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Linking the conjectures of Artin-Tate and Birch-Swinnerton-Dyer

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1. Introduction

In 1965 Birch and Swinnerton-Dyer suggested a conjecture about the arithmetic of elliptic curves defined over \( \mathbb{Q} \). During the next few years various people gradually extended this conjecture to abelian varieties of any dimension, and defined over any global field. In final form (for objects associated to a given abelian variety) it describes the behaviour near 1 of the \( L \)-series in terms of the Mordell-Weil and Shafarevich-Tate groups, two arithmetic objects.

To some algebraic surfaces equipped with additional structure one can naturally associate abelian varieties defined over function fields; Artin noticed that in this case the Brauer group of the surface is closely related to the Shafarevich-Tate group. This observation led him and Tate to try to rephrase the original conjecture in terms related to the surface. As Tate explained [25], they were led to make a geometric conjecture; of course they expected the equivalence of their conjecture and the arithmetic one of Birch and Swinnerton-Dyer. The main result of this paper is Theorem 6.1, which establishes the equivalence (under mild restrictions on the surface).

In the remainder of this section we introduce our notation and state the conjectures. Global properties of these surfaces equipped with additional structure are explored in Section 2 and again, in terms of the Néron-Severi group, in section 4. Local properties are studied in Section 3. An explicit calculation of an intersection matrix in Section 5 links the local and global properties. In the final section a study of the \( \zeta \)-function of the surface by means of the Weil conjectures is
combined with the explicit form of this matrix to prove the equivalence of the conjectures.

**Notation.** Through this paper unless otherwise indicated we use the following symbols as indicated here:

- \( k \) = a finite field of characteristic \( p \), with \( q \) elements
- \( \ell \) = any prime distinct from \( p \);
- \( V \) = a nonsingular geometrically irreducible one-dimensional \( k \)-scheme;
- \( N_v \) = the number of elements of the finite field \( k(v) \);
- \( K \) = the function field \( k(V) \);
- \( X_K \) = a smooth proper geometrically irreducible curve defined over \( K \);
- \( A \) = the Jacobian of the curve \( X_K \);
- \([G]\) = the order of the finite group \( G \);
- \( \bar{S} = S \times_k \bar{k} \), the base extension of a \( k \)-scheme to the algebraic closure \( \bar{k} \) of \( k \);
- \( F \) = the geometric Frobenius morphism on such a scheme \( \bar{S} \), defined as identity on \((k\text{-rational})\) closed points; \( q \)th-power on structure sheaves
- \( \varphi \) = arithmetic Frobenius, the lifting to \( \bar{S} \) of the \( q \)th-power map on \( \bar{k} \)
- \( H^i_\ell(S) = \lim H^i(\bar{S}, \mathbb{Z}/\ell^n \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \), the \( i \)th \( \ell \)-adic cohomology group;
- \( P_i(S, T) = \det(1 - F_i T) \), the characteristic polynomial for the action induced by \( F \) on \( H^i_\ell(S) \);
- \( G_m \) or \( G_X \) = the sheaf multiplicative group (on the scheme \( X \)).

Sheaves and cohomology groups are, unless otherwise indicated, taken in the étale sense.

1. **The conjectures.** Both the arithmetic and geometric conjectures are stated carefully in [25]. Here we run through them quickly to establish more notation.

Let \( A \) be an abelian variety of dimension \( d \) defined over the global field \( K \). At almost all places \( v \) of \( K \), \( A \) has good reductions to abelian varieties \( A_v \) defined over the finite residue fields \( k(v) \). At any such place write the characteristic polynomial for the action of the (arithmetic) Frobenius element \( \varphi \), the generator of \( G(\bar{k}/k(v)) \), on the Tate module \( T_v(A_v) = \lim \rho(A_v(k(v))) \) as

\[
P_v(A, T) = \det(1 - \varphi T).
\]
This polynomial, known to have integral coefficients independent of the choice of \(\ell\), has complex reciprocal roots \(\{\alpha_1, \ldots, \alpha_{2d}\}\) of absolute value \((N_v)^{1/2}\) and characterized by the equality

\[
[A_v(k_n)] = \prod (1 - \alpha_i^n) \quad 1 \leq i \leq 2d
\]

where \(k_n\) is the extension field of \(k(v)\) of degree \(n\).

Up to multiplication by a nonzero element of \(K\) there is a unique nonvanishing top order holomorphic differential form \(\omega\) on \(A\). For all places \(v\) of \(K\) – even for places where \(A\) has bad reduction – we can define an integral:

\[
\int_{A(K_v)} |\omega|_v \mu_v^d
\]

where, to explain the symbols, \(K_v\) denotes the completion of \(K\) at \(v\), \(|\cdot|_v\) is the \(v\)-adic valuation on \(K_v\), and \(\mu_v\) is the Haar measure on \(K_v\) giving measure 1 to the ring of \(v\)-integers. (Since \(K = k(V)\) all places \(v\) of \(K\) are discrete.)

Finally, when \(\mu_v\) is normalized as above for all \(v\), \(\prod \mu_v\) defines a Haar measure on the adele ring \(A_K\). We write \(|\mu|\) for the volume of the quotient group \(A_K/K\) of \(A_K\) by the discrete subring \(K\).

Let \(S\) be a finite set of places \(v\) of \(K\) containing all “bad” places \(v\), at which either the abelian variety \(A\) has bad reduction, or else the reduced differential form \(\omega_v\) is not nonzero, nonpolar. Birch and Swinnerton–Dyer have associated to \(A\) an \(L\)-series:

\[
L^\#: (A, s) = |\mu|^d \left/ \left( \prod_{v \in S} P_v(A, N_v^{-1}) \prod_{v \in S} \int_{A(K_v)} |\omega|_v \mu_v^d \right) \right.
\]

The asymptotic behaviour of \(L^\#(A, s)\) as \(s \to 1\) is independent of the choice of \(S\), since for \(v\) good, both \(P_v(A, N_v^{-1})\) and the integral over \(A(K_v)\) equal \([A_v(k(v))]/(N_v)^d\).

By the Mordell–Weil Theorem the group \(A(K)\) is finitely generated; we always write a basis for \(A(K)\) modulo torsion as \(\{\alpha_1, \ldots, \alpha_r\}\), where of course \(r\) is the rank of \(A(K)\) modulo torsion. For \(A\) self-dual (e.g. a Jacobian) Néron and Tate have defined a height pairing from \((A(K)/\text{tor}) \times (A(K)/\text{tor}) \to \mathbb{R}\):

\[
|\det(\alpha_i, \alpha_j)| \quad 1 \leq i, j \leq r
\]
denotes the absolute value of the determinant of this pairing. The last ingredient is the Shafarevich–Tate group \( \Sha(A, K) \), the set of everywhere locally trivial principal homogeneous spaces for \( A \); equivalently, we can also define it by

\[
\Sha(A, K) = \bigcap_{v \in |\mathcal{V}|} \ker(H^1(K, A) \to H^1(K_v, A))
\]

where \( K_v \) is the ordinary henselization of \( K \) at \( v \in |\mathcal{V}| \). Now we state the arithmetic conjecture of Birch and Swinnerton-Dyer, for a self-dual abelian variety \( A \):

\[
\text{BSD}(A, K): \text{The group } \Sha(A, K) \text{ is finite. The } L\text{-series } L_s(A, s) \text{ has a zero of order exactly } r = \text{rank } A(K) \text{ at } s = 1. \text{ As } s \to 1,
\]

\[
L_s(A, s) \sim (s - 1)^r \frac{[\Sha(A, K)]\left|\det(\alpha_i, \alpha_j)\right|}{[A(K)_{\text{tor}}]^2}.
\]

To state the geometric conjecture, consider \( X \) a smooth surface defined over \( k \). The Néron–Severi group \( \text{NS}(X) \) of algebraic equivalence classes of divisors on \( X \) is, by the Theorem of the Base [12], known to be finitely generated as abelian group. We write \( \rho(X) = \rho \) for its rank and \( \{D_1, \ldots, D_\rho\} \) for a basis of \( \text{NS}(X) \) modulo torsion.

\[
|D_i \cdot D_j| \quad 1 \leq i, j \leq \rho(X)
\]

denotes the absolute value of the determinant of the intersection matrix.

