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DOUBLE POINT RESOLUTIONS OF DEFORMATIONS
OF RATIONAL SINGULARITIES

Joseph Lipman

Let $Y_0$ be a normal algebraic surface over an algebraically closed field $K$, and let $f_0 : Z_0 \to Y_0$ be the minimal resolution of singularities. There is a unique way to factor $f_0$ as $Z_0 \stackrel{g_0}{\to} X_0 \to Y_0$ with a proper birational $g_0$ and a normal surface $X_0$, such that a reduced irreducible curve $C$ on $Z_0$ which blows down to a point on $Y_0$ already blows down on $X_0$ iff $C$ is non-singular and rational with self-intersection $C^2 = -2$ (cf. [1, p. 493, (2.7)]; or, for greater generality, [7, p. 275, (27.1)]). The singularities of $X_0$ are all Rational Double Points. $X_0$, which is uniquely determined by $Y_0$, will be called the RDP-resolution of $Y_0$.

Suppose now that $Y_0$ is affine and has just one singularity $y$, $y$ being rational. There is a conjecture of Wahl [9], and Burns-Rapoport [2, 7.4] to the effect that the Artin component in the (formal or henselian) versal deformation space of $y$ is obtained from the deformation space of $Z_0$ by factoring out a certain Coxeter group. According to Wahl [10], this would imply that the Artin component is smooth, and is in fact formally identical with the deformation space of $X_0$ (at least after things are suitably localised). Wahl also shows how the conjecture reduces to the statement that under “blowing down”, the set of first order infinitesimal deformations of $X_0$ maps injectively into that of $Y_0$. Our purpose here is to outline a proof that this injectivity (and so the conjecture) does indeed hold.

Wahl has previously established some special cases of the conjecture by showing that a certain cohomology group vanishes [10]. I am grateful to him for all the information and motivation he has provided.

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Let $g : Y \to S$ be a flat map of finite type of noetherian schemes, such that for each $s \in S$ the fibre $Y_s$ is a normal surface having only rational singularities. We assume - for simplicity only - that every closed point of $Y$ maps to a closed point of $S$ whose residue field is algebraically closed. By an RDP-resolution of $Y \to S$ is meant a proper map $f : X \to Y$ such that $g \circ f : X \to S$ is flat, and for each $s$ the map $f_s : X_s \to Y_s$ is the RDP-resolution of $Y_s$ (see above).

**Theorem:** Given $Y \to S$ as above, there exists at most one (up to $Y$-isomorphism) RDP-resolution.

(For $S = \text{Spec}(k[t]/t^2)$ we get the above-indicated injectivity statement.)

In fact we show the following:

Let $U \subseteq Y$ be the open set where $g$ is smooth and let $i : U \to Y$ be the inclusion map. Set

$$\omega = i_* \Omega^2_{U/S} \quad (\Omega^2 = \text{relative 2-differentials})$$

and for all $n \geq 0$, let $\omega^n$ be the image of the natural map $\omega^\otimes n \to i_*((\Omega^1_{U/S})^\otimes n)$. Then, if an RDP-resolution $f : X \to Y$ exists, we must have

$$X = \text{Proj}(\bigoplus_{n \geq 0} \omega^n)$$

(with $f$ the canonical map). In other words $X$ is the scheme-theoretic closure of $U$ in the projective bundle $P(\omega)$ [4, II, (4.1.1)].

**Example** (Suggested by Riemenschneider). Let $Y_0$ be the cone over a non-singular rational curve of degree 4 in $P^4$ (say over the complex numbers $C$). The versal deformation of the vertex has a one-dimensional non-Artin component, found by Pinkham: the corresponding deformation is $Y \to S = \text{Spec}(C[t])$, where $Y \subseteq C^4 \times S$ is the zero-set of the $2 \times 2$ minors of

$$\begin{pmatrix} x_1 & x_2 & x_3 + t \\ x_2 & x_3 & x_4 \\ x_3 + t & x_4 & x_5 \end{pmatrix}$$

Here, of course, no RDP-resolution can exist. One computes that $\omega^n$ is isomorphic to $J^n$, where $J$ is the fractionary ideal generated by
(1/x_2, 1/(x_3 + t), 1/x_4); and \text{Proj}(\bigoplus \omega^*) (\equiv \text{blow-up of } J) \text{ is non-singular, but not an RDP-resolution of } Y. (\text{The fibre over } t = 0 \text{ has two components!})

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After outlining the proof of some preliminary facts from duality theory (Lemma 1), we prove the Theorem in Lemmas 2 and 3 below.

Let \( f : X \to Y \) be an RDP-resolution of \( g : Y \to S \).

**Lemma 1:** (cf. [5, p. 298] or 5', Prop. 9) (i) Since \( g : Y \to S \) is flat, with Cohen-Macaulay fibres, the relative dualizing complex \( g^! O_S \) has just one non-zero cohomology sheaf \( \omega_{Y/S} \), and \( \omega_{Y/S} \) is flat over \( S \). Since \( g \circ f \) is flat, with Gorenstein fibres, the similarly defined \( O_X \)-module \( \omega_{X/S} \) is invertible.

