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APPLICATIONS OF UNSTABLE NORMAL INVARIANTS – PART I

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Let $h: M^n \to N^n$ be a homotopy equivalence between two oriented, closed, smooth manifolds such that $h^*\tau_N \simeq \tau_M$. Since most differential invariants of manifolds (e.g., characteristic classes) are defined in terms of the tangent bundle, one might expect that it would be very difficult to find invariants which would detect whether h is homotopic to a diffeomorphism. In the early 60's, however, Novikov [14] defined a new, simple invariant called the *normal invariant* which has been used by Browder [1, 3], Novikov [14], Sullivan [15], and Wall [18] to solve this problem (see Chapter 10 of [18]). Let $F: M^n \to S^{n+k} (k > n+1)$ be an embedding with normal bundle γ^k. The normal invariant $c_M \in \pi_{n+k}(T(\gamma^k))$ is then represented by the collapse map $S^{n+k} \to D(\gamma)/S(\gamma) = T(\gamma)$. Since $\pi_{n+k}(T(\gamma^k))$ is a stable homotopy group, we shall call c_M the *stable* normal invariant of M. In Chapters 10 and 11 of [18] the classification of manifolds up to diffeomorphism and the classification of smooth embedding up to concordance are reduced to algebra, i.e., Wall surgery obstruction groups and homotopy theory, i.e., Poincaré spaces and Poincaré embeddings. The goal of this paper is to show how *unstable normal invariants* can be used in the classification of Poincaré spaces and Poincaré embeddings.

Main Results

Let$(Y,B)^m$ be an oriented Poincaré pair, and let ξ^l be an oriented spherical fibration over Y with fibre S^{l-1}. $\mathrm{Emb}(D(\xi),S(\xi))$ is the set of concordance classes of Poincaré embeddings of $(D(\xi),S(\xi))$ in S^{m+l}. $\Omega\mathrm{Emb}(D(\xi),S(\xi))$ is the set of concordance classes of Poincaré

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embeddings of $(D(\xi \oplus \epsilon^1),\ S(\xi \oplus \epsilon^1))$ in S^{m+l+1} with the additional structure of a trivialization on the "link" of the nonzero cross section $s_0: Y \to S(\epsilon^1) \subset S(\xi \oplus \epsilon^1)$. (See Section 1 for details.)

THEOREM A: *Assume* $\pi_1(S(\xi) \cup D(\xi|_B)) \underset{\sim}{\to} \pi_1(D(\xi))$.

(i) The *unstable normal invariant* map N: $\mathrm{Emb}(D(\xi), S(\xi)) \to D =$ *subset of deg 1 elements in* $\pi_{m+l}(T(\xi)/T(\xi|_B)$ *factors through a bijection*

$$N_1 : \Omega\mathrm{Emb}(D(\xi), S(\xi)) \underset{\sim}{\to} D.$$

(ii) *Assume* $m + l \geqq 6$ *and* Y *is homotopy equivalent to a CW complex of dimension* k. *If* $2(m + l) \geqq 3(k + 1)$, *then* N *is onto. If* $2(m + l) > 3(k + 1)$, *then* N *is a bijection.*

$\mathfrak{T}^n(K)$ *is the set of n-dimensional Poincaré complex thickenings of a CW complex* K *of dimension* k. $\mathrm{Sec}\mathfrak{T}^n(K)$ *is the set of Poincaré complex thickenings with the additional structure of a cross section (see Section 2 for details). If* $K \simeq \Sigma \bar{K}$ *and* K *is* $2k - n + 1$ *connected, then we define a map* $J^t : \mathfrak{T}^n(K) \to \pi_{n-1}(D_nK)$, *where* D_nK *is the n-Spanier–Whitehead dual of K. If* $K = S^k$ *and* $n - k > 2$, *then* $\mathfrak{T}^n(K) = \pi_{k-1}(SG(n-k))$ *and* J^t *is a generalization of the classical J-homomorphism. Theorem A and Theorem 2.2 of Hodgson* [9] *imply the following result.*

THEOREM B: *Assume* K *is* $2k - n + 3$ *connected,* $n \geqq 6$, *and* $K \simeq \Sigma \bar{K}$
(i) J^t *factors through a bijection*

$$J_1^t : \mathrm{Sec}\mathfrak{T}^{n+1}(K) \underset{\sim}{\to} \pi_{n-1}(D_nK)$$

(ii) *If* $3k \leqq 2n - 3$, *then* J^t *is onto. If* $3k < 3n - 3$, *then* J^t *is* $1 - 1$.