\( \text{Pic}^0_X \), the connected component of identity of the Picard scheme of \( X \), is an abelian variety defined over \( k \) (and \( \text{Pic}^0(\overline{k}) = \text{Pic}^0(X) \) is the set of linear equivalence classes of divisors algebraically equivalent to zero on \( X \)). Write

\[
\alpha(X) = \chi(X, 0_X) - 1 + \dim_k \text{Pic}^0_X
\]

and define the (cohomological) Brauer group of \( X \) by

\[
\text{Br}(X, k) = H^2(X, G_m)
\]

(Artin has shown that for \( X \) smooth of dimension \( \leq 2 \), \( \text{Br}(X, k) \) is a birational invariant.) We now state the geometric conjecture of Artin and Tate:

\[
\text{AT}(X, k): \text{Br}(X, k) \text{ is a finite group. The polynomial } P_2(X, T) \text{ has}
\]
$q^{-1}$ as a root of multiplicity exactly $\rho(X)$; as $s \to 1$,

\[
P_2(X, q^{-s}) \sim (1 - q^{(1-s)}(\log X))\frac{\prod_{i,j} D_i \cdot D_j}{q^{\sigma(X)}[\text{NS}(X)_{\text{tor}}]^2}
\]

2. A lemma. To calculate characteristic polynomials we frequently quote the following lemma of linear algebra, presented here for want of a better place. Define (not in the standard fashion) the characteristic polynomial of an endomorphism $\varphi$ of $S$, a finite-dimensional vector space over an arbitrary field $E$, as $P(\varphi, T) = \det(1 - \varphi T)$ for $\varphi$ considered as a matrix. Consider \{V_1, \ldots, V_r\} a set of isomorphic finite-dimensional $E$-vector spaces with a collection of $E$-linear maps $f_i : V_i \to V_{i+1}$ cyclically permuting the $V_i$:

The $f_i$ induce an endomorphism $f = \bigoplus f_i$ of $V = \bigoplus V_i$ and $f'$ induces endomorphisms $F_i$ of each $V_i$. The following pleasant result holds.

**Lemma 1.1:** Let $f = \bigoplus f_i$ and $F = \bigoplus F_i = f'$ be as described. Let $P(f, T)$ denote the characteristic polynomial of $f$ acting on $V$, and $Q(F_i, T)$ that of $F_i$ acting on $V_i$. The polynomial $Q(F_i, T)$ is independent of choice of $i$, and

\[
P(f, t) = Q(F_i, t')
\]

**Proof:** Since formation of characteristic polynomial is unaffected by base field extension, assume $k$ algebraically closed. Choose a basis $B^1$ of $V_1$. Choose bases $B^2$ of $V_2$, $B^3$ of $V_3$, \ldots, and finally $B^r$ of $V_r$ seriatim, so that the matrices of $f_1$ relative to $B^1$ and $B^2$, of $f_2$ relative to $B^2$ and $B^3$, \ldots, and finally of $f_{r-1}$ relative to $B^{r-1}$ and $B^r$ are upper triangular. With respect to the basis $\{B^1, B^2, \ldots, B^r\}$ of $V = \bigoplus V_i$:

\[
P(f, t) = \det \begin{vmatrix}
I & 0 & \cdots & 0 & -tT_r \\
-tT_1 & I & 0 & \cdots & 0 \\
0 & -tT_2 & I & \cdots & \cdot \\
\cdot & 0 & -tT_3 & \cdots & \cdot \\
\cdot & \cdot & 0 & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & -tT_{r-1} & I
\end{vmatrix}
\]
where $T_1, T_2, \ldots, T_{r-1}$ are upper triangular. Row reduction changes this to an upper triangular matrix with $r - 1$ blocks $I$ on the diagonal, and a last diagonal block

$$I - t'T_{r-1}T_{r-2}\ldots T_1T_r = I - t'F_r;$$

this matrix has as its determinant $Q(F_r, t')$. A different sequence of row and column reductions (which cannot change the determinant) gives as $i$th diagonal block $I - t'F_i$, with all other blocks $I$. ■

This lemma will usually be applied to the $\ell$-adic cohomology groups. When $X$ is a $k$-irreducible variety which splits over $\overline{k}$ into $r$ (conjugate) irreducible varieties, by choosing notation carefully we can write

$$\tilde{X} = \sum X_i \quad 1 \leq i \leq r$$

where $F(X_i) = X_2, \ldots, F(X_r) = X_1$. By functoriality $F$ induces maps on cohomology $F_i : H^i(X_i) \rightarrow H^i(X_{i-1})$. Using the lemma on the map $\oplus F_i$ gives the corollary

**Corollary:** Let $X_i$ denote any fixed component of the cycle $\tilde{X}$. For all $i$,

$$P_i(X, T) = P_i(X_i, T').$$

### 2. Global properties of fibrations

1. **Definitions.** To relate the Artin–Tate conjecture for a smooth surface to the Birch–Swinnerton-Dyer conjecture, we need to furnish the surface with additional structure.

   **Definition:** A fibration (defined over the field $k$) is a triple $(X, V, f)$ of objects (defined over $k$) with
   
   (1) $V$ a complete (smooth and geometrically irreducible by our notation) curve;
   (2) $X$ a smooth surface; and
   (3) $f : X \rightarrow V$ a proper flat morphism with generic fiber $X_K$ a curve (smooth, proper, and geometrically irreducible) defined over $K = k(V)$. 


Further, we assume that $f$ is cohomologically flat (in dimension zero) as well.

Given a fibration $(X, V, f)$, a $k$-rational (Weil) divisor $D$ on $X$ is mapped by $f$ either to a divisor on $V$ or onto all of $V$, with some multiplicity $m(D)$. Set $\alpha = \gcd\{m(D)\}$ as $D$ runs through all the $k$-rational divisors on $X$. We say that the fibration $(X, V, f)$ has degree $\alpha$; we choose once and for all a fixed $k$-rational divisor $\Omega$ of degree $m(\Omega) = \alpha$ consisting only of "horizontal" components (we think of $\Omega$ as an "$\alpha$-fold section"). The fibration is said to be of genus $d$ if $X_K$, considered as curve over $K$, has genus $d$, or equivalently [4] if for all complete fibers $X_v$,

$$d = \dim_{k(v)} H^1(X_v, \mathcal{O}_{X_v}).$$

**Remarks:**

1. $\alpha = \gcd\{\text{degrees of cycles on } X_K(K)\}$. If $f$ admits a section, $\alpha$ is 1.
2. In fact we can compute $\alpha$ as $(\Omega \cdot X_v)$, the intersection degree of $\Omega$ and any complete vertical geometric fiber of $f$. Immediately, each closed fiber $X_v$ must, as a curve defined over the residue field $k(v)$, be of multiplicity dividing $\alpha$ (see Section 3 for a definition of multiplicity).
3. The condition of cohomological flatness (in dimension zero), which can be restated as the condition that universally

$$f_* \mathcal{O}_X = \mathcal{O}_V,$$

allows us to use Raynaud's precise numerical results, stated in Section 5. Raynaud gives conditions [22; §7.2.1] which show that for a flat proper morphism $f : X \to V$ admitting a section (or even with all fibers of multiplicity one) cohomological flatness (in dimension zero) is equivalent to the single equality

$$f_* \mathcal{O}_X = \mathcal{O}_V.$$

And this equality is guaranteed by the first sentence of (3) in the definition.