(ii) For any map \( S' \to S \), with \( S' \) noetherian, if \( Y' = Y \times_S S' \) and \( \pi : Y' \to Y \) is the projection, then \( \omega_{Y'/S} = \pi^*(\omega_{Y/S}) \).

**Proof:** We may assume \( S = \text{Spec}(A) \), \( Y = \text{Spec}(B) \), with \( B \) a homomorphic image of a polynomial ring \( R = A[\xi_1, \ldots, \xi_n] \) (cf. [5, p. 383, p. 388]). Then the first part of (i) states that

(i*): \( \text{Ext}_R^i(B, R) = 0 \) for \( i \neq n - 2 \), and \( \text{Ext}_R^{n-2}(B, R) \) is \( A \)-flat;

and (ii) says that

(ii*): for any noetherian \( A \)-algebra \( A' \), if \( R' = R \otimes_A A' \) and \( B' = B \otimes_A A' \), then the natural map

\[
\text{Ext}_R^{n-2}(B, R) \otimes_A A' \to \text{Ext}_R^{n-2}(B', R')
\]

is an isomorphism.

(i*) and (ii*), together with Nakayama's lemma, reduce the proof of the last assertion in (i) to the well-known case where \( S \) is the spectrum of a field [5, p. 296, Prop. 9.3].

We first show that \( B \) has homological dimension \( n - 2 \) over \( R \). For this we may replace \( R \) by its localization at an arbitrary maximal ideal \( \mathfrak{m} \), and \( A \) by its localization at \( \mathfrak{m} \cap A \) [8, p. 188, Thm. 11]. Let \( \mathfrak{m} \) be the maximal ideal of \( A \), let \( \tilde{R} = R \otimes_A (A/\mathfrak{m}) \) and \( \tilde{B} = B \otimes_A (A/\mathfrak{m}) \), so that \( \tilde{B} \) is a normal two-dimensional homomorphic image of the regular \( n \)-dimensional local ring \( \tilde{R} \), and so \( \tilde{B} \) has homological dimen-
sion $n - 2$ over $\bar{R}$. (What matters here and subsequently is that $\bar{B}$ is Cohen–Macaulay.) Let $K$ be the residue field of $R$. Since $R$ and $B$ are $A$-flat, we have for any $R$-projective (hence $A$-flat) resolution $P.$ of $B$ that the homology $H_j(\bar{P} \otimes_A (A/m)) = \text{Tor}_j^R(\bar{B}, A/m) = 0$ for $j > 0$, i.e. $\bar{P} \otimes_A (A/m)$ is an $\bar{R}$-projective resolution of $\bar{B}$, whence, for all $i$,

$$\text{Tor}_i^R(\bar{B}, K) = \text{Tor}_i^R(\bar{B}, K);$$

and our assertion follows from [8, p. 193, Thm. 14].

Thus $B$ has a finitely generated $R$-projective (hence $A$-flat) resolution

$$P.: 0 \to P_{n-2} \to P_{n-3} \to \cdots \to P_0 \to 0.$$

Let $Q.$ be the complex with

$$Q_i = \text{Hom}_R(P_{n-2-i}, R) \quad (i \in \mathbb{Z}).$$

For any $A$-algebra $A'$ and any $i$ we have

$$H_{n-2-i}(Q \otimes_A A') = \text{Ext}_i^R(\bar{B}, R') = \text{Ext}_i^R(B', R').$$

(For the second equality cf. the proof of (#) above.)

Now (i*) results from the following Lemma. (We assume again, as we may, that $A$ and $R$ are local, and let $m$ be the maximal ideal of $A$.)

**Lemma 1a:** Let $Q.$ be an $A$-flat complex of finitely-generated $R$-modules, with $Q_i = 0$ for $i < 0$, and such that the homology $H_j(Q \otimes_A (A/m)) = 0$ for all $j > 0$. Then $H_0(Q.)$ is $A$-flat, and $H_j(Q.) = 0$ for all $j > 0$.

The proof is left to the reader.

Finally, applying [4, III', (7.3.1)(c) and (7.3.7)] to the homological functor $T.$ of $A$-modules $M$ given by

$$T_p(M) = H_p(Q \otimes_A M) = \text{Ext}_R^{n-2-p}(B, R \otimes_A M) \quad (p \in \mathbb{Z})$$

(Q. as above) we see that for every $A$-module $M$ and every $i$, there is a natural isomorphism

$$\text{Ext}_i^R(B, R) \otimes_A M \cong \text{Ext}_R^i(B, R \otimes_A M).$$

In view of (##), taking $M = A'$ we get (ii*). Q.E.D.
LEMMA 2: Let $\mathcal{L}$ be the invertible $\mathcal{O}_X$-module $\omega_{X/S}$. Then:

(i) $\mathcal{L}$ is very ample for $f$.

(ii) For every $n > 0$, the canonical map $(f_*\mathcal{L})^n \to f_*(\mathcal{L}^n)$ is surjective.

