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THE HAWKINS SIEVE AND BROWNIAN MOTION

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1. Introduction and summary

1.1. The complete story of the Hawkins sieve can now be told. (But see §1.3.) Even though we follow Neudecker–Williams [5] closely, we shall for the reader's convenience make the present paper a self-contained treatment of the Hawkins sieve.

The Hawkins sieve (see [9] for 'Eratosthenes' motivation and some of the history) operates inductively as follows. Let

$$A_1 \equiv \{2, 3, 4, 5, \dots\}.$$

(We use the symbol '$\equiv$' to signify 'is *defined* to be equal to'.)

Stage 1. Declare $X_1 \equiv \min A_1$. From the set $A_1 \backslash \{X_1\}$, each number in turn (and each independently of the others) is deleted with probability X_1^{-1}. The set of elements of $A_1 \backslash \{X_1\}$ which remain is denoted by A_2.

Stage n. Declare $X_n \equiv \min A_n$. From the set $A_n \backslash \{X_n\}$, each number in turn (and each independently of the others) is deleted with probability X_n^{-1}. The set of elements of $A_n \backslash \{X_n\}$ which remain is denoted by A_{n+1}.

We regard X_n as the nth 'random prime' and introduce the 'Mertens' product

$$Y_n \equiv \prod_{1 \le k \le n} (1 - X_k^{-1})^{-1}.$$

Wunderlich ([10]) proved that, almost surely,

$$X_n \sim n \log n, \quad Y_n \sim \log n \quad (n \to \infty). \tag{1.1}$$

Neudecker and Williams ([5]) proved that, almost surely, the (*random*) limit

$$L \equiv \lim X_n \exp(-Y_n) \tag{1.2}$$

exists in $(0, \infty)$, showed that

$$E_n \equiv L \operatorname{li}(X_n/L) - n \tag{1.3}$$

should be regarded as the 'error in the prime-number theorem', and established that, almost surely, the 'Riemann hypothesis':

$$E_n = O(n^{1/2+\epsilon}) \tag{1.4}$$

holds.

Convention. The constant repetition of 'almost surely' phrases would become tiresome. We shall therefore suppress them and regard exceptional null sets as deleted from the sample space. Readers concerned with 'rigour' can do a preliminary check at this stage to see that the number of results on the Hawkins sieve in this paper is at most countable.

We now state our main result in direct intuitive form, mention some corollaries, and then clarify what is meant by 'expanding the sample space'.

THEOREM 1: *The sample space on which the Hawkins sieve is defined may be expanded so as to carry a Brownian motion* $B = \{B_t : t \geq 0\}$ *such that*

$$E_n = B_n + O([n \log \log n]^{1/2}/\log n). \tag{1.5}$$

It really is hard to think what more needs to be said, except for discussion (see §1.3) of behaviour conditional on L. Theorem 1 is of course the appropriate 'Strassen invariance principle' for the Hawkins sieve. The $O(\cdot)$ term in (1.5) is the 'natural' one, but we have to work quite hard to obtain the very crucial $(\log n)$ in the denominator. The main point of Theorem 1 is of course that it allows us to transfer to the E process nearly all of the known big theorems for Brownian motion – and especially Strassen's momentous improvement of the classical iterated-logarithm law.

Set $E_0 \equiv 0$ and extend E_t from $\{0, 1, 2, \ldots\}$ to $[0, \infty]$ by linear interpolation. Introduce the Banach space $C[0, 1]$ with its usual supremum norm. For each $n \geq 3$, the formula

$$f_n(t) \equiv (2n \log\log n)^{-1/2} E_{nt} \quad (0 \leq t \leq 1)$$

defines an element f_n of $C[0, 1]$. Because of Theorem 1, Strassen's law for B now implies that (almost surely) the set of accumulation points in $C[0, 1]$ of the sequence $(f_n\colon n \geq 3)$ is *exactly* the compact subset K of $C[0, 1]$ consisting of functions g in $C[0, 1]$ such that

(i) $g(0) = 0$,

(ii) g is absolutely continuous (with derivative g' existing almost everywhere),

(iii) $\int_{[0,1]} g'(t)^2 dt \leq 1$.