In [16] Wall studied $\mathfrak{T}^n_{\mathrm{Diff}}(K) =$ diffeomorphism classes of smooth thickenings of K. He shows that $\mathfrak{T}^n_{\mathrm{Diff}}(\ )$ has many properties formally analogous to $\pi_{n-1}(\ )$. For example, he proved a Hilton–Milnor theorem, defined Hopf invariants, and found an EHP sequence for $\mathfrak{T}^n_{\mathrm{Diff}}(\ )$. His results can now be "explained" by Theorem B.

1. *Poincaré Embedding Theory*

Let $(Y,B)^m$ be an oriented, finite Poincaré pair of formal dimension

m such that Y and B are connected. Let ξ be an oriented spherical fibration over Y of fibre dimension $l-1$. Let $S(\xi)$ be the total space of ξ and let $D(\xi)$ be the associated disc fibration. Let $(X,A) = (D(\xi), S(\xi) \cup D(\xi|_B))$. Then (X, A) is an oriented, finite Poincaré pair of formal dimension $n = m + l$. We assume that the inclusion map $i: A \to X$ induces an isomorphism on π_1. If $l > 2$, this is always true. We assume that X is homotopy equivalent to a finite complex of dimension k and $k < n-2$.

Definition: A *Poincaré embedding of* (X,A) *in* S^n is a map f from A to a finite CW complex C such that if M_f is the mapping cylinder of f, then $X \cup_A M_f$ is homotopy equivalent to S^n.

Proposition 2.7(ii) of Wall [18] implies that (M_f,A) is an oriented Poincaré pair. $C \simeq M_f$ is called the *complement* of the embedding.

Two Poincaré embeddings $f_1: A \to C_1$ and $f_2: A \to C_2$ are concordant if there exists an orientation preserving homotopy equivalence g: $(M_{f_1},A) \to (M_{f_2},A)$ such that $g|_A \simeq Id_A$.

Emb(X,A) is the set of concordance classes of Poincaré embeddings of (X,A) in S^n.

Proposition 1.1:

(i) If $n-k > 2$, then there exists a bijection
$\alpha_1: \pi_k(SG(n-k)) \to \mathrm{Emb}(S^k \times D^{n-k}, S^k \times S^{n-k-1})$.

(ii) If $n-p_1$, $n-p_2 > 2$, then there exists a bijection
$\alpha_2: \pi_{p_1}(SG(n-p_1)) \oplus \pi_{p_2}(SG(n-p_2)) \oplus L^n(p_1,p_2) \to$
$\mathrm{Emb}(S^{p_1} \times D^{n-p_1} \# S^{p_2} \times D^{n-p}, \partial(S^{p_1} \times D^{n-p_1} \# S^{p_2} \times D^{n-p_2}))$ where $L^n_{(p_1,p_2)}$ is the group of *smooth* links of type (p_1,p_2) as defined by Haefliger [5] and $\#$ is connected sum along the boundary.

If $h: S^k \to SG(n-k)$ represents an element in $\pi_k(SG(n-k))$, then $\alpha_1(h)$ is represented by the adjoint $ad(h): S^k \times S^{n-k-1} \to S^{n-k-1}$. The proof that α_1 is a bijection is straightforward.

1.1(ii) is implicit in Theorem 4.3 and Section 4.3 of Haefliger [5].

Links of Cross Sections

Suppose $(M,\partial M)^m$ is a minifold, η^l is a vector bundle, and $g: D(\eta \oplus \epsilon^1) \to S^{m+l+1}$ is an embedding. Since the normal bundle of $g|_M$ has a nonzero cross section $s_0: M \to S(\epsilon^1) \subset S(\eta \oplus \epsilon^1)$, Smale–Hirsh theory implies that $g|_M$ is regular homotopic to an immersion into S^{m+l}. Hirsh [7] has called the map $l(g,s_0): M \xrightarrow{s_0} S(\eta \oplus \epsilon^1) \subset (S^{m+l+1} - \mathrm{int(image}\ g)) = C$ the *link of* s_0 *and* g. If $g|_M$ is isotopic to an embedding of Y into S^{m+k} then $l(g,s_0)$ is homotopy trivial and $S(\eta \oplus \epsilon^1) \subset C$ extends to a map $S(\eta \oplus \epsilon^1) \cup_{s_0} \mathrm{Cone}\ M \to C$. This motivates the following:

Definition: An element in $\mathfrak{L}\mathrm{Emb}(X,A)$ is represented by a map $\bar{f}:\partial(X\times I)\cup_{s_0}\mathrm{Cone}\ X\to C$ such that $\partial(X\times I)\overset{p}{\hookrightarrow}\partial(X\times I)\cup_{s_0} \mathit{Cone}\ X\overset{\bar{f}}{\to}C$ represents an element in $\mathrm{Emb}(X\times I,\partial(X\times I))$.