Rather than starting with the surface $X$, we could have adopted a different attitude, beginning with $k$, $V$, and a curve $X_K$ (of course smooth, etc.) defined over $K = k(V)$. 
DEFINITION: A triple \((X, V, f)\) is a model for \(X_K\) over \(V\) (defined over \(k\)) if \(X\) is a two-dimensional \(k\)-scheme and \(f : X \to V\) is a proper flat morphism (defined over \(k\)) with generic fiber \(X_K\). The model is smooth or regular if \(X\) is, and (strictly) local if \(V\) is \(\text{Spec}(R)\) for \(R\) a (strictly) henselian ring.

2. Cohomology. In studying a fibration we play off the local properties—typified by the structure of the degenerate fibers, studied in the next section—against the global properties, principally expressed in terms of the Néron-Severi group and the \(\zeta\)-function. Much global information is provided cohomologically by the Leray Spectral Sequence:

\[
H^p(V, R^q f_* \mathbb{G}_m) \Rightarrow H^{p+q}(X, \mathbb{G}_m)\]

The vanishing theorem of Artin cited here makes the spectral sequence quite simple to use (Artin’s results come from [9, Br III], where Theorem 2.1 is numbered as 3.2, and Theorem 2.3 can be extracted from the fourth section of the article).

THEOREM 2.1 (Artin): Let \(S\) and \(V\) be locally Noetherian regular schemes, and let \(f : X \to V\) be a proper flat morphism with all its fibers curves. Assume \(V\) is also a curve and that all the local rings of \(V\) are Japanese. Then for all \(q > 1\),

\[
R^q f_* \mathbb{G}_m = 0.
\]

PROPOSITION 2.2: Let \(f : X \to V\) be a proper flat morphism from a surface onto a curve (all defined over \(k\)) for which \(f_* \mathcal{O}_X = \mathcal{O}_V\), and assume that there is a \(k\)-rational divisor \(D \subseteq X\) such that \(f : D \to V\) is a flat surjective \(n\)-to-1 covering. Then the kernel of each edge homomorphism \(H^p(V, \mathcal{G}_V) \to H^p(X, \mathcal{G}_X)\) is contained in the kernel of \(H^p (\text{nth-power})\) acting on \(H^p(V, \mathcal{G}_V)\).

PROOF: In the diagram
combine the edge homomorphisms for \( f \) and \( i \) to get a map

\[
H^p(V, G_V) \xrightarrow{f^*} H^p(X, G_X) \xrightarrow{i^*} H^p(D, G_D).
\]

As \( D \to V \) is flat and surjective of degree \( n \), \( \mathcal{O}_V \) is an \( \mathcal{O}_V \)-module locally free of rank \( n \). The norm map \( N: \mathcal{O}_V^\times \to \mathcal{O}_V^\times \) induces \( n \)th-power on \( \mathcal{O}_V^\times \). In other words, the long composition

\[
H^p(V, G_V) \xrightarrow{f^*} H^p(X, G_X) \xrightarrow{i^*} H^p(D, G_D) \xrightarrow{H^p(N)} H^p(V, G_V)
\]

is just \( H^p \) (\( n \)th-power).

When \((X, V, f)\) is a fibration, one relates the groups \( \text{Br}(X, k) \) and \( \text{III}(A, K) \), where \( A \) denotes the Jacobian of the generic fiber, by comparing both with \( H^1(V, R^1f_*G_X) \). By Proposition 2.2, since \( \text{Br}(V) \) is zero (because \( V \) is a complete curve over a finite field) \( \text{Br}(X) \) and \( H^1(V, R^1f_*G_X) \) differ only by a finite group. In [9, Br III, §4] it is proven that \( H^1(V, R^1f_*G_X) \) and \( \text{III}(A, K) \) also differ only by finite groups. Consequently, we have Theorem 2.3.

**Theorem 2.3 (Artin):** Let \((X, V, f)\) be a \( k \)-fibration with \( A \) the Jacobian of generic fiber. Finiteness of \( \text{Br}(X, k) \) and of \( \text{III}(A, K) \) are equivalent. If \((X, V, f)\) has degree 1, in fact the two groups are isomorphic.

### 3. Néron models

For the moment, \( k \) will denote either a finite or an algebraically closed field, and \( V \) will be a (nonsingular geometrically irreducible) \( k \)-scheme of dimension 1 with function field \( K \). Write the generic fiber of a \( V \)-scheme \( S \to V \) as \( S_K \to K \). (At least when \( V \) is either a complete curve over \( k \) a finite field or \( \text{Spec}(D) \) for a discrete valuation ring \( D \)) Néron has shown that any abelian variety \( A \) defined over \( K \) has a minimal model \( A_V \to V \), a smooth commutative group scheme over \( V \) which represents, on the category of smooth \( V \)-schemes, the functor:

\[
T \mapsto \text{hom}_{\text{Spec}K}(T_K, A).
\]

We write \( A^\circ_V \) for the open subscheme of \( A_V \) characterized by having for its fiber at the closed point \( v \) of \( V \) the connected component of identity \( (A_v)^\circ \) of the \( v \)-fiber of the group scheme \( A_V \).
Consider a regular model \((X, V, f)\) of a generic curve \(X_K\) of genus \(d\). The Jacobian of \(X_K\), is an abelian variety of dimension \(d\) defined over \(K\); each fiber \(A_v\) of the Néron model is an algebraic group of dimension \(d\) also, and so fiber by fiber the space of invariant differential \(d\)-forms on \(A_v\) is one dimensional; these patch to give a line bundle on \(V\).

**Lemma 2.4:** Let \((X, V, f)\) be a regular model for the generic curve \(X_K\) of genus \(d\), and let \(w\) denote the line bundle on \(V\) whose fiber \(w_v\) is the space of invariant differential \(d\)-forms on \(A_v\). \(R^1f_*\mathcal{O}_X\) is a locally free sheaf of rank \(d\), and

\[
\omega^{-1} = \Lambda^d (R^1f_*\mathcal{O}_X).
\]

If \((X, V, f)\) is a fibration the line bundle \(\omega\) has

\[
\deg(\omega) = d \cdot \chi(V, \mathcal{O}_V) - \chi(V, R^1f_*\mathcal{O}_X).
\]

**Proof:** The stalk at a geometric point of \(V\) is given by

\[
(R^1f_*\mathcal{O}_X)_\mu = \lim_{\longrightarrow} H^1(U \times_V X, \mathcal{O}_X)
\]

where the limit is taken over \(\mu\)-punctured spaces \(U\) étale over \(V\). Since \(\mathcal{O}_X\) is quasi-coherent, we can calculate using Zariski cohomology rather than the étale cohomology. But for the Zariski cohomology, we known that for all geometric points \(\mu\) of \(V\), the groups \(H^2(X_\mu, \mathcal{O}_{X_\mu})\) are zero, and that the groups \(H^1(X_\mu, \mathcal{O}_{X_\mu})\) as \(k\)-vector spaces all have dimension \(d\). By the base-change theorem for Zariski cohomology [cf. 19, lecture 7.3°] the sheaf \(R^1f_*\mathcal{O}_X\) is locally free of rank \(d\).

By the Riemann–Roch Theorem, a locally free sheaf \(E\) of rank \(d\) on a complete curve \(V\) induces a line bundle \(\Lambda^d E\) on \(V\) satisfying

\[
\chi(V, E) = \deg(\Lambda^d E) + d \cdot \chi(V, \mathcal{O}_V)
\]

As \(R^1f_*\mathcal{O}_X\) is isomorphic to the tangent space at the zero-section to \(Pic_{X/V}\), \(\Lambda^d R^1f_*\mathcal{O}_X\) is dual to \(\omega\). \(\square\)
3. Degenerate fibers

1. Structure of the fibers. The arithmetic properties of a smooth surface $X$ which is part of the fibration $(X, V, f)$ depend on the structures of the degenerate fibers of $X$. Here we localize to study the fibers one by one.

**DEFINITION:** Let $(X, V, f)$ be a fibration and $V(v) \hookrightarrow V$ be the inclusion of the ordinary henselization of $V$ at $v$. The base-extended model $(X(v), V(v), f(v) = f)$ is the associated local model (for $X_K$) at $v$; for base extension over the inclusion of a strict henselization $V(\mu) \hookrightarrow V$ at a geometric point $\mu$ above $v$ we refer to the associated strictly local model.