Freedman's book [1] has a nice account of Strassen's law. For reference, let us record the law via the shorthand:

$$\partial(f_n) = K. \tag{1.6}$$

Equation (1.6) gives us very precise information on the fluctuations of the sequence (E_n). In particular, it allows us to improve the result (1.4) to the form:

$$\limsup([2n \log\log n]^{-1/2} E_n) = +1, \liminf(\ldots) = -1. \tag{1.7}$$

Of course, (1.7) follows from Theorem 1 and the *classical* iterated logarithm law, but it is best seen as following from (1.6) because of the easily verified fact that

$$\sup\{g(1) : g \in K\} = 1.$$

CLARIFICATION OF THEOREM 1: Results like (1.6) and (1.7) are of course true without 'expanding the sample space'. As far as those results are concerned, Theorem 1 is only a means to an end. However Theorem 1 is the story of the Hawkins sieve in a nutshell and needs to be properly explained. The idea of expanding the sample space is crucial in all Skorokhod /.../ Strassen invariance-principle theory, and seems to be an essential means for getting to (1.6).

The sequence $(X_n\colon n \in \mathbb{N})$ of Hawkins primes is defined on some probability triple $(\Omega, \mathcal{F}, P)$. Theorem 1 states that there exists a new probability triple $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ supporting both a Brownian motion $\{\hat{B}_t\colon t \geq 0\}$ and a sequence $(\hat{X}_n\colon n \in \mathbb{N})$ of random variables such that

(i) the $\hat{P}$ law of the sequence $(\hat{X}_n\colon n \in \mathbb{N})$ is identical to the P law of the sequence $(X_n\colon n \in \mathbb{N})$;

(ii) $\hat{E}_n = \hat{B}_n + O([n \log\log n]^{1/2}/\log n)$.

Here $\hat{E}_n$ is the 'error in the prime-number theorem for $\hat{X}_n$' defined via the 'hatted' versions of Equations (1.2), (1.3).

High-falutin' ('pull-back') considerations allow us to reinterpret this formulation in the more intuitive language of Theorem 1, but you can always switch to the 'hatted' formulation if you prefer.

1.2. It was noted in [9] that the stochastic structure of the Hawkins sieve may be conveniently described as follows. Write

$$\mathcal{F}_n \equiv \sigma\{X_m, Y_m): m \leq n\}$$

for the smallest σ-algebra with respect to which all variables X_m, Y_m with $m \leq n$ are measurable. Then it is easily verified that the process $\{(X_n, Y_n): n \in \mathbb{N}\}$ is *Markovian* with

$$X_1 = Y_1 = 2, \tag{1.8.i}$$

$$P[X_{n+1} - X_n = j \mid \mathcal{F}_n] = Y_n^{-1}(1 - Y_n^{-1})^{j-1} \quad (j \in \mathbb{N}), \tag{1.8.ii}$$

$$Y_{n+1} = Y_n(1 - X_{n+1}^{-1})^{-1}. \tag{1.8.iii}$$

From elementary properties of the geometric distribution, we now read off:

$$E[X_{n+1} - X_n \mid \mathcal{F}_n] = Y_n, \tag{1.9.i}$$

$$\mathrm{Var}[X_{n+1} - X_n \mid \mathcal{F}_n] = Y_n^2 - Y_n. \tag{1.9.ii}$$

Set

$$U_{n+1} \equiv Y_n^{-1}(X_{n+1} - X_n), \tag{1.10.i}$$

$$S_n \equiv \sum_{k=2}^{n} (U_k - 1). \tag{1.10.ii}$$

Because of (1.9.i), the process $\{S_n: n \in \mathbb{N}\}$ is a martingale. *We derive all of our results from the 'Strassen' properties of this martingale.* We shall be able to check very easily that Theorem 1.3 of Strassen [7] implies that the sample space of the Hawkins sieve may be expanded so as to carry a Brownian motion $\{B_t: t \geq 0\}$ such that

$$S_n = B_n + o(n^{1/4}[\log n]^{3/2}). \tag{1.11}$$

Theorem 1 then follows from the following result.

