > 0$. Then μ is given by $\mu = |(1/2^m m!)\Lambda^m \omega|$.

THEOREM 7: *Let us fix the character χ_0 of Ω in such a way that $\chi_0 = 1$ on $\mathscr{O}$, $\chi_0 \neq 1$ on P^{-1}. Let O be any G-orbit in $\mathscr{G}'$ of positive dimension. Then the invariant measure* $\mathrm{d}v$ *and the canonical measure μ on O coincide.*

The proof is essentially the same as in the real case ([12], Theorem).

§8. Square-integrable representations mod Z

Let G and Z be as in §5. An irreducible unitary representation π of G on $\mathscr{H}$ is called *square-integrable mod Z* if there exist $\xi, \eta \in \mathscr{H} - (0)$ such that

$$\int_{G/Z} |\langle \pi(x)\xi, \eta\rangle|^2\,\mathrm{d}\dot{x} < \infty.$$

Such representations are extensively discussed by C.C. Moore and J. Wolf for real unipotent groups [10]. For p-adic unipotent groups, see [13]: the restriction of π to the space $\mathscr{H}_\infty$ of π-smooth vectors is a supercuspidal representation. Our main goal is to find a closed formula for the multiplicity of the trivial representation of well-chosen open and compact subgroups K of G in the restriction of π to K.

Let $f \in \mathscr{G}'$. By O_f we denote the G-orbit of f in $\mathscr{G}'$ and by π_f an irreducible unitary representation of G, corresponding to f (more precisely: to O_f) by Kirillov's theory (§6). Let $\mathscr{H}_f$ denote the space of π_f. Then we have, similar to ([10], Theorem 1):

THEOREM 8: *The following four statements are equivalent*:

(i) π_f *is square-integrable* mod Z,

(ii) dim O_f = dim G/Z,

(iii) $O_f = f + \mathscr{Z}^\perp$,

(iv) B_f *is a non-degenerate bilinear form on* $\mathscr{G}/\mathscr{Z}$.

Here $\mathscr{Z}^\perp = \{X' \in \mathscr{G}' : \langle X', \mathscr{Z}\rangle = 0\}$.

Now assume π_f to be square-integrable mod Z. The orbit O_f carries the canonical measure μ. We shall define another G-invariant measure ν on O_f. Let us fix a G-invariant differential form ω on $\mathscr{G}/\mathscr{Z}$ of maximal degree. Let σ denote the adjoint representation of G on $\mathscr{G}$ and let ρ be the representation of G contragredient to σ. Fix $p \in O_f$. We have $\mathrm{Stab}_G(p) = Z$ and $g \mapsto \rho(g)h$ is an isomorphism[3] of G/Z onto O_f. Call β_p the differential of this map at e; β_p: $\mathscr{G}/\mathscr{Z} \to T_h$. Define

$$\omega_p(\beta_p(X_1), \ldots, \beta_p(X_n)) = \omega(X_1, \ldots, X_n)$$
$$(n = \dim \mathscr{G}/\mathscr{Z};\ X_1, \ldots, X_n \in \mathscr{G}/\mathscr{Z}).$$

In this way we get a n-form ω' on O_f. We claim that ω' is G-invariant:

$$\omega_p(\beta_p(X_1), \ldots, \beta_p(X_n)) = \omega_q(\mathrm{d}\rho_p(a)\beta_p(X_1), \ldots, \mathrm{d}\rho_p(a)\beta_p(X_n))$$

if $p, q \in O_f$, $q = \rho(a)p$ $(X_1, \ldots, X_n \in \mathscr{G}/\mathscr{Z})$. This is a simple exercise:

$$\omega_q(\mathrm{d}\rho_p(a)\beta_p(X_1), \ldots, \mathrm{d}\rho_p(a)\beta_p(X_n)) = \omega_q(\beta_q(\sigma(a)X_1), \ldots, \beta_q(\sigma(a)X_n))$$
$$= \omega(\sigma(a)X_1, \ldots, \sigma(a)X_n) = \omega(X_1, \ldots, X_n) = \omega_p(\beta_p(X_1), \ldots, \beta_p(X_n)).$$