($I=[0,1]$, $\partial(X\times I)=X\times 0\cup A\times I\cup X\times 1$, and $s_0(x)=x\times 0$).
Two maps $\bar{f}_1:\partial(X\times I)\cup_{s_0}\mathrm{Cone}\ X\to C_1$ and $\bar{f}_2:\partial(X\times I)\cup_{s_0}\mathrm{Cone}\ X\to C_2$ represent the same element in $\mathfrak{L}\mathrm{Emb}(X,A)$ if there exists an orientation preserving homotopy equivalence g: $(M_{\bar{f}_1\circ p},\partial(X\times I))\to(M_{\bar{f}_2\circ p},\partial(X\times I))$ such that $g|_{\partial(X\times I)}=$ identity and such that the following diagram commutes

$$\begin{array}{ccc} & \partial(X\times I)\cup_{s_0}\mathrm{Cone}\ X & \\ \swarrow^{\bar{f}_1} & & \searrow^{\bar{f}_2} \\ C_1 & & C_2 \\ \downarrow & & \downarrow \\ M_{\bar{f}_1\circ p} & \xrightarrow{\quad g\quad} & M_{\bar{f}_2\circ p}\end{array}$$

Proposition 1.2: If $n-k>2$, then there exists a bijection $\beta:\pi_k(SF(n-k-1))\to\mathscr{L}\mathrm{Emb}(S^k\times D^{n-k-1}, S^k\times\partial D^{n-k-1})$.

If f: $S^k\to SF(n-k-1)$ represents an element in $\pi_k(SF(n-k-1))$, $\bar{f}:S^k\times S^{n-k-1}\to S^{n-k-1}$ is the map which sends (x,y) to $(f(x))\ (y)$. Since $\bar{f}(S^k\times *)=*$, $\bar{f}$ extends to a map $(S^k\times S^{n-k-1})\cup\mathrm{Cone}\ (S^k\times *)\to S^{n-k-1}$ which represents $\beta_1(f)$. The proof that β_1 is a bijection is straightforward.

Definition:

(i) F: $\mathfrak{L}\mathrm{Emb}(X,A)\to\mathrm{Emb}(X\times I,\partial(X\times I))$ is the forgetful map

(ii) $S_1:\mathrm{Emb}(X,A)\to\mathfrak{L}\mathrm{Emb}(X,A)$ sends $f:A\to C$ to the element represented by the composition

$$\partial(X\times I)\cup_{s_0}\mathrm{Cone}\,X=\frac{\partial(X\times I)}{X\times 0}\to\frac{\partial(X\times I)}{X\times 0\cup *\times I\cup X\times 1}\simeq\Sigma A\overset{\Sigma f}{\to}\Sigma C,$$

where * is a base point in A.

The definition of S_1 is motivated by the following two easy propositions.

Proposition 1.3: If $(M,\partial M)$ is a smooth manifold, η is a vector bundle, $g:D(\eta)\to S^n$ is a smooth embedding, and $g|$ is the underlying Poincaré embedding; then $F\circ S_1(|g|)$ is the underlying Poincaré embedding of the smooth embedding given by

$$D(\eta) \times I \xrightarrow{g \times Id} S^n \times I \subset S^{n+1}.$$

PROPOSITION 1.4: If $n - k > 2$, then the following diagram commutes up to sign

$$\begin{array}{ccccc}
\mathrm{Emb}(S^k \times D^{n-k}, S^k \times S^{n-k-1}) & \xrightarrow{S_1} & \mathscr{L}\,\mathrm{Emb}(S^k \times D^{n-k}, S^k \times S^{n-k-1}) & \xrightarrow{F} & \mathrm{Emb}((S^k \times D^{n-k}, S^k \times S^{n-k-1}) \times (I, \partial I)) \\
\alpha_1 \uparrow \wr & \square_2 & \beta \uparrow \wr & \square_1 & \alpha_1 \uparrow \wr \\
\pi(SG(n-k)) & \xrightarrow{\gamma_1} & \pi_k(SF(n-k)) & \xrightarrow{\gamma_2} & \pi_k(SG(n-k+1)
\end{array}$$

where γ_1 and γ_2 are induced by the standard inclusions $SG(n-k) \subset SF(n-k)$ and $SF(n-k) \subset SG(n-k+1)$.