The following standard notation will apply to any (degenerate) closed fiber $X_s$ of any smooth model $(X, V, f)$. We write the closed fiber $X_s$, which is a curve defined over $k(s)$, as a sum of $k(s)$-irreducible components with multiplicities

$$X_s = \sum p_a X_a \quad 1 \leq a \leq h_s$$

The integer $d_s = \gcd\{p_a\}$ is the *multiplicity* of the fiber $X_s$ and we call the fiber non-reduced if $d_s > 1$. If so, writing $p_a = d_s m_a$ we define a reduced divisor by

$$R_s = \sum m_a X_a \quad ; \quad X_s = d_s R_s$$

and $R_s$ is the least positive rational multiple of the complete fiber $X_s$. The $k(s)$-components $X_a$ are in general not absolutely irreducible. If, over $k(s)$, $X_a$ splits as a cycle of $q_a$ distinct irreducible components

$$X_a \times_{k(s)} k(\tau) = \sum X_{aj} \quad 1 \leq j \leq q_a$$

we say that $X_a$ has separable fiber multiplicity $q_a$. (A sum over $a$ always denotes $k(s)$-components, which are not necessarily absolutely irreducible; sums over $j$ denote completely split—e.g. geometric—components.) For $\tau$ a geometric point over $s$, we see that the geometric fiber $X_\tau$ has $n_s = \sum p_a q_a$ components, multiplicities
counted. On the base-extended scheme $\tilde{X}$ the base-extension of the closed fiber $X_s$, as the disjoint union of $\text{deg}(s)$ geometric fibers, has a total of $n_s \cdot \text{deg}(s)$ components. We associate to the closed fiber an anonymous integer defined by the equality

$$d_s \Delta_s = \gcd\{p_a q_a\}$$

which expresses how much of the structure of the fiber on the base-extended surface $\tilde{X}$ was not apparent over $k$.

**Proposition 3.1:** Any $k$-rational divisor $D$ on $X$ intersects the closed fiber $X_s$ with total intersection multiplicity divisible by $d_s \Delta_s$. If $(X, V, f)$ is a fibration of degree 1, each closed (or geometric) fiber has multiplicity 1. If $f$ admits a section $\sigma$, each fiber has a component of multiplicity 1.

**Proof:** By definition $D$ intersects the closed fiber $X_s$ with total intersection multiplicity $(D \cdot X_s) = \Sigma (D \cdot X_\tau)$ as $\tau$ runs through the geometric points above $s$. $(D \cdot X_\tau) = (D \cdot \Sigma p_a \Sigma X_{aj}) = \Sigma p_a (D \cdot \Sigma X_{aj})$. For each $a$, the components $X_{aj}$ are conjugate over $k$; if $D$ intersects one of the $X_{aj}$ it must intersect $X_{aj}$ for each $j$, $1 \leq j \leq q_a$. We see $(D \cdot X_{aj}) = q_a (D \cdot X_{aj})$ for any particular choice of $j$. This shows that $(D \cdot X_s)$ and so $(D \cdot X_s)$ is divisible by $d_s \Delta_s = \gcd\{p_a q_a\}$. “Degree 1” means that $(\Omega \cdot X_s) = 1$, so $d_s \Delta_s$ is 1, and $X_s$ has multiplicity 1. If there is a section, $\sigma(V)$ hits each closed fiber in a single point and with multiplicity 1. This point must be on a component of multiplicity 1.

Normalization allows us to read off the precise structure of certain étale sheaves on a (completely split) fiber $X_s$.

**Lemma 3.2:** Let $X_s$ be a completely split fiber. Write $(X_s)_\text{red} = \bigcup X_j$ as a union of distinct absolutely irreducible components and denote by $\tilde{X}_j$ the normalization of the (reduced) component $X_j$. Write $\pi : \tilde{X}_j \to (X_s)_\text{red}$ for the obvious map. Let $\mathcal{F}$ denote either $\mathbb{G}_m$ or $\mu_r$, considered as étale sheaf on $X_s$, and $\tilde{\mathcal{F}}$ be the corresponding sheaf on $\tilde{X}_s$. Then

$$0 \to \mathcal{F} \to \pi_* \tilde{\mathcal{F}} \to \mathcal{C} \to 0$$

is an exact sequence of sheaves with the cokernel $\mathcal{C}$ a skyscraper sheaf.
concentrated on the singular points of $X_s$. If $k(s)$ is (separably)
algebraically closed the higher cohomology of $\mathcal{C}$ vanishes.

**PROOF:** At a nonsingular point $Q$ of $X_s$, $\mathcal{F}$ and $\pi_* \mathcal{F}$ have the same stalk since near $Q$, $X_j = X_j$. This shows that $\mathcal{C}$ is a skyscraper concentrated on the singular points of the fiber; $\mathcal{C} = \bigoplus i_* \mathcal{C}_P$. For $k(s)$ algebraically closed (e.g. in the strictly local case) each singular point $P$ is Spec $k$ for $k$ an algebraically closed field. As inclusion of closed point is a finite morphism, we see that

$$H^i(X_s, \mathcal{C}) = H^i(X_s, \bigoplus i_* \mathcal{C}_P) = \bigoplus H^i(P, \mathcal{C}_P).$$

For $i > 0$, this last group is zero since étale cohomology over Spec $k$ is Galois cohomology, which is zero as $k$ is algebraically closed. □

2. $\zeta$-functions. Following Deligne [5], for $X$ an algebraic $k$-scheme and $x \in |X|$ a closed point of $X$, write $N_x$ for the number of elements in the finite field $k(x)$; this number happens to be $q^{\deg(x)}$. Define

$$\zeta(X, s) = \prod (1 - (N_x)^{-s})^{-1} \quad x \in |X|.$$

As this product depends only on the underlying set of closed points of $X$, $\zeta(X, s) = \zeta(X_{\text{red}}, s)$ and for a fibration $(X, V, f)$, $\zeta(X, s) = \prod \zeta(X_v, s)$, the product over $v \in |V|$. Write $q^{-s} = T$; since $N_x$ is just $T^{\deg(x)}$, we can rewrite $\zeta(X, s)$ as $Z(X, T)$. In this form the Weil conjectures provide a rational expression, valid even for $X$ singular,

$$Z(X, T) = \frac{P_1(X, T) \cdots P_{2n-1}(X, T)}{P_0(X, T) \cdots P_{2n}(X, T)}$$

where $P_i(X, T)$ denotes the characteristic polynomial for the action of $F$ on the $\ell$-adic cohomology group $H^i_{\text{ét}}(X)$, and $n = \dim X$. $P_i(X, T)$ is known to have integral coefficients independent of the choice of $\ell \neq p$, provided that $X$ is smooth and proper. Whenever $X$ is smooth Poincaré duality provides a perfect pairing for all $i$, $0 \leq i \leq 2n$,

$$H^i(X)(d) \otimes H^{2n-i}(X)(n - d) \to \mathbb{Q}_\ell$$

which we can use to calculate the polynomials $P_i(X, T)$; recall that the maps induced by $F$ and $\varphi$ on the $\ell$-adic cohomology groups are
(cohomological) inverses. Since by definition $\varphi$ acts on $H^1(\mathcal{O})$ as $q^d\varphi$ does on $H^1(X)$, $F$ acts on $H^1(\mathcal{O})$ as $q^dF$ does on $H^1(X)$. Calculating for the action induced on $H^1(\mathcal{O})$, we get

$$P_t(X, T) = \det(1 - q^dF_T).$$

**Proposition 3.3:** The $\zeta$-function of the complete curve $X_v$ of genus $d$ defined over the finite field $k(v)$ and with $(X_v)_{\text{red}} = \bigcup X_a$ is

$$Z(X_v, T) = \frac{P_t(X_v, T)}{(1 - T) \cdot \prod(1 - (N_v T))^a} \quad 1 \leq a \leq h_v$$

If we write $A_i^o$ for the connected component of identity of the Picard scheme of $X_v$ we get the precise value

$$P_t(X_v, N_v^{-1}) = [A_i^o(k(v))]/(N_v)^d.$$  

**Proof:** Without changing the $\zeta$-function we replace $X_v$ by $X_v$ red. Write this reduced curve still as $X_v$, and write $q$ for $N_v$.

$$P_0(X_v, T) = \det(1 - F_0 T).$$

Since $F$ acts trivially on $H^0(\mathcal{O}) = \mathcal{O}_v$, we conclude that $P_0(X_v, T) = (1 - T)$.