Call ν the measure on O_f corresponding to ω'; ν is uniquely determined by the choice of the volume form ω on $\mathscr{G}/\mathscr{Z}$. Let $|P(f)|$ denote

[3] Here isomorphism is meant in the sense of algebraic geometry.

the constant relating μ and ν: $\mu = |P(f)|\nu$.[4] The volume form ω fixes, on the other hand, a Haar measure $d\dot{g}$ on G/Z. It is obvious that ν is the image of $d\dot{g}$ under the mapping $g \mapsto \rho(g)f$ of G/Z onto O_f. From the definition of ν we see that the same is true for the mapping $g \mapsto \rho(g)h$ of G/Z onto O_f, for any $h \in O_f$.

Let us denote by $d(\pi_f)$ the *formal degree* of π_f:

$$\int_{G/Z} |\langle \pi_f(g)\xi, \xi\rangle|^2 \, d\dot{g} = d(\pi_f)^{-1}\langle \xi, \xi\rangle \qquad (\xi \in \mathcal{H}_f).$$

THEOREM 9: *$d(\pi_f)$ is a positive real number, which satisfies the following identity: $d(\pi_f) = |P(f)|$.*

This is proved exactly the same way as in the real case ([10], Theorem 4).

§9. Multiplicities

Let G be as usual, $f \in \mathcal{G}'$ such that π_f is square-integrable mod Z. Let K be an open and compact subgroup of G. We shall call K a *lattice subgroup* if $L = \log K$ is a lattice in $\mathcal{G}$, i.e. an open and compact, $\mathcal{O}$-submodule of $\mathcal{G}$.

THEOREM 10: *Let K be a lattice subgroup of G, $L = \log K$. Normalize Haar measures dg on G and dz on Z such that $\int_K dg = \int_{K\cap Z} dz = 1$. Choose a Haar measure $d\dot{g}$ on G/Z such that $dg = dz\, d\dot{g}$. Then the trivial representation of K occurs in the restriction of π_f to K if and only if $f(L \cap \mathcal{Z}) \subset \mathcal{O}$; moreover, its multiplicity $m(\pi_f, 1)$ is $1/d(\pi_f)$.*

The proof of Theorem 10 is rather long and proceeds by a careful induction on dim G. The theorem is obvious if dim $G = 1$. So assume dim $G = n > 1$. Put $\mathcal{Z}^0 = \operatorname{Ker} f \cap \mathcal{Z}$ and $Z^0 = \exp \mathcal{Z}^0$. We have two cases:

1. dim $\mathcal{Z}^0 \neq 0$. Replace $\mathcal{G}$ by $\mathcal{G}/\mathcal{Z}^0$ and G by G/Z^0. The center of G/Z^0 is Z/Z^0 (cf [13], proof of Theorem, (i)). Replace also K by $K^0 = KZ^0/Z^0$. K^0 is a lattice subgroup of G/Z^0: $\log K^0 = L/L \cap \mathcal{Z}^0$. Let f^0, π_f^0 be the pull down of f, π_f to $\mathcal{G}/\mathcal{Z}^0$ and G/Z^0 respectively. It is well-known that π_f^0 is equivalent to π_{f^0}. Hence $m(\pi_f, 1) = m(\pi_{f^0}, 1)$.

[4] $P(f)$ actually is the Pfaffian of the canonical differential form, defining μ, relative to ω ([1], §5, no. 2).

Furthermore, $f(L \cap \mathscr{Z}) = f^0(L^0 \cap \mathscr{Z}/\mathscr{Z}^0)$. Normalizing the Haar measures on G/Z^0, Z/Z^0 and $G/Z^0/Z/Z^0$ as prescribed in the theorem, one obtains $d(\pi_f) = d(\pi_{f^0})$. The assertion for G now follows immediately from the result for G/Z^0, which is of smaller dimension.