Let A and B be simply connected spaces and let f: $A \to B$ a map with cofibre C_f. Let g: $C_f \to S^n$ be a homotopy equivalence. The relative Hurewicz theorem implies there exists an isomorphism $l: \pi_n(f) \underset{\sim}{\to} H_n(f) \simeq H_n(C_f) \underset{\sim}{\overset{g^*}{\to}} H_n(S^n)$. Let $k = \partial(l^{-1}[S^n]) \in \pi_{n-1}(A)$.

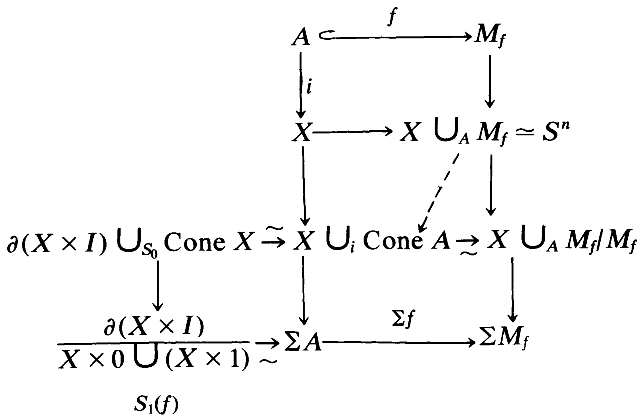
LEMMA 1.5: $B \simeq A \bigcup_k e^n$.

DEFINITION: N: $\mathrm{Emb}(X,A) \to D =$ subset of degree 1 elements in $\pi_n(X/A)$. If f: $A \to C$ represented an element in $\mathrm{Emb}(X,A)$, then $N([f])$ is represented by the composition $S^n \simeq X \bigcup_A M_f \to X \bigcup_A M_f/M_f \simeq X/A$.

PROOF OF THEOREM A:

(*i*) Let $\tilde{f}$: $X/A \simeq \partial(X \times I) \bigcup_{s_0} \mathrm{Cone}\, X \to C$ represent an element in $\mathscr{L}$ $\mathrm{Emb}(X,A)$. A van Kampen and homology argument shows that $C_{\tilde{f}} \simeq S^{n+1}$. There exists a unique homotopy equivalence g: $C_{\tilde{f}} \to S^{n+1}$ such that the isomorphism $H_n(X/A) \underset{\sim}{\leftarrow} H_{n+1}(C_{\tilde{f}}) \underset{\sim}{\overset{g^*}{\to}} H_{n+1}(S^{n+1})$ sends $[X,A]$ to $[S^{n+1}]$. Lemma 1.5 yields a degree one element $k \in \pi_n(X/A)$ such that $X/A \bigcup_k e^{n+1} \simeq C$. Let $N_1([f]) = k$. It is easily seen that N_1 is well defined and is a bijection.

We are done with (i) if we can show that $N_1 \circ S_1([f]) = N(f)$ for any Poincaré embedding f: $A \to C$. Consider the following commutative diagram where the columns are cofibrations:



If we view $N(f)$ as a map into $\partial(X \times I) \cup_{s_0}$ Cone X, the cofibre is $S_1(f)$ but that is the defining property of $N_1 \circ S_1(f)$.

(ii) We shall give two proofs. The first one is essentially just Levitt's proof of his Corollary 3.4 in [12]. (See also Hirsh [7].) The second proof is based upon the criterion of [11] for when a nonproper immersion of a bounded manifold is homotopic to an embedding.

Proof I: Theorem A(i) implies that it suffices to show (1) $n \geqq 6$ and, $2n \geqq 3(k+1)$ imply S_1 is onto and (2) $n \geqq 6$ and $2n > 3(k+1)$ imply S_1 is $1-1$.