Recall that $P_t(X_v, T) = \det(1 - qF_1 T)$ for the action of $F_1$ on $H^1(X_v)(1)$. By the Riemann Hypothesis all the inverse roots of $F_1$ have absolute value $q^{1/2}$, so we calculate that $P_t(X_v, T)$ is also $\det(1 - \varphi T)$ for the action induced on $H^1(\mathcal{O})(1)$ by $\varphi$. By Kummer Theory, $H^1(\mathcal{O})(1)$ is just $T_{\mathcal{C}}(\text{Pic}_{\mathcal{E}}(k(v))) = T_{\mathcal{C}}(A^0_{\mathcal{E}}(k(v)))$ via the canonical identification of $H^1(\mathcal{E}_v, G_m)$ with $\text{Pic}_{\mathcal{E}}(k(v))$. Since $A_i^o$ is a smooth connected group scheme over $k(v)$ a field of $q = N_v$ elements, a lemma of Milne [15, page 182] shows that the claimed value of $P_t(X_v, T) = \det(1 - \varphi T)$ is correct.

For $P_2(X_v, T)$ we need to work harder. Base extension over $k(v) \to \overline{k(v)}$ replaces $X_v$ by $\tilde{X}_v = \bigcup X_{aj}$. Write $\tilde{X}_{aj} \to X_{aj}$ for the normalizations. By Lemma 3.2 applied to $\tilde{X}_v$ and the sheaf $\mu_{e^r}$ we see that

$$H^2(\tilde{X}_v, \mu_{e^r}) = \bigoplus_{a} \bigoplus_{j} H^2(\tilde{X}_{aj}, \mu_{e^r}).$$

Taking the projective limit over $n$ and tensoring with $\mathcal{O}_v$, find

$$H^2(\mathcal{O}_v)(1) = \bigoplus H^2(\tilde{X}_{aj})(1).$$
By Poincaré duality on the smooth schemes $\tilde{X}_{aj}$ we see that

$$H^2(X, \mathbb{Q}) \cong \bigoplus H^0(\tilde{X}_{aj}) \cong \bigoplus a(\mathbb{Q}) \oplus \mathbb{Q}.$$ 

$F$ acts by permuting cyclically the components in each $\bigoplus \mathbb{Q}$, and $F^{a_0}$ acts as identity. Lemma 1.1 tells us that $P_2(X, T)$ is $\Pi(1 - (qT)^{a_0})$, as claimed, with the product for $1 \leq a \leq h_v$. 

4. The Néron–Severi group

Henceforth, unless otherwise stated, $(X, V, f)$ is a fibration of degree $\alpha$ with $\Omega$ the chosen $k$-rational divisor of degree $\alpha$; $A$ is the Jacobian of the generic fiber $X_K$; and $B$ is the $K/k$ trace of $A$, as explained below.

1. The group of divisors. Since $X$ is a smooth surface all its local rings are regular, hence factorial; this shows that on $X$ (or $\tilde{X}$) the notions of Cartier (locally principal) and Weil (codimension one) divisors coincide. We identify a $k$-rational divisor $D$ on $X$ with its base extension $\tilde{D}$ on $\tilde{X}$; via this identification we view $\text{Div} X$ as the subgroup of $k$-rational divisors in the divisor group $\text{Div} \tilde{X}$. Notions of algebraic and linear equivalence are always taken with reference to $\tilde{X}$. The map $f^*$ defines injections of the groups $\text{Div} V$ and $\text{Div} \tilde{V}$ of divisors on $V$ and $\tilde{V}$ into $\text{Div} X$, $\text{Div} \tilde{X}$. We view the groups $\text{Div} V$, etc. as subgroups of $\text{Div} X$, etc.

By specifying generating sets of divisors on $\tilde{X}$ we define several subgroups of $\text{Div} \tilde{X}$. Namely, $\text{Div}_{\text{vert}}(\tilde{X})$ is generated by the irreducible curves $C$ on $\tilde{X}$ for which $\tilde{f}(C)$ is a single point; $\text{Div}_{\text{hor}}(\tilde{X})$ by the irreducible curves $C$ for which $\tilde{f}: C \to \tilde{V}$ is surjective; $\text{Div}_0(\tilde{X})$ by those $C$ which intersect each complete vertical fiber with total intersection multiplicity zero; and $\text{Div}^0(\tilde{X})$ by those $C$ which are algebraically equivalent to zero on $\tilde{X}$. (The image in $\text{Div} \tilde{X}$ of) $\text{Div}^0(\tilde{V})$, the set of divisors on $\tilde{V}$ algebraically equivalent to zero, is the collection of all divisors of the form $\sum a_v X_v$ with the $X_v$ complete fibers and the $a_v$ integers with $\sum a_v = 0$. We also define $\text{Div}_0(\tilde{V})$ to be the set of divisors on $\tilde{X}$ which can be expressed as a rational linear combination of complete fibers. Intersecting with $\text{Div} X$ these subgroups define similar subgroups of $\text{Div} X$; e.g. $\text{Div}_{\text{vert}}(X)$ is generated by the $k(v)$-irreducible components of the various fibers $X_v$. Reading these subgroups modulo $D_v(\tilde{X})$, the group of divisors linearly equivalent to zero, we deduce subgroups of $\text{Pic} \tilde{X}$ and $\text{Pic} X$; e.g. $\text{Pic}_{\text{vert}}(X)$ is the
group of divisor classes representable by divisors in $\text{Div}_{\text{vert}}(X)$. Working modulo $\text{Div}^0(\bar{X})$ we define subgroups $\text{NS}_{\text{vert}}(X)$ and $\text{NS}_0(X)$ of $\text{NS}(X)$ the group of algebraic equivalence classes of divisors on $X$.

**Lemma 4.1:** Write $\mathcal{D}$ for the subgroup $\text{Div}_{\text{vert}}(X) \oplus \mathbb{Z}\Omega$ of $\text{Div} X$. We have a filtration of $\text{Div} X$ by subgroups

$$0 \rightarrow f^* \text{Div} V \hookrightarrow \text{Div}_{\text{vert}}(X) \hookrightarrow \mathcal{D} \hookrightarrow \text{Div} X$$

**Proof:** Identifying Cartier and Weil divisors on $\bar{X}$ we see that $\text{Div} \bar{X} = \text{Div}_{\text{vert}}(\bar{X}) \oplus \text{Div}_{\text{hor}}(\bar{X})$. Since $f$, and therefore $\bar{f}$, is defined over $k$, this is a direct sum decomposition of $G(\bar{k}/k)$-modules. Galois-invariant groups give the filtration.

We read this filtration modulo algebraic equivalence to arrive at a similar one for $\text{NS}(X)$. To do so we need to know the quotient groups, which we can analyze by putting the divisors in a canonical form. Any vertical divisor $F$ can be written as a sum

$$F = \sum F_v$$

of divisors each supported on a single complete vertical fiber. Any divisor $E$ supported on $X_v$ can be written uniquely as a $\mathbb{Z}$-linear combination

$$E = rX_v + \sum s_aX_v$$

(Such a statement is clear for the base-extended divisors $\bar{E}$ and $\bar{X}_v$, and descends by Galois-invariance.) Any $G \in \text{Div} X$ intersects the generic fiber with degree $(G \cdot X_K) = \alpha \cdot b(G)$ a multiple of $\alpha$; by replacing $G$ by $G - b(G)\Omega$, we can force $G$ into $\text{Div}^0(X)$.