2. $\dim \mathscr{Z} = 1$ and $f \neq 0$ on $\mathscr{Z}$. $L \cap \mathscr{Z}$ is a lattice of rank one. Let $\underline{Z}$ be a generator of $L \cap \mathscr{Z}$. Choose $\underline{X} \notin \mathscr{Z}$ such that $[\underline{X}, \mathscr{G}] \subset \mathscr{Z}$. Put $\mathscr{G}_0 = \{U: [U, \underline{X}] = 0\}$. $\mathscr{G}_0$ is an ideal in $\mathscr{G}$ of codimension one with center $\mathscr{Z}_0 = \mathscr{Z} + (\underline{X})$ (cf [13], p. 149). $\mathscr{Z}_0 \cap L$ is a lattice of rank two; $\mathscr{Z}_0 \cap L/\mathscr{Z} \cap L$ is a lattice of rank one. We may assume that $\underline{X}$ is chosen in such a way that $\underline{X} \bmod(\mathscr{Z} \cap L)$ generates $\mathscr{Z}_0 \cap L/\mathscr{Z} \cap L$. Then obviously,

$$\mathscr{Z}_0 \cap L = \mathcal{O}\underline{X} + \mathscr{Z} \cap L = \mathcal{O}\underline{X} + \mathcal{O}\underline{Z}.$$

Since $L/L \cap \mathscr{G}_0$ is a lattice of rank one, we can choose $\underline{Y} \in L$, $\underline{Y} \notin \mathscr{G}_0$ such that $L = \mathcal{O}\underline{Y} + L \cap \mathscr{G}_0$. Put $G_0 = \exp \mathscr{G}_0$, $G_1 = (\exp s\underline{Y})_{s \in \Omega}$. Then $G = G_0 \cdot G_1$ and $G_0 \cap G_1 = \{e\}$.

Now choose a basis $\underline{Z}$, $\underline{X}$, $e_1, \ldots, e_{n-3}$ of $\mathscr{G}_0$ such that $L \cap \mathscr{G}_0 = \mathcal{O}\underline{Z} + \mathcal{O}\underline{X} + \mathcal{O}e_1 + \cdots + \mathcal{O}e_{n-3}$ and such that $e_1, \ldots, e_{n-3}$ is a supplementary basis of $\mathscr{Z}_0$ in the sense of Pukanszky ([12], section 3). One easily checks that this is possible. Given $X_0 \in \mathscr{G}_0$, write

$$X_0 = z\underline{Z} + t\underline{X} + t_1 e_1 + \cdots + t_{n-3}e_{n-3}$$

and choose $(z, t, t_1, \ldots, t_{n-3})$ as coordinates of the second kind on G_0. Then $\mathrm{d}g_0 = \mathrm{d}z\,\mathrm{d}t\,\mathrm{d}t_1 \ldots \mathrm{d}t_{n-3}$ is a Haar measure on G_0 and $\mathrm{d}s\,\mathrm{d}g_0$ is a Haar measure on G. Moreover, if $Z_0 = \exp \mathscr{Z}_0$, $K_0 = K \cap G_0$, we now have:

$$\mathrm{vol}(K) = \mathrm{vol}(K_0) = \mathrm{vol}(K \cap Z) = \mathrm{vol}(K_0 \cap Z_0) = 1$$ [5]

Let f_0 denote the restriction of f to $\mathscr{G}_0$. It is part of the Kirillov theory that π_f is equivalent to $\mathrm{ind}_{G_0 \uparrow G}\, \pi_{f_0}$. Moreover, π_{f_0} is square-integrable mod Z_0 ([13], p. 149). We need a relation between $d(\pi_f)$ and $d(\pi_{f_0})$. The Haar measures on G/Z and G_0/Z_0 should be chosen as prescribed in the theorem. The following lemma is proved by computations, similar to those given in ([13], Section 5).

LEMMA 5: *Let* $r = f[\underline{X}, \underline{Y}]$. *Furthermore, put for any* $s \in \Omega$, $f_s(X_0) = f(Ad(\exp - s\underline{Y})X_0)$ $(X_0 \in \mathscr{G}_0)$ *and* $\pi_s = \pi_{f_s}$. *Then* π_s *is square-*

[5] We take $\mathrm{d}z$ and $\mathrm{d}z\,\mathrm{d}t$ as Haar measures on Z and Z_0 respectively.

integrable mod Z_0 *and*

$$d(\pi_s) = \frac{1}{|r|} d(\pi_f)$$

for all $s \in \Omega$.