Suppose $\tilde{f}$: $\partial(X \times I) \cup_{s0}$ Cone $X \to C$ represents an element in $\mathfrak{L}$ Emb(X,A). Theorem 12.1 of [18] implies $F([\tilde{f}])$ determines (up to concordance) an embedded submanifold $(N,\partial N)^{n+1} \subset S^{n+1}$ such that $(N,\partial N) \simeq (X,A) \times (I,\partial I)$. Furthermore ∂N contains a co-dimension 0 submanifold M^m such that there is a homotopy equivalence of triads $(\partial N; M, \partial N\text{-int } M) \to (\partial(X \times I),\ X \times 0,\ \partial(X \times I)\text{-int}(X \times 0))$. $\tilde{f}$ determines a map $\tilde{\tilde{f}}$: (Cone $X,X) \to (S^{n+1} - \text{int}(N),\partial N)$. The relative induced thickening theorem of Hodgson (Theorem 2.3 of [8]) implies $S^{n+1} - \text{int}(N)$ contains a disc D^{n+1} such that $M \subset \partial D^{n+1} = S^n$. The smooth embedding $(M,\partial M) \subset S^n$ determines a Poincaré embedding f of (X,A) in S^n such that $S_1([f]) = \tilde{f}$.

If $2n > 3(k+1)$, then the uniqueness of the relative induced thickening (Cone $X,X) \to (D^{n+1},M)$ implies that the above procedure yields an inverse to S_1.

Proof II: (outline, for details see proof of Theorem C in [19]) Let $c \in D$. Theorem 3.3 of Wall [18] implies that (X,A) is homotopy equivalent to a π-manifold $(M,\partial M)$ such that c corresponds to an

element $c' \in \pi_n(M/\partial M)$ which when stabilized is a stable normal invariant of M. Smale–Hirsh theory implies there exists an immersion of $(M,\partial M)$ in S^n. The main theorem of [11] implies this immersion is regular homotopic to an embedding g such that $N(g) = c'$. This smooth embedding then determines a Poincaré embedding of (X,A). If $2n > 3(k+1)$, then c' determines a unique smooth embedding up to concordance and the above procedure yields an inverse to N.
Theorem A(i) implies the following two fundamental questions are in fact equivalent.

Question 1: When is a deg 1 element in $\pi_n(X/A)$ the unstable normal invariant of an embedding?

Question 2: When does a null homotopy on the link of an embedding f and a cross section s_0 determine a "desuspension" of that embedding?

The proof of A(i) was motivated by the proof of the corollary in [2].

Theorem a(i), Proof I of A(ii), and the Browder–Sullivan–Casson–Wall Embedding Theorem can be used to give new proofs of Theorems 1 and 2 in [13] and the main theorem of [11].

2. Poincaré Complex Thickenings

Definition: If K^k is a connected, finite CW complex of dimension k with base point $*$, then an *oriented Poincaré n-thickening of* K is a map ψ: $A^{n-1} \to K$ where $n - k > 2$, further A^{n-1} is an oriented Poincaré complex of dimension $n-1$ such that

(1) $\psi_{\#}$: $\pi_1 A \to \pi_1 K$ is an isomorphism.
(2) (M_ψ, A) is an oriented Poincaré pair (where M_ψ is the mapping cylinder of ψ). And
(3) The fundamental class $[A]$ is the image of $[M_\psi, A]$ under the boundary map.

Two Poincaré thickenings ψ_1: $A_1 \to K$ and ψ_2: $A_2 \to K$ are equivalent if there exists an orientation preserving homotopy equivalence $g: (M_{\psi_1}, A_1) \to (M_{\psi_2}, A_2)$ such that the following diagram commutes up to homotopy

$$\begin{array}{ccc} A_1 & \xrightarrow{g|A_1} & A_2 \\ & {\scriptstyle\psi_1} \searrow \quad \swarrow {\scriptstyle\psi_2} & \\ & K & \end{array}$$

$\mathfrak{T}^n(K)$ shall denote the set of equivalence classes of oriented Poincaré n-thickenings of K.

DEFINITION: An oriented Poincaré n-thickening with section of K is an oriented Poincaré n-thickening ψ: $A^{n-1} \to K$ plus a map s: $K \to A$ such that $\psi \circ s \simeq Id_K$.

Two thickenings with section $A_1^{n-1} \underset{s_1}{\overset{\psi_1}{\rightleftarrows}} K$ and $A_2^{n-1} \underset{s_2}{\overset{\psi_2}{\rightleftarrows}} K$ are equivalent if there exists an equivalence of thickenings g: $(M_{\psi_1}, A_1) \to (M_{\psi_2}, A_2)$ such that the following diagram commutes up to homotopy

$$\begin{array}{ccc} A_1^{n-1} & \xrightarrow{g|A_1} & A_2^{n-1} \\ & {\scriptstyle s_1}\searrow \quad \swarrow{\scriptstyle s_2} & \\ & K & \end{array}$$

Sec $\mathfrak{T}^n(K)$ shall denote the set of equivalence classes of oriented Poincaré n-thickenings with section of K.