THEOREM 2:

$$E_n = S_n + O([n \log \log n]^{1/2}/\log n). \tag{1.12}$$

1.3. The complete story of a piece of mathematics can never be told.

The fact that L is *random* and involves the entire history of the process X makes Theorem 1 rather unsatisfactory. An exactly similar situation occurs in branching-process theory in connection with the 'lag' L of the Galton–Watson process. The very nice solution there (see Kendall [2]) is to condition the process by the value of L. It would be plain silly to attempt this for the Hawkins sieve itself. The thing to do is to switch attention to the diffusion sieve of [9]. By now, we are fully entitled to expect even delicate results for the diffusion sieve to transfer to the Hawkins sieve. But perhaps we should reverse our thinking! Kronecker himself would be prepared on this occasion to regard the integer Hawkins sieve as a rather clumsy approximation to the diffusion sieve. Equation (1.11) shows how hard the Hawkins sieve strives for perfection.

The reader who wishes to proceed directly to the proofs of Theorems 1 and 2 can do so at this stage. The remainder of this section takes (say) Chapters 1–3 of McKean [4] for granted.

From (1.8) and (1.10), we obtain

$$X_{n+1} - X_n = Y_n(S_{n+1} - S_n) + Y_n, \tag{1.13.i}$$

$$Y_{n+1} - Y_n = Y_n/(X_{n+1} - 1). \tag{1.13.ii}$$

These equations suggest that (as in [9]) we take for the 'diffusion sieve' a solution

$$\boldsymbol{D} = \{(X_t, Y_t): 0 \le t < \zeta\}$$

of the stochastic differential equations:

$$(X_0, Y_0) = (x, y), \tag{1.14.i}$$

$$dX = YdB + Ydt, \tag{1.14.ii}$$

$$dY = (Y/X)dt, \tag{1.14.iii}$$

where (x, y) is a fixed point of $(0, \infty)^2$, B *is a given Brownian motion*, and ζ denotes the escape-time from $(0, \infty)^2$.

Let us now quote some of the main results of [9] and supplement them with the analogue of Theorem 2 (the proof of which the reader can easily supply once he has seen the proof of Theorem 2 itself). The limit

$$L \equiv \lim_{t \uparrow \zeta} X_t \exp(-Y_t)$$

exists in $[0, \infty)$, and one of the following two alternatives (I) and (II) occurs.

(I) $L = 0$, $\zeta < \infty$, $\lim_{t \uparrow \zeta} X_t = 0$, $\lim_{t \uparrow \zeta} Y_t < \infty$.

(II) $L > 0$, $\zeta = \infty$ and, if $E_t \equiv L \operatorname{li}(X_t/L) - t$, then

$$E_t = B_t + O([t \log \log t]^{1/2}/\log t). \tag{1.15}$$

(Note that the Brownian motion B in (1.15) is the same Brownian motion as is given in (1.14). Thus (1.15) is the analogue of Theorem 2, the distinction between S and B having evaporated).

Further

$$P[L = 0] = \exp(-\mu),$$

and on $(0, \infty)$, the distribution of L has the density

$$h_L(x, y) \equiv \exp(-\mu - \alpha L)(\mu \alpha L^{-1})^{1/2} I_1(2[\alpha \mu L]^{1/2}) \tag{1.16}$$

where we have used the shorthand

$$\mu \equiv 2x(1 + y)^{-1}, \quad \alpha \equiv 2e^y(1 + y)^{-1},$$

and where I_1 is the usual Bessel function of imaginary argument. Equation (1.16) follows on inverting the Laplace transform in Equation (1.6) of [9].

From the point of view of diffusion theory, Equation (1.15) looks rather strange because E_t is *not* non-anticipating and therefore can not be 'differentiated stochastically'. This difficulty is removed by the operation of 'conditioning on L'.

It is easily checked (recall Doob's $h^{-1}\mathscr{D}h$ formula) that the law of the process $\boldsymbol{D}$ conditioned by the value (assumed strictly positive) of L is described by the new stochastic differential equations:

$$dX = Y d\tilde{B} + Y dt + Y^2\left[\frac{\partial}{\partial x} \log h_L(X, Y)\right] dt,$$

$$dY = (Y/X) dt;$$

here $\tilde{B}$ denotes 'some' Brownian motion.