**Lemma 4.2:** Write $D_\ell$ for the subgroup of $\text{Div} X$ of divisors linearly equivalent to zero. Then (via the map $\bar{f}$ defined in the proof)

$$\text{Div} X/(\mathcal{D} + D_\ell) = A(K).$$

**Proof:** As we saw above, $\text{Div} X/Z \cdot \Omega = \text{Div}_0(X)$ so by an elementary isomorphism theorem

$$\text{Div} X/(\mathcal{D} + D_\ell) = \text{Div}_0(X)/(\text{Div}_{\text{vert}}(X) + D_\ell).$$
Define a map \( \tau : \text{Div}_0(X) \to A(K) \) by \( \tau(D) = [DK] \), the linear equivalence class as divisor on \( X_K \) of the degree-zero divisor \( D_K = D \cap X_K \). Notice that \( \tau(D) = \tau(E) \iff D_K \sim E_K \). By regularity of the surface \( X \) this holds \( \iff D \sim E + W \) for some vertical divisor \( W \). We have shown \( \tilde{\tau} \) injective; surjectivity is obvious, as if \( \delta = [D_K] \in A(K) \), \( \delta = \tau(D_K) \) for \( \tilde{D}_K \in \text{Div}_0(X) \) the Zariski closure on \( X \) of \( D_K \).

**Lemma 4.3:** \( \text{Div}^0(X) \cap \text{Div}_{\text{vert}}(X) = \text{Div}^0(V) \).

**Proof:** The inclusion \( \text{Div}^0(V) \subset \text{Div}^0(X) \cap \text{Div}_{\text{vert}}(X) \) is clear. Say that the vertical divisor \( W \) is in \( \text{Div}^0(X) \). Then for all \( v \) and \( a \) we have \( 0 = (W \cdot X_v) \), which implies by negative semi-definiteness of the intersection matrices of partial fiber components that \( W = \Sigma s_v X_v \) is a rational linear combination of complete fibers, i.e. \( W \in \text{Div}_0(V) \). Also \( 0 = (W \cdot \Omega) \), which shows that \( \Sigma s_v = 0 \). Fix a particular degenerate fiber \( X_0 \). Modifying \( W \) by divisors in \( \text{Div}^0(V) \) of the form \( X_v - X_0 \), we may assume that \( W = \sum r_v X_v \), \( 0 \leq v \leq h \), is a rational linear combination of the degenerate fibers with \( 0 \leq r_v < 1 \) for all \( v > 0 \). Consider

\[
\pi : \tilde{X} \to X \otimes V \Omega \to X
\]

where \( \tilde{X} \) is a desingularization of \( X \otimes V \Omega \) achieved by blowings-up, with no contraction of exceptional fibers of the first kind. Since \( W \approx 0 \), \( \pi^* W = \pi^*(\Sigma r_v X_v) \) is also equivalent to zero. If \( 0 < r_v \), \( |r_v X_v| = |X_v| \). Because of this, for \( 0 < r_v < 1 \), \( |\pi^*(r_v X_v)| = |\pi^*(X_v)| = |\sum \tilde{X}_w| \), the sum of fibers at points \( w \in \Omega \) over \( v \). The divisor \( \pi^* W \) is then a rational linear combination of the degenerate fibers \( \tilde{X}_w \); in fact because \( \tilde{X} \to \Omega \) admits a section (already \( X \otimes V \Omega \to \Omega \) does) by Proposition 3.1 we see that the divisor \( \pi^* W \) is an integral linear combination of the \( \tilde{X}_w \).

Consider a component \( C \) of multiplicity \( p \) in \( X_v \). \( \pi^* C = \Sigma \tilde{C}_w \); \( \tilde{C}_w \) occurs with multiplicity \( p \) in each fiber \( \tilde{X}_w \) over \( X_v \). If \( 0 \leq r_v < 1 \), in the divisor \( W = \Sigma r_v X_v \), \( C \) appears with multiplicity \( a = r_v p \), and we see that \( 0 \leq a < p \). In the divisor \( \pi^* W \), the lifted component \( \tilde{C}_w \) occurs in the fiber \( \tilde{X}_w \) with the multiplicity \( a \). For \( \pi^* W \) to contain \( \tilde{X}_w \) with integral multiplicity, \( p \) must divide \( a \); \( a \) must be zero, whence \( r_v = 0 \). This shows that for \( 0 < v \leq h \), \( r_v = 0 \). By the sum formula, \( r_0 = 0 \) also. In other words, \( W \) was in \( \text{Div}^0(V) \).

2. The \( K/k \) trace. Given a field \( k \) - arbitrary for the moment - we say that the extension of fields \( K/k \) is *primary* if the algebraic closure of \( k \) inside \( K \) is purely inseparable over \( k \). This happens if \( K \) is a function
field with field of constants $k$. Given a primary extension $K/k$ and an
abelian variety $A$ defined over $K$, we consider triples $(J, A, j)$ of an
abelian variety $J$ defined over $k$, the given $K$-abelian variety $A$, and
a map defined over $K$

$$j : J \to A.$$ 

With the obvious morphisms, such triples form the category of
$k$-abelian varieties over $A$.

**Definition:** A $K/k$ trace of $A$ is a final object $(B, A, \tau)$ of this
category, provided that $\tau : B(\bar{k}) \to A(K\bar{k})$ has finite kernel.

By Chow's theorem that an abelian variety contains no moving
family of subvarieties, one sees that the $K/k$ trace always exists and
is unique [11, §8.3]. Further it is independent of ground field in the
sense that for $E/k$ an extension independent of $K/k$, $\text{Tr}_{K/k}A = \text{Tr}_{E/K}\text{Tr}_{K/k}A$
for $A$ considered as abelian variety over $EK$. Writing this trace as $B$, we also have $A(K) \cap \tau(B(E)) = \tau(B(k))$. [12, §5.1, Prop. 2].

**Proposition 4.4:** The abelian variety $A = \text{Jac}(X_K)$ has for $K/k$
trace (an abelian variety which is purely inseparably isogenous to) the
quotient abelian variety $\text{Pic}^\times_X/\text{Pic}^\circ_Y$.

**Proof:** By stability of trace under base-field extension we may
work over $\bar{k}$ on the base-extended varieties $\bar{X}$, $\bar{V}$. (Sloppily, we still
write them as $X$, $V$.) The map $\tau : \text{Div}_0(X) \to A(K\bar{k})$ in the proof of
Lemma 4.2 defines (by Lemma 4.3) an injection of groups
$\text{Div}^\circ(X)/(\text{Div}^\circ(V) + D_\ell) \hookrightarrow A(K\bar{k})$. By the universal mapping property
of the Picard variety, $\tau$ defines a morphism of abelian varieties from
$\text{Pic}^\times_X$ into $A$, which clearly factors through a morphism

$$\lambda : \text{Pic}^\times_X/\text{Pic}^\circ_Y \to A.$$ 

Notice that $\lambda$ induces an injection on $\bar{k}$-valued points. We also define
a map in the other direction. By regularity of $X$,

$$\pi : A(K\bar{k}) \to \text{Div} X/(\text{Div}_{\text{vert}}(X) + D_\ell) \cong \text{Pic} X/\text{Pic}_{\text{vert}} X$$

is well-defined by taking $\delta = [D_\beta] \in A(K\bar{k})$ to the class represented by
$\bar{D}_\beta$, the Zariski closure on $X$ of the $X_K$-divisor $D_\beta$.