PROOF: The space $\mathcal{H}_f$ of π_f may be identified with $L^2(\Omega, \mathcal{H}_{f_0})$. Fix a smooth vector $v \in \mathcal{H}_{f_0}$, $v \neq 0$. Choose $\psi \in C_c^\infty(\Omega)$, $\psi \neq 0$ and put $\psi_v(x) = \psi(x)v$ $(x \in \Omega)$.

Then $\psi_v \in \mathcal{H}_f$. Furthermore, the computations in ([13], Section 5), show

$$\int_{G/Z} |\langle \pi_f(g)\psi_v, \psi_v\rangle|^2 \, d\dot{g}$$

$$\frac{1}{|r|}\int_\Omega \int_\Omega |\psi(s+s_1)\bar{\psi}(s)|^2 \Big\{\int_{G_0/Z_0} |\langle \pi_s(g_0)v, v\rangle|^2 \, d\dot{g}_0\Big\} \, ds \, ds_1.$$

Moreover,

$$\begin{aligned} &\int_{G_0/Z_0} |\langle \pi_s(g_0)v, v\rangle|^2 \, d\dot{g}_0 \\ &= \int_{G_0/Z_0} |\langle \pi_0(\exp s\underline{Y} \cdot g_0 \cdot \exp -s\underline{Y})v, v\rangle|^2 \, d\dot{g}_0 \\ &= \int_{G_0/Z_0} |\langle \pi_0(g_0)v_1 v\rangle|^2 \, |\det_{\mathfrak{g}_0/\mathfrak{z}_0} Ad(\exp - s\underline{Y})| \, d\dot{g}_0 \\ &= \int_{G_0/Z_0} |\langle \pi_0(g_0)v, v\rangle|^2 \, d\dot{g}_0 \qquad \text{for all } s \in \Omega. \end{aligned}$$

Hence, π_s is square-integrable mod Z_0 and $d(\pi_s) = d(\pi_0)$ for all $s \in \Omega$. In addition:

$$\langle \psi_v, \psi_v\rangle d(\pi_f)^{-1} = \frac{1}{|r|}\langle v, v\rangle\langle \psi, \psi\rangle d(\pi_0)^{-1},$$

$$\text{or } d(\pi_0) = \frac{1}{|r|} d(\pi_f).$$

This completes the proof of the lemma.

Let ϕ, ϕ_0 denote the characteristic functions of K, K_0 respectively. Given $\psi \in L^2(\Omega, \mathcal{H}_{f_0})$, we have

$$
\begin{aligned}
\pi_f(\phi)\psi(\xi) &= \int_G \pi_f(g)\phi(g)\psi(\xi)\,\mathrm{d}g \\
&= \int_\Omega \int_{G_0} \phi(g_0 \cdot \exp s\underline{Y})\pi_{f_0}(\exp \xi\underline{Y} \cdot g_0 \cdot \exp -\xi\underline{Y})\psi(s+\xi)\,\mathrm{d}g_0\,\mathrm{d}s \\
&= \int_\Omega \left\{\int_{G_0} \phi(g_0 \cdot \exp(s-\xi)\underline{Y})\pi_{f_0}(\exp \xi\underline{Y} \cdot g_0 \cdot \exp -\xi\underline{Y})\,\mathrm{d}g_0\right\}\psi(s)\,\mathrm{d}s \\
&\qquad (\xi \in \Omega).
\end{aligned}
$$

Hence, by a p-adic analogue of Mercer's theorem,

$$\operatorname{tr} \pi_f(\phi) = \int_\Omega \operatorname{tr}\left\{\int_{G_0} \phi(g_0)\pi_{f_0}(\exp s\underline{Y} \cdot g_0 \cdot \exp -s\underline{Y})\,\mathrm{d}g_0\right\}\mathrm{d}s.$$

So, we obtain the following relation:

$$\operatorname{tr} \pi_f(\phi) = \int_\Omega \operatorname{tr} \pi_s(\phi_0)\,\mathrm{d}s.$$

Equivalently:

LEMMA 6: $m(\pi_f, 1) = \int_\Omega m(\pi_s, 1)\,\mathrm{d}s$.