We content ourselves now with pointing out that the new 'conditioning' term involving h_L is awkward not only in appearance but also in substance. We find that for the conditioned diffusion,

$$dE = \frac{Y}{\log X - \log L}\,d\tilde{B} + \frac{Y - \log X + \log L}{\log X - \log L}\,dt$$
$$+ (\text{Itô term})dt + (\text{conditioning } h_L \text{ term})dt.$$

As $t \to \infty$,

$$\frac{Y}{\log X - \log L} \to 1, \quad \frac{Y - \log X + \log L}{\log X - \log L} = o(t^{-1/2})$$

and this represents good behaviour from the point of view of random-evolution theory.(See, for example, Pinsky [6], Stroock [8].) The usual 'second-order' Itô correction term characteristic of stochastic differential calculus proves of order $o(t^{-1/2})$ and is therefore harmless. However, one can check from the well-known fact:

$$\frac{d}{du}\log I_1(u) = 1 + O(u^{-1}) \quad (u \uparrow \infty)$$

that the conditioning h_L term is definitely *not* $o(t^{-1/2})$. Hence, if the conditioning proves harmless in the sense that E_t remains 'approximately Brownian', it is because of cancellation effects. We leave this point aside for now.

The above discussion should have made the reader aware of the fact that *stochastic calculus seems incapable of providing a neat proof of* (1.15). The reader should therefore take the trouble to prove (1.15) for himself, by transferring our proof of Theorem 2 or otherwise.

2. Proof of Theorems 1 and 2

2.1. We need to (re-)establish first the estimates:

(2.1.i) $$X_n \sim n \log n, \quad Y_n \sim \log n,$$

(2.1.ii) $$X_{n+1} - X_n = O([\log n]^2), \quad Y_{n+1} - Y_n \sim n^{-1},$$

and for these, we simply paraphrase the arguments of [5]. As remarked earlier, Wunderlich was the first to prove (2.1.i), but his method is unnecessarily complicated. The results (2.1.i) follow immediately from the following two propositions and (1.13.ii).

PROPOSITION 1: *$Y_n \uparrow \infty$ and $X_n \sim nY_n$.*

PROPOSITION 2: *Set*

$$K_n \equiv \log X_n - Y_n.$$

Then the (random) limit

$$C \equiv \lim K_n$$

exists.

For $x > 0$ and $n \in \mathbb{N}$,

$$P[U_{n+1} > x \mid \mathcal{F}_n] \le (1 - Y_n^{-1})^{-1}(1 - Y_n^{-1})^{xY_n} \le 2e^{-x}. \tag{2.2}$$

(Recall that $Y_n \ge 2$, $\forall n$.) By the (elementary!) Borel–Cantelli Lemma,

$$U_n = O(\log n). \tag{2.3}$$

It is now clear that once (2.1.i) is established, (2.1.ii) will follow. (See (1.10.i), (1.13.ii).) Because the martingale-differences $\{(U_{n+1} - 1): n \in \mathbb{N}\}$ are *orthogonal* and

$$E[(U_{n+1} - 1)^2 \mid \mathcal{F}_n] = 1 - Y_n^{-1} \le 1, \tag{2.4}$$

Theorem 33B(ii) of Loéve [3] implies that

$$\sum_{k \le n} (U_{k+1} - 1) = O(n^{1/2+\epsilon}), \quad \forall \epsilon > 0. \tag{2.5}$$

PROOF OF PROPOSITION 1, ([5]): If $Y_n \uparrow Y < \infty$, then we could conclude from (1.8.iii) that $\Sigma X_n^{-1} < \infty$ and from (1.10.i) and (2.5) that (in contradiction) $n^{-1}X_n \to Y$.

Thus $Y_n \uparrow \infty$ and

$$(X_{n+1} - 1)Y_{n+1}^{-1} - (X_n - 1)Y_n^{-1} = U_{n+1} - Y_{n+1}^{-1} = U_{n+1} + o(1),$$

so that, from (2.5), $X_n \sim nY_n$.