Now let $(J, A, j)$ be a $\bar{k}$-abelian variety over $A$. Write $c$ for the map
defined on $J(\bar{k})$ by $\pi \circ j$. Under specialization we get for each $v \in |V|$ a
morphism

\[ j_v : J \rightarrow A \xrightarrow{\text{sprn}} A_v \]

of \( J \) into the algebraic group scheme \( A_v = \text{Pic}_{X_v} \); by connectedness \( j_v \) maps \( J \) into the connected component of identity \( A_v^0 \), whose \( \bar{k} \)-valued points correspond to divisor classes on \( X_v \) having degree 0 as divisor on each partial fiber component. Since \( j_v(\alpha) = [c(\alpha) \cap X_v] \), for any point \( \alpha \) in \( J(\bar{k}) \) we see that \( c \) actually maps \( J(\bar{k}) \) into \( \text{Pic}_* X/(\text{Pic}_{\text{vert}}(X) \cap \text{Pic}_* X) \), where \( \text{Pic}_* X \) denotes the set of divisor classes intersecting each partial fiber component with degree 0. But \( \text{Pic}_{\text{vert}}(X) \cap \text{Pic}_* X \) is just \( \text{Pic}_{\mathcal{O}}(V) \), the image under \( \text{Div} X \rightarrow \text{Pic} X \) of \( \text{Div}_\mathcal{O}(V) \). We have defined a map

\[ c : J(\bar{k}) \longrightarrow \frac{\text{Pic}_* X}{\text{Pic}_{\mathcal{O}}(V)} \subset \frac{\text{Pic} X}{\text{Pic}_{\mathcal{O}}(V)}. \]

Consider the diagram below, in which the middle row (degree-2 terms of the Leray Spectral Sequence for \( f_* G_m \)) is exact because \( \text{Br} V \) is zero for a complete curve \( V \) over an algebraically closed field, and the cokernel in the middle column is, by the Theorem of the Base [12] a finitely generated abelian group.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Pic}^0 V & \leftarrow & \text{Pic}^0 X & \rightarrow & \text{Pic}^0 X/\text{Pic}^0 V & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Pic} V & \xrightarrow{i^*} & \text{Pic} X & \rightarrow & H^0(V, R^1 f_* G_m) = \text{Pic}(X/V) & \rightarrow & 0 \\
\downarrow & \text{deg} & \downarrow & q & & & & & \\
0 & \rightarrow & Z & \xrightarrow{i} & Z^\gamma/\mathcal{K} & \rightarrow & \mathcal{E} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

The map \( i : Z \rightarrow Z^\gamma/\mathcal{K} \) is injective. In fact, choosing any fixed point \( v \in |V| \), we see that the (dotted) map \( n \mapsto [n \cdot v] \in \text{Pic} V \) splits the first column. For \( i(n) = 0 \) we would need \( q(f^*[n \cdot v]) = 0 \), or that \( f^*[n \cdot v] \in \text{Pic}^0 X \). That would imply that \( 0 = (\Omega \cdot nX_v) = n\alpha \), so \( n = 0 \). By the snake lemma we get the exact sequence of cokernels

\[
0 \rightarrow \text{Pic}^0 X/\text{Pic}^0 V \rightarrow \text{Pic}(X/V) = \text{Pic} X/\text{Pic} V \rightarrow \mathcal{E} \rightarrow 0
\]
with \( \mathcal{C} \) a discrete group. Since the map \( \text{Pic} X/\text{Pic} V \to \text{Pic} X/\text{Pic}_0(V) \) is surjective and, by Lemma 4.3, \( \text{Pic}^0 X \cap \text{Pic}_0(V) = \text{Pic}^0 V \) we get a similar exact sequence

\[
0 \to \text{Pic}^0 X/\text{Pic}^0 V \to \text{Pic} X/\text{Pic}_0(V) \to \mathcal{C}' \to 0
\]

with another discrete group \( \mathcal{C}' \). As the group \( J(\bar{k}) \) is divisible, \( c \) maps \( J(\bar{k}) \) into \( \text{Pic}^0 X/\text{Pic}^0 V = (\text{Pic}^0\mathcal{X}/\text{Pic}^0\mathcal{Y})(\bar{k}) \).

Apply all this to \((B, A, \rho)\), a \( K/k \) trace of \( A \). By definition of trace, there is a morphism \( \alpha: \text{Pic}^0\mathcal{X}/\text{Pic}^0\mathcal{Y} \to B \) for which \( \rho \circ \alpha = \lambda \). Writing, e.g., \( \bar{\alpha} \) for the map induced on \( k \)-valued points by a morphism of abelian varieties \( \varphi \), we see that \( \bar{\lambda} = \bar{\rho} \circ \bar{\alpha} \). Since \( \bar{\lambda} \) is injective so is \( \bar{\alpha} \); i.e. \( \alpha \) is purely inseparable. By the argument above, the map \( \bar{\rho}: B(\bar{k}) \to A(K\bar{k}) \) factors through \( \bar{\lambda} \), i.e. there is a map \( \bar{\beta}: B(\bar{k}) \to (\text{Pic}^0\mathcal{X}/\text{Pic}^0\mathcal{Y})(\bar{k}) \) for which \( \bar{\rho} = \bar{\lambda} \circ \bar{\beta} \). This shows that \( \bar{\rho} = \bar{\rho} \circ \bar{\alpha} \circ \bar{\beta} \), i.e. that \( \bar{1}_B - \bar{\alpha} \circ \bar{\beta} \) maps \( B(\bar{k}) \) into the finite group \( \text{ker} \bar{\rho} \). By divisibility of \( B(\bar{k}) \) we see that \( \bar{1}_B - \bar{\alpha} \circ \bar{\beta} \) is the zero map. Thus \( \bar{\alpha} \) is also surjective, and we conclude that \( \alpha \) is an isogeny. ■

3. The structure of \( \text{NS}(X) \). Recall that \( \text{NS}(X) \) denotes the group of algebraic equivalence classes of divisors on \( X \); \( \text{NS}_{\text{vert}}(X) \) is the image in \( \text{NS}(X) \) of \( \text{Div}_{\text{vert}}(X) \subset \text{Div} X \).

**Proposition 4.5:** Write \( \mathcal{N} \) for \( \text{NS}_{\text{vert}}(X) \oplus \mathbb{Z} \cdot \Omega \). \( \text{NS}(X) \) has a filtration by subgroups

\[
0 \to f^* \text{NS}(V) \subset \text{NS}_{\text{vert}}(X) \subset \mathcal{N} \subset \text{NS}(X)
\]

with respective quotients \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{Z} \), and \( A(K)/B(k) \), where \( Q \) is the group of partial vertical fibers,

\[
Q = \bigoplus_v \frac{\mathbb{Z} X_v}{X_v}
\]

By the Mordell–Weil Theorem \( A(K) \) is finitely generated as an abelian group because \( K \) is finitely generated over the prime field. As \( B \) is an abelian variety defined over a finite field, \( B(k) \) is a finite group.

**Corollary:** \( \text{NS}(X) \) is a finitely generated abelian group of rank \( \rho(X) = 2 + r + \Sigma(h_v - 1) \), where \( r = \text{rank of } A(K) \) and \( h_v \) is the number of \( k(v) \)-rational components of the fiber \( X_v \).
Proof of Proposition 4.5: We break up the filtration of Lemma 4.1 into short exact sequences which we read modulo algebraic equivalence.

1. \( f^* \text{Div} V \approx \approx \mathbb{Z} \).

We prove that for \( E \) a divisor on \( V \), \( f^* E \approx 0 \) on \( X \) if and only if \( E \approx 0 \) on \( V \). But \( f^* E \approx 0 \Rightarrow (f^* E \cdot \Omega) = 0 \Rightarrow \alpha \deg E = 0 \Rightarrow \deg E = 0 \), i.e. \( E \approx 0 \). By pullback, \( E \approx 0 \Rightarrow f^* E \approx 0 \).

2. The following exact sequence holds:

\[
0 \to \mathbb{Z} = f^* \text{NS}(V) \to \text{NS}_{\text{vert}}(X) \to Q \to 0.
\]

In fact our favorite group isomorphism shows

\[
\frac{\text{NS}_{\text{vert}}(X)}{f^* \text{NS}(V)} \approx \frac{\text{Div}_{\text{vert}}(X)}{(f^* \text{Div} V + \text{Div}^\circ X) \cap \text{Div}_{\text{vert}}(X)}.
\]

By Lemma 4.3 the denominator is \( f^* \text{Div} V + f^* \text{Div}^\circ V = f^* \text{Div} V \), so the cokernel is as claimed.