Now assume $m(\pi_f, 1) > 0$. Then $m(\pi_s, 1) > 0$ for some $s \in \Omega$. By induction, $f_s(L_0 \cap \mathscr{L}_0) \subset \mathcal{O}$, where $L_0 = L \cap \mathscr{G}_0$. Hence

$$f(L \cap \mathscr{L}) = f_s(L \cap \mathscr{L}) \subset f_s(L_0 \cap \mathscr{L}_0) \subset \mathcal{O}.$$

Conversily, assume $f(L \cap \mathscr{L}) \subset \mathcal{O}$. Let $s \in \Omega$. Then $f_s(L_0 \cap \mathscr{L}_0) \subset \mathcal{O}$ if and only if $f_s(\underline{X}) \subset \mathcal{O}$. We have:

$$f_s(\underline{X}) = f(\underline{X}) + sf[\underline{X}, \underline{Y}] = f(\underline{X}) + sr.$$

Hence, by induction, $m(\pi_s, 1) > 0$ if and only if $s \in (1/r)(-f(\underline{X}) + \mathcal{O})$. Moreover, again by induction, applying Lemma 5 and 6,

$$
\begin{aligned}
m(\pi_f, 1) &= \int_{(1/r)(-f(\underline{X})+\mathcal{O})} \frac{1}{d(\pi_s)}\,\mathrm{d}s = \frac{|r|}{d(\pi_f)} \operatorname{vol}\left(\frac{1}{r}(-f(\underline{X}) + \mathcal{O})\right) \\
&= \frac{|r|}{d(\pi_f)} \cdot \frac{1}{|r|} = \frac{1}{d(\pi_f)}.
\end{aligned}
$$

This completes the proof of Theorem 10.

§10. Multiplicities and *K*-orbits

Let K be a lattice subgroup of G, $L = \log K$. Choose a basis $e_1, \ldots, e_p$ of $\mathscr{Z}$ and let $e_{p+1}, \ldots, e_n$ be a supplementary basis of $\mathscr{Z}$ such that $L = \Sigma_{i=1}^{n} \mathcal{O} e_i$ ($n = \dim \mathscr{G}$). Choose $(t_1, \ldots, t_n)$ as coordinates on $\mathscr{G}$. Then $(t_1, \ldots, t_n)$ can also be used as coordinates of the second kind on G. Similarly $(t_1, \ldots, t_p)$ will denote coordinates on Z. Choose corresponding Haar measures on G and Z, as usual. Then $\mathrm{vol}(K) = \mathrm{vol}(K \cap Z) = 1$. Moreover, fix a volume form ω on $\mathscr{G}/\mathscr{Z}$ by $\omega = \mathrm{d}t_{p+1} \wedge \cdots \wedge \mathrm{d}t_n$.

Let ϕ denote the characteristic function of K. Fix $f \in \mathscr{G}'$. To compute $m(\pi_f, 1)$ we can apply the character formula (§7). We obtain:

$$m(\pi_f, 1) = \mathrm{tr}\ \pi_f(\phi) = \int_{O_f} \hat{\phi}_1(v)\, \mathrm{d}\mu_f(v),$$

where μ_f is the canonical measure on O_f.

Observe that $\hat{\phi}_1$ is the characteristic function of the lattice L', dual to L; $L' = \{l \in \mathscr{G}' : l(L) \subset \mathcal{O}\}$. Hence $m(\pi_f, 1) = \mu_f$-measure of $L' \cap O_f$. K acts on $L' \cap O_f$; $L' \cap O_f$ is a disjoint union of finitely many, say l_f, K-orbits.