PROOF: We first show that $R^1f_*(\mathcal{O}_X) = 0$: for any closed point $y$ of $Y$, let $\mathfrak{m}_y$ be the maximal ideal of $\mathcal{O}_{Y,y}$; then for all $r \geq 0$, $\mathfrak{m}_y^r\mathcal{O}_X/\mathfrak{m}_y^{r+1}\mathcal{O}_X$ is an $\mathcal{O}_{f^{-1}(y)}$-module generated by its global sections; since $Y_s$ (as $g(y)$) has rational singularities, therefore $H^1(\mathcal{O}_{f^{-1}(y)}) = 0$, and since $f^{-1}(y)$ has dimension $\leq 1$, we get $H^1(\mathfrak{m}_y^r\mathcal{O}_X/\mathfrak{m}_y^{r+1}) = 0$; conclude with [4, III, (4.2.1)].

Now by [7, p. 220, (12.1) and p. 211, proof of (7.4)] it will be enough to show for any reduced irreducible curve $C \subseteq f^{-1}(y)$ that $(\mathcal{L}, C)$ (the degree of $0$ pulled back to $C$) is $> 0$. Lemma 1 (ii) reduces us to the case $S = \text{Spec}(k)$, $k$ an algebraically closed field. Let $p: Z \to X$ be a minimal resolution of singularities, and let $\tilde{C}$ be the component of $p^{-1}(C)$ which maps onto $C$. Then $(\mathcal{L}, C) = (p^{-1}\mathcal{L}, \tilde{C})$. But $p^{-1}\mathcal{L}$ is a dualizing sheaf on $Z$ [1, p. 493, (2.7)]; since (by definition of RDP-resolution) $\tilde{C}$ is not a nonsingular rational curve with $C^2 \geq -2$, therefore $(p^{-1}\mathcal{L}, \tilde{C}) > 0$. Q.E.D.

Lemma 2(i) implies that $X = \text{Proj}(\bigoplus_{n=0} f_*(\mathcal{L}^n))$, cf. [4, II, (4.6.2), (4.6.3), (5.4.4)] or [4, III, (2.3.4.1)]; so we need only show that $f_*(\mathcal{L}^n) = \omega^n (n > 0)$. This is given by (ii) of Lemma 2 and by:

LEMMA 3: Let $\mathcal{L} = \omega_{X/S}$. Then for all $n > 0$ there is a natural injective map

$$\theta_n: f_*(\mathcal{L}^n) \to i^*((\Omega^2_{U/S})^n);$$

and $\theta_1$ is even bijective (i.e. $f_*\mathcal{L} = \omega$).

PROOF: Let $U \subseteq Y$ be as before, and let $j:f^{-1}(U) \to X$ be the inclusion map. $f$ induces a proper map $f^{-1}(U) \to U$ which is fibrewise (over $S$) an isomorphism; hence $f^{-1}(U) \to U$ is an isomorphism. Now

$$\Omega^1_{U/S} = \omega_{U/S} = j^*\mathcal{L},$$

and $\theta_n$ is obtained by applying $f_*$ to the natural map $\mathcal{L}^n \to j_*j^*\mathcal{L}^n$. For the injectivity, it suffices that the $(X - f^{-1}(U))$ depth of $\mathcal{L}$ be $\geq 1$ [3, 1.9 and 3.8]. [4, IV, (11.3.8)] and (ii) of Lemma 1 above enable us to verify this fibrewise (over $S$); in other words, we need only check
the simple case where $S = \text{Spec}(k)$, $k$ a field.

Similarly we see that the $(Y - U)$-depth of $\omega_{Y|S}$ is $\geq 2$, and conclude that

$$\omega_{Y|S} = i_* i^* \omega_{Y|S} = \omega.$$ 

For the surjectivity of $\theta_1$, it suffices (by Nakayama's Lemma) that $f_* \mathcal{L} \to \omega_s = \omega_{Y|S} \otimes \mathcal{O}_{Y_s}$ be surjective for all $s \in S$. Since $f^* f_* \mathcal{L} \to \mathcal{L}$ is surjective (Lemma 2), so is $f^* f_* \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is the kernel of $\mathcal{L} \to \mathcal{L}_s = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s}$; since $R^1 f_* \mathcal{O}_X = 0$ and $f$ has fibres of dimension $\leq 1$, therefore $R^1 f_* \mathcal{H} = 0$, and $f_* (\mathcal{L}) \to f_* (\mathcal{L}_s)$ is surjective. But $\mathcal{L}_s$ (resp. $\omega_s$) is a dualizing sheaf on $X_s$ (resp. $Y_s$), and since $Y_s$ has rational singularities, therefore $f_*(\mathcal{L}_s) = \omega_s$, ([6, p. 606, 3.5] or, for greater generality, [7', §2]). Q.E.D.

**Remark:** For $S = \text{Spec}(k[t]/t^2)$, Wahl has a proof of Lemma 3 which avoids duality theory [10, §2].

**References**