PROOF OF PROPOSITION 2, ([5]): Some elementary algebra shows that

$$\begin{aligned}K_{n+1}-K_n &= \log(1+Y_nU_{n+1}X_n^{-1})-Y_n(X_n+Y_nU_{n+1}-1)^{-1}\\ &= Y_nX_n^{-1}(U_{n+1}-1)+O(Y_nU_{n+1}X_n^{-1})^2.\end{aligned}$$

Thus, from (2.3) and Proposition 1,

$$K_{n+1}=G_{n+1}+H+O(n^{-1}[\log n]^2) \tag{2.6}$$

where G_{n+1} is the 'stochastic integral'

$$G_{n+1}\equiv\sum_{r\le n}Y_rX_r^{-1}(U_{r+1}-1) \tag{2.7}$$

and H is some random variable. (We shall often use the fact that a 'sum to n terms' is the 'sum to infinity' minus the 'tail'.) An easy exercise in partial summation, utilising Proposition 1, (2.3) and (2.5), now shows that

$$G\equiv\lim G_n \tag{2.8}$$

exists. Then C exists and $C=G+H$.

2.2. PROOF OF (1.11): Let V be the variance process of the martingale S; that is, put

$$V_n\equiv\sum_{k=2}^{n}E[(U_k-1)^2\mid\mathcal{F}_{k-1}].$$

Then, by (2.4),

$$V_{n+1}=n-\sum_{k=1}^{n}Y_k^{-1}\sim n. \tag{2.9}$$

Use the notation

$$E[\xi;\Lambda\mid\mathcal{F}_{n-1}]\equiv E[\xi I_\Lambda\mid\mathcal{F}_{n-1}]$$

where ξ is a random variable and Λ is a subset of Ω with indicator function I_Λ. Theorem 1.3 of Strassen [7] states that if $f(u)\uparrow$ on (b,∞),

$u^{-1}f(u) \downarrow$ on (b, ∞) (for some b) and

$$(2.10) \qquad \sum f(V_n)^{-1} E[(U_n - 1)^2; (U_n - 1)^2 > f(V_n) \mid \mathcal{F}_{n-1}] < \infty,$$

then the sample space may be extended to support a Brownian motion B such that

$$S_n = B_n + o([nf(n)]^{1/4} \log n).$$

Because of the Schwarz inequality, the trivial estimate

$$E[(U_n - 1)^4 \mid \mathcal{F}_{n-1}] = O(1),$$

and estimate (2.2), we see that (2.10) is guaranteed provided that

$$(2.11) \qquad \sum f(V_n)^{-1} \exp[-\tfrac{1}{2} f(V_n)^{1/2}] < \infty.$$

Because of (2.9), (2.11) will hold if $f(u) = 5(\log u)^2$.

Property (1.11) therefore holds. Note the important consequence that

$$(2.12) \qquad S_n = O([n \log \log n]^{1/2}).$$

2.3. All that remains is to prove Theorem 2. The next step is to prove a result which improves both Proposition 2 of this paper and Proposition 2 of Neudecker–Williams [5] in which the error term was $O(n^{-1/2+\epsilon})$.

PROPOSITION 3:

$$(2.13) \qquad K_n \equiv \log X_n - Y_n = C + O(n^{-1/2}[\log \log n]^{1/2}).$$

NOTE: This result is the exact analogue of the diffusion result (2.8) in [9] which is best possible.

PROOF: It is 'clear' that Proposition 3 must follow on applying partial summation to (2.7) using the improvement (2.12) of (2.5). However, the 'obvious' attempt *fails* because the estimate

$$Y_r X_r^{-1} - Y_{r+1} X_{r+1}^{-1} = O(r^{-2} \log r)$$

is best possible *and is 'too big' by an all-important factor* $\log r$. The source of the trouble is that though $X_n \sim n \log n$, the best bound for $X_{n+1} - X_n$ is $O(\log n)^2$, while the derivative of $x \log x$ is $O(\log x)$. What we must do therefore is to use partial summation *twice.*

Starting with (2.7) and (2.8), we have (the reader should check that the upper limit ∞ causes no trouble)

$$G - G_n = \sum_{r\geq n} Y_r X_r^{-1}(S_{r+1} - S_r)$$

$$= -Y_n X_n^{-1} S_n + \sum_{r\geq n} S_{r+1}(Y_r X_r^{-1} - Y_{r+1} X_{r+1}^{-1})$$

$$= -Y_n X_n^{-1} S_n - \sum_{r\geq n} H_r + \sum_{r\geq n} H_r(X_{r+1} - X_r),$$

where

$$H_r \equiv Y_r S_{r+1}[X_r(X_{r+1} - 1)]^{-1}.$$

Introduce the shorthand

$$g_n \equiv n^{-1/2}[\log \log n]^{1/2}, \quad h_n \equiv n^{-3/2}[\log \log n]^{1/2}/\log n.$$