DEFINITION: S^t: $\mathfrak{T}^n(K) \to \text{Sec}\ \mathfrak{T}^{n+1}(K)$.

If ψ: $A \to K$ represents an element in $\mathfrak{T}^n(K)$, then $S^t([\psi])$ is represented by

$$K \times -1 \cup_\psi (A \times [-1,1]) \cup_\psi K \times 1 \underset{\times 1}{\overset{Id_K \cup \psi \circ \pi_A \cup Id_K}{\rightleftarrows}} K$$

where π_A: $A \times [-1,1] \to A$ is the projection map. F^t: Sec $\mathfrak{T}^n(K) \to \mathfrak{T}^n(K)$ is the forgetful map.

PROPOSITION 2.1: Assume $n - k > 2$. There exist bijections θ_1: $\mathfrak{T}^n(S^k) \xrightarrow{\approx} \pi_k(\text{BSG}(n-k))$ and θ_2: Sec $\mathfrak{T}^{n+1}(S^k) \xrightarrow{\approx} \pi_k(\text{BSF}(n-k))$ such that the following diagram commutes up to sign

$$\begin{array}{ccccc} \mathfrak{T}^n(S^k) & \xrightarrow{S^t} & \text{Sec}\ \mathfrak{T}^{n+1}(S^k) & \xrightarrow{F^t} & \mathfrak{T}^{n+1}(S^k) \\ \wr\downarrow \theta_1 & & \wr\downarrow \theta_2 & & \wr\downarrow \theta_1 \\ \pi_k(\text{BG}(n-k)) & \xrightarrow{B_{\gamma_1}} & \pi_k(\text{BF}(n-k)) & \xrightarrow{B_{\gamma_2}} & \pi_k(\text{BG}(n-k+1)) \end{array}$$

PROOF: θ_1 is essential due to Spivak, see 1.2 of [6]. θ_2 is defined similarly. The proof is then straightforward.

DEFINITION: J^t: $\mathfrak{T}^n(K) \to \pi_{n-1}(D_n K)$ assuming $K \simeq \Sigma \bar{K}$ and K is $2k - n + 2$ connected.

Let ψ: $A^{n-1} \to K$ represent an element in $\mathfrak{I}^n(K)$. Let γ_ψ^q be the Spivak fibration of $(M_\psi,A)^n$. Since $M_\psi \simeq \Sigma\bar{K}$, there exists a map χ: $\bar{K} \to F(q)$ such that if $\hat{\chi}$: $\Sigma^q\bar{K} \to S^q$ is the adjoint of χ, then $T(\gamma_\psi) \simeq S^q \cup_\chi$ Cone $\Sigma^q\bar{K}$. (See 3.5 in [17].) Atiyah has shown that $T(\gamma_\gamma) \simeq D_{n+q}(M_\psi/A)$. The Freudenthal suspension theorem implies there exists a space D_nK unique up to homotopy which is a Spanier–Whitehead dual of K. Let $\hat{\psi}$: $D_nK \dashrightarrow M_\psi/A$ be the stable map which is dual to the map $T(\gamma_\psi) \to T(\gamma_\psi)/S^q \simeq \Sigma^qK$. The Freudenthal suspension theorem implies $\hat{\psi}$ is in fact unstable. Since $\hat{\psi}$ is dual to $T(\gamma_\psi) \to T(\gamma_\psi)/S^q$, $C_{\hat{\psi}}$ is stably homotopy equivalent to S^n. The van Kampen theorem implies $C_{\hat{\psi}}$ is homotopy equivalent to S^n. There exists a unique homotopy equivalence g: $C_{\hat{\psi}} \to S^n$ such that the map $H_n(M_\psi/A) \to H_n(C_{\hat{\psi}}) \xrightarrow{g_*} H_n(S^n)$ sends $[M_\psi,A]$ to $[S^n]$. Lemma 1.5 yields an element $k \in \pi_{n-1}(D_nK)$ such that $M_\psi/A \simeq D_nK \cup_k e^n$. Let $J^t(\psi) = k$. It is easily seen that k is determined by the equivalence class of ψ.

Proof of Theorem B: We first give a proof of B(ii) which uses A(ii). Then we use A(i) to prove B(i). Finally we give a second proof of B(ii) which uses B(i) but not A(ii).