Now assume π_f to be square-integrable mod Z. Then we have the measure ν, relative to ω, (§8) on O_f. It follows from its construction, that all K-orbits in $L' \cap O_f$ have the same ν-measure, namely, one. Since $\mu_f = d(\pi_f)\nu$ (§8), we get:

$$m(\pi_f, 1) = l_f \cdot d(\pi_f).$$

On the other hand, $m(\pi_f, 1) = 1/d(\pi_f)$, provided $m(\pi_f, 1) > 0$ (Theorem 10). So we have the following result:

THEOREM 11: *Let K be a lattice subgroup of G, $L = \log K$ and $L' = \{l \in \mathscr{G}' : l(L) \subset \mathcal{O}\}$. Fix $f \in \mathscr{G}'$ and let O_f denote the G-orbit of f. Let l_f be the number of K-orbits in L'. Then $m(\pi_f, 1) > 0$ if and only if $l_f > 0$. Moreover, if π_f is square-integrable* mod Z, *then $m(\pi_f, 1) = \sqrt{l_f}$.*

This theorem is related to work of C.C. Moore [9]. Actually, Moore proves the inequality:

$$m(\pi_f, 1) \leq l_f$$

for all $f \in \mathscr{G}'$.

§11. An example

We consider the p-adic Heisenberg group H_3, consisting of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $x,\ y,\ z \in \mathbb{Q}_p$, $p \neq 2$. Put

$$K = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z}_p \right\}.$$

K is easily seen to be a lattice subgroup of H_3 and

$$\log K = L = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{Z}_p \right\}.$$

Choosing Haar measures $dx\,dy\,dz$ on G and dz on the center Z of H_3,

$$Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{Q}_p \right\}$$

we have $\mathrm{vol}(K) = \mathrm{vol}(K \cap Z) = 1$. Normalize the Haar measures on G/Z and $\mathcal{G}/\mathcal{Z}$ in the usual way.

Given $f \in \mathcal{G}'$, we shall write $f = \{\alpha, \beta, \gamma\}$ if

$$f\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \alpha x + \beta y + \gamma z \qquad (x, y, z, \alpha, \beta, \gamma \in \mathbb{Q}_p).$$

Similar to the real case, we have $|P(f)| = |\gamma|$ ([10]). Put $f_0 = \{0, 0, \lambda\}$, $\lambda \neq 0$. Then π_{f_0} is square-integrable $\operatorname{mod} Z$ and $d(\pi_{f_0}) = |\lambda|$. The G-orbit of f_0 consists of all triples

$$\{y\lambda, -x\lambda, \lambda\} \qquad (x, y \in \mathbb{Q}_p\}.$$

Assume $|\lambda| \leq 1$. $L' = \{\{\alpha, \beta, \gamma\}: \alpha, \beta, \gamma \in \mathbb{Z}_p\}$ and

$$L' \cap O_{f_0} = \left\{ \{y\lambda, -x\lambda, \lambda\} : x, y \in \frac{1}{\lambda}\mathbb{Z}_p \right\}.$$

K acts on $L' \cap O_{f_0}$; if

$$k = \begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$k \cdot \{y\lambda, -x\lambda, \lambda\} = \{y\lambda + u\lambda, -x\lambda - v\lambda, \lambda\};$$

therefore $l_{f_0} = 1/|\lambda|^2$.

On the other hand, π_{f_0} is given on $L^2(\mathbb{Q}_p)$ by:

$$\pi_{f_0}\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\phi(t) = \chi_0(\lambda(z + ty))\phi(t + x).$$

We have

$$m(\pi_{f_0}, 1) = \dim\{\phi \in C_c^\infty(\mathbb{Q}_p) : \chi_0(\lambda ty)\phi(t + x) = \phi(t)$$

for $t \in \mathbb{Q}_p$; $x, y \in \mathbb{Z}_p\} = \dim\{\phi \in C_c^\infty(\mathbb{Q}_p)$: Supp $\phi \subset (1/\lambda)\mathbb{Z}_p$, ϕ $\mathbb{Z}_p$ – periodic$\} = 1/|\lambda|$.

Similar computations can be done for the higher dimensional Heisenberg groups.

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Mathematisch Instituut
Wassenaarseweg 80
Leiden, The Netherlands