From (2.1) and (2.12),

$$Y_n X_{n+1}^{-1} S_n = O(g_n), \quad H_r = O(h_r)$$

so that

$$\sum_{r\geq n} H_r = O(g_n/\log n) = O(g_n).$$

Because $X_{r+1} - X_r$ *is always positive* and $H_r = O(h_r)$,

$$\sum_{r\geq n} H_r(X_{r+1} - X_r) = O\Big(\sum_{r\geq n} h_r(X_{r+1} - X_r)\Big).$$

Finally,

$$\sum_{r\geq n} h_r(X_{r+1} - X_r) = -h_n X_n + \sum_{r\geq n} X_{r+1}(h_r - h_{r+1})$$

$$= O(g_n) + O\Big\{\sum_{r\geq n} (r \log r) r^{-5/2}[\log \log r]^{1/2}/\log r\Big)$$

$$= O(g_n) + O(g_n),$$

as required.

2.4. We now complete the proof of Theorem 2. That Theorem 2 implies Theorem 1 is already proved. Of course,

$$L \equiv \exp(C).$$

By Taylor's Theorem, we have, for $x > 1$ and $h > 0$,

$$|\mathrm{li}(x+h) - \mathrm{li}(x) - h(\log x)^{-1}| \leq \tfrac{1}{2}h^2 x^{-1}(\log x)^{-2}.$$

Hence

$$L\,\mathrm{li}(X_{n+1}/L) - O(1) = \sum_{r=1}^{n} [L\,\mathrm{li}(X_{r+1}/L) - L\,\mathrm{li}(X_r/L)]$$

$$= \sum_{1}^{n} [\log X_r - C]^{-1}(X_{r+1} - X_r) + O(\log n)^2$$

$$= \sum_{2}^{n} \{Y_r^{-1} + O(r^{-1/2}[\log\log r]^{1/2}/[\log r]^2)\}\,(X_{r+1} - X_r) + O(\log n)^2,$$

from Proposition 3. By using the positivity of $X_{r+1} - X_r$ in exactly the same way as in the proof of Proposition 3, we obtain

$$\sum_{2}^{n} O(r^{-1/2}[\log\log r]^{1/2}/[\log r]^2)(X_{r+1} - X_r)$$
$$= O(n^{1/2}[\log\log n]^{1/2}/\log n).$$

Since

$$\sum_{1}^{n} Y_r^{-1}(X_{r+1} - X_r) = \sum_{1}^{n} U_{r+1} = S_{n+1} + n,$$

the proof of Theorem 2 is finished.

REFERENCES

[1] D. Freedman: *Brownian Motion and Diffusion* (Holden-Day, San Francisco, 1971).

[2] D.G. Kendall: Branching processes since 1873, *J. London Math. Soc. 41* (1966) 385–406.

[3] M. Loéve: *Probability Theory* (van Nostrand, Princeton, N.J., 1963).

[4] H.P. McKean: *Stochastic Integrals* (Academic Press, New York–London, 1969).

[5] W. Neudecker and D. Williams: The 'Riemann hypothesis' for the Hawkins random sieve, *Compositio Math. 29* (1974) 197–200.

[6] M. Pinsky: Differential equations with a small parameter and the central limit theorem for functions defined on a Markov chain, *Z. Wahrscheinlichkeitstheorie 9* (1968) 101–111.

[7] V. STRASSEN: Almost sure behavior of sums of independent random variables and martingales, *Proc. 5th Berkeley Symp.*, Vol. 2, part 1 (1966) 315–343.
[8] D.W. STROOCK: Two limit theorems for random evolutions having non-ergodic driving processes, (to appear in proceedings of Park City, Utah conference on stochastic differential equations).
[9] D. WILLIAMS: A study of a diffusion process motivated by the sieve of Eratosthenes, *Bull. London Math. Soc.* *6* (1974) 155–164.
[10] M.C. WUNDERLICH: The prime number theorem for random sequences, *J. Number Theory* *8* (1976) 369–371.

(Oblatum 24-I-1977)

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Added in proof

We are very pleased to report that this paper has helped motivate a much deeper one by H. Kesten entitled "The speed of convergence of a martingale" (to appear).