Proof I of B(ii): Let M^{n-1} be a submanifold of S^{n-1} such that $\pi_1\partial M \underset{\sim}{\to} \pi_1 M$ and such that $M \simeq \bar{K}$. The Embedding Theorem in [16] implies that such a M exists and is unique up to concordance. We construct a map μ: $\mathrm{Emb}(M,\partial M) \to \mathfrak{I}^n(K)$. Suppose f: $\partial M \to C$ represents an element in $\mathrm{Emb}(M,\partial M)$. Then $\mu([f])$ is represented by $(\psi_f: A_f \to K) = (S^{n-1} - \mathrm{int}(M)) \cup_{\partial M} M_f \hookrightarrow D^n \cup_M \mathrm{Cone}\ (M \cup_{\partial M} M_f) \simeq K$. Hodgson has shown that if K is $2k - n + 3$ connected then μ is a bijection (see 2.2 of [Ho2], 4.2 of [8], and [10]). Let d: $M/\partial M \to D^n/S^n - \mathrm{int}\ M \simeq \Sigma D_{n-2}\bar{K} \simeq D_nK$ be the map induced by inclusion maps. $C_d \simeq S^n$, and the composition $D \subset \pi_{n-1}(M/\partial M) \xrightarrow{d_\#} \pi_{n-1}(D_nK)$ is a bijection.

Theorem A(ii) implies we are done if we can show that $J^t \circ \mu = d_\# \circ N$.

Consider the cofibre square

$$
\begin{array}{ccc}
M/\partial M & \xrightarrow{\quad d \quad} & D^n/S^{n-1} - \mathrm{int}M \simeq D_nK \\
\Big\downarrow j_2 & & \Big\downarrow j_1 \\
\mathrm{Cone}\ (M \cup_{\partial M} M_f)/M_f & \to & \mathrm{Cone}\ (M \cup_{\partial M} M_f) \cup_M D^n/A_f \simeq M_{\psi_f}/A_f
\end{array}
\tag{2.2.1}
$$

where the maps are induced by inclusions. The cofibre of j_1 is the degree 1 map $M_{\psi_f}/A_f \to S^n$ and j_1 is dual to $T(\gamma^q(M_{\psi_f})) \to T(\gamma(M(\psi_f))/S^q \simeq \Sigma^q K$.
Thus we are done if we can show that j_1 is the cofibre of $d \circ N(f)$. Thus it suffices to show j_2 is the cofibre of $N(f)$. This follows from the following commutative diagram where the rows are up to homotopy cofibrations.

$$\text{(2.2.2)} \qquad \begin{array}{ccccccc} M_f & \to & M \cup_{\partial M} M_f & \to & M \cup_{\partial M} M_f/M_f & \to & \text{Cone } (M \cup_{\partial M} M_f)/M_f \\ \wr\| & & & & \wr\uparrow & \nearrow & \\ S^n & & \xrightarrow{N(f)} & & M/\partial M & & j_2 \end{array}$$

Proof of B(i): We shall construct a map μ_1: $\mathfrak{L}\,\text{Emb}(M,\partial M) \to \text{Sec}\,\mathfrak{I}^{n+1}(K)$ such that the following diagram commutes:

$$\text{(2.2.3)} \qquad \begin{array}{ccc} \text{Emb}(M,\partial M) & \xrightarrow{\mu} & \mathfrak{I}^n(K) \\ \downarrow S_1 & & \downarrow S^t \\ \mathfrak{L}\,\text{Emb}(M,\partial M) & \xrightarrow{\mu_1} & \text{Sec}\,\mathfrak{I}^{n+1}(K) \\ \downarrow F & & \downarrow F^t \\ \text{Emb}((M,\partial M)\times(I,\partial I)) & \xrightarrow{\mu} & \mathfrak{I}^{n+1}(K) \end{array}$$

Then we shall construct a map J_1^t: $\text{Sec}\,\mathfrak{I}^{n+1}(K) \to \pi_{n-1}(D_n K)$ such that $J_1^t \circ S^t = J^t$ and such that $J_1^t \circ \mu_1 = d_{\#} \circ N_1$. Theorem A(i) will then imply B(i).

Definition μ_1: $\mathfrak{L}\,\text{Emb}(M,\partial M) \to \text{Sec}\,\mathfrak{I}^{n+1}(K)$. $M \times I \subset S^{n-1} \times I \subset D^n_0 \cup S^{n-1} \times I \cup D^n_1 = S^n$ and we view $M \times I$ as a submanifold of S^n. Let g_0: (Cone $M,M) \to (S^n - \text{int}(M \times I))$ be the trivialization of $l(f \times Id, s_0)$ which factors through D^n_0. Suppose $\tilde{f}$: $\partial(M \times I) \cup_{s_0}$ Cone $M \to C$ represents an element in $\mathfrak{L}$ Emb$(M,\partial M)$. Let $f = \tilde{f}|_{\partial(M\times I)}$. Then

s_f: $K \simeq \text{Cone}_0\, M \cup_M \text{Cone}_1\, M \xrightarrow{g_0 \cup \tilde{f}|\text{Cone } M} (S^n - \text{int}(M \times I)| \cup_{\partial(M\times I)} M_f$

is a section of $\mu(f)$.
Let $\mu_1([\tilde{f}]) = (\mu(f), s_{\tilde{f}})$. The proof that μ_1 is well defined is straightforward. By varying the extension of f to $\partial(M \times I) \cup_{s_0} M$ one can get all possible sections of $B(f)$ (up to homotopy). Different extensions yield different sections. Thus μ_1 is a bijection.

Definition J_1^t: Sec $\mathfrak{I}^{n+1}(K) \to \pi_{n-1}(D_n K)$. Suppose ψ: $A^n \to K$ and s: $K \to A$ represent an element in Sec $\mathfrak{I}^{n+1}(K)$. As in the definition of J^t we have a map $\hat{\psi}$: $D_{n+1}K \to M_\psi/A$. $\Sigma(A \cup_s \text{Cone } K)$ is canonically homotopy equivalent to M_ψ/A and the Freudenthal suspension theorem implies $\hat{\psi}$ desuspends uniquely (up to homotopy to a map θ: $D_nK \to A \cup_s \text{Cone } K$. $C_\theta \simeq S^n$ and there exists a unique homotopy equivalence g: $C_\theta \to S^n$ such that the isomorphism $H_n(A) \underset{\sim}{\to} H_n(A \cup_s \text{Cone } K) \underset{\sim}{\to} H_n(C_\theta) \overset{g*}{\to} H_n(S^n)$ sends $[A]$ to $[S^n]$. Lemma 1.5 then yields a map K: $S^{n-1} \to D_nK$ such that $D_nK \cup_k e^n \simeq A \cup_s \text{Cone } K$. Let $J_1^t([\psi,s]) = [k]$. The proofs that J_1^t is well defined and that $J_1^t \circ S^t$ are straightforward. *Claim* $J_1^t \circ \mu_1 = d_\# \circ N_1$.

Consider the cofibre square

$$\begin{array}{ccc}
M/\partial M & \longrightarrow & D_nK \\
\| & & \| \\
\partial(M \times I)/M \times 0 & \longrightarrow & S^n - \text{int}(M \times I)/D_0 \\
\Big\downarrow j_2' & & \Big\downarrow j_1' \\
C/\tilde{f}(\text{Cone } M \times 0) & \longrightarrow & (S^n - \text{int}(M \times I)) \cup_{\partial(M\times I)} C/D_0 \cup_{\tilde{f}} (\text{Cone } M \times 0)
\end{array}$$

j_2' is the cofibre of $N_1(\tilde{f})$ and j_1' is the dual of $T(M_{\psi_f}/A_f) \to \Sigma^q K$. Thus $J_1^t \circ \mu_1(\tilde{f}) = d_\# \circ N_1(\tilde{f})$.

Proof II of B(ii): Theorem B(i) implies that we are done if we can show $3k \leqq 2n-3$ implies that S^t is onto and that $3k < 2n-3$ implies S^t is 1.1. Let (ψ: $A^n \to K$, s: $K \to A$) represent an element of Sec $\mathfrak{T}^{n+1}(K)$. The induced Poincaré complex thickening theorem of Hodgson (see 2.3 in [9] and [10]) implies that if $3k \leqq 2n-3$ s can be made an "embedding up to homotopy," i.e., there exists a Poincaré pair $(Q,B)^n$, and homotopy equivalences, h: $A \xrightarrow{\simeq} Q \cup_B Q$, Φ: $K \to Q$ such that $hs \simeq \Phi$, and $\pi_1 B \simeq \pi_1 Q$. Then Φ represents an element in $\mathfrak{T}^n(K)$ such that $S^t(\Phi) = (\psi,s)$. If $3k < 2n-3$, then we get a unique induced thickening and we have described an inverse to S^t.

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