3. \( \mathcal{N} \), the image in \( \text{NS}(X) \) of \( \text{Div}_{\text{vert}}(X) \oplus \mathbb{Z} \Omega \), is the direct sum \( \text{NS}_{\text{vert}}(X) \oplus \mathbb{Z} \Omega \).

In fact, for \( W \in \text{Div}_{\text{vert}}(X) \), \( W + n\Omega \approx 0 \) on \( X \Rightarrow ((W + n\Omega) \cdot X_K) = 0 \). But this degree is \( n\alpha = n(\Omega \cdot X_K) \), so \( n = 0 \) and already \( W \approx 0 \).

4. In the exact sequence

\[
0 \to \mathcal{N} \to \text{NS}(X) \to \text{Cokernel} \to 0
\]

the cokernel is exactly \( A(K)/B(k) \).

By the isomorphism theorem

\[
\frac{\text{NS}(X)}{\mathcal{N}} \approx \frac{\text{Div} X}{\text{Div}_{\text{vert}}(X) + \mathbb{Z} \Omega + \text{Div}^\circ X} \approx \frac{\text{Div}_0(X)}{\text{Div}_{\text{vert}}(X) + \text{Div}^\circ X + D_\mathcal{E}}
\]

since \( D_\mathcal{E} \subset \text{Div}^\circ(X) \). This last group is

\[
\frac{\text{Div}_0(X)/(\text{Div}_{\text{vert}}(X) + D_\mathcal{E})}{\text{Div}^\circ(X)/(\text{Div}_{\text{vert}}(X) + D_\mathcal{E})} \approx \frac{A(K)}{B(k)}
\]

by the results of Lemma 4.2 and Proposition 4.4.

As a complement we can give, up to an explicit finite index, a set of generators of \( \text{NS}(X) \) modulo torsion. For \( \alpha \in A(K) \) we will say that a divisor \( \mathcal{A} \) on \( X \) represents \( \alpha \) if in \( A(K) \), \( \alpha = [\mathcal{A} \cap X_K] \).
Proposition 4.6: Let $r$ be the rank of $A(K)$ and choose divisors $A_1, \ldots, A_r$ representing $\alpha_1, \ldots, \alpha_r$, a basis of $A(K)$ modulo torsion. Let $F$ be any complete fiber of $f$. Write $N'$ for the subgroup of $N$ generated by $F$, $\Omega$, and $\{X_{va} \text{ for all } v, \text{ and, for each } v, \text{ all } a > 1\}$. Then $N'$ is a free subgroup of $N$, and $N'$ and $\{A_i\}$ generate a $\mathbb{Z}$-submodule of finite index

$$\frac{[A(K)_{\text{tor}}]}{[B(k)]} \cdot \prod_{A_i} \frac{d_{m_{a1}}}{[\text{NS}(X)_{\text{tor}}]}$$

in $\text{NS}(X)$ modulo torsion.

Proof: The fiber $F$ is a generator of $f^* \text{NS}(V) = \mathbb{Z}$. To show freedom for $N'$ it suffices to show that if $D = aF + b\Omega + \sum \sigma_{a>1} c_{va} X_{va} = 0$ then $a$, $b$, and all the $c_{va}$ are zero. Let $G$ be any complete fiber. $D - G = 0 \Rightarrow (D - G) = 0$. But $(D - G) = b(\Omega - G) = b\alpha$, so $b = 0$. Next fix a particular degenerate fiber $X_v$. There is a partial fiber component $X_{va}$ intersecting $\Sigma c_{va} X_{va}$ with strictly negative multiplicity since the sum is not a rational multiple of the fiber $X_v$. But $(D \cdot X_{vr}) = (\Sigma c_{va} X_{va}) \cdot X_{vr}$, a negative number, a contradiction unless the $c_{va}$ are all zero. We have shown that $D$ must be just $aF$. Intersecting with $\Omega$, we see $a$ is also zero.

Define another subgroup $N''$ by the equation $N'' \oplus \mathbb{Z} \Omega = N'$. Then

$$\frac{N}{N''} \cong \frac{\text{NS}_{\text{vert}}(X)}{\text{NS}_{\text{vert}}(X)/f^* \text{NS}(V)} \cong \frac{Q}{\text{NS}''/f^* \text{NS}(V)}$$

where, as earlier, $Q$ is the group of partial fibers. Since $N''$ misses the components $X_{v1}$, $\text{NS}''/f^* \text{NS}(V)$ has index $\prod d_{m_{a1}}$ in $Q$.

Define a new group $A'$ by exactness of row $A$ of the following diagram; row $B$ is exact by Proposition 4.5.

\[
\begin{array}{cccccc}
A. & 0 & \rightarrow & N' & \rightarrow & \text{NS}(X) & \rightarrow & A' & \rightarrow & 0 \\
& & & \downarrow & & \| & & \downarrow & & \\
B. & 0 & \rightarrow & N & \rightarrow & \text{NS}(X) & \rightarrow & A(K)/B(k) & \rightarrow & 0 \\
\end{array}
\]

By the snake lemma we find that $A'$ is an extension of $A(K)/B(k)$ by the finite group $N'/N''$. Consequently the basis $\alpha_1, \ldots, \alpha_r$ of $A(K)$ modulo torsion is also a basis of $A'$ modulo torsion. Applying
Tor(*, Q/Z) to row A gives the long exact sequence

\[ 0 \to N'_{\text{tor}} \to NS(X)_{\text{tor}} \to A'_{\text{tor}} \]
\[ \to N' \otimes Q/Z \to NS(X) \otimes Q/Z \to A' \otimes Q/Z \to 0, \]

which demonstrates that \( N' \) and \( \{A_i\} \) generate a submodule of \( NS(X) \) modulo torsion of index

\[
\frac{[A'_{\text{tor}}]}{[NS(X)_{\text{tor}}]} = \prod_v d_v m_v \cdot \frac{[A(K)_{\text{tor}}]}{[NS(X)_{\text{tor}}]} \cdot \frac{[B(K)]}{[NS(X)_{\text{tor}}]},
\]

where the value \([A'_{\text{tor}}]\) comes from the diagram. ■

5. The intersection matrix

Given any set \( \{F_1, \ldots, F_r\} \) of divisors on \( X \), we write \( (F_1, \ldots, F_r) \) for the absolute value of the determinant of the associated intersection matrix. In this section, we will approximate \( (D_1, \ldots, D_{p(X)}) \) for a basis \( \{D_1, \ldots, D_{p(X)}\} \) of \( NS(X) \) modulo torsion. To do this, we interpret Proposition 4.6 as providing a set of approximate generators of \( NS(X) \) modulo torsion. Throughout, unless otherwise specified, \( |X_{v_a} \cdot X_{v_b}| \) is to be understood as \( (X_{v_2}, \ldots, X_{v_n}) \).

1. The product formulation. Call a divisor \( D \in \text{Div}_v(X) \) good if it intersects each \( k(v) \)-irreducible component of each closed fiber \( X_v \) with total multiplicity 0. Tate has found a characterization of the height pairing on \( A(K) \) modulo torsion:

Let \( A, B \) be good divisors on \( X \) representing \( \alpha, \beta \in A(K) \) modulo torsion. Then \( \langle \alpha, \beta \rangle = -(A \cdot B) \log q \).

PROPOSITION 5.1: We have the equality

\[
\langle \Omega; F; A_1, \ldots, A_r; \{X_{v_a}\} \rangle = \alpha^2 \frac{|\det(\alpha_i, \alpha_j)|}{(\log q)^{2}} \cdot \prod_v |X_{v_a} \cdot X_{v_b}|.
\]

PROOF: Set \( d = \text{lcm} \{|X_{v_a} \cdot X_{v_b}|\} \) as \( v \) ranges over the degenerate fibers. The divisors \( A_i \) representing the basis elements \( \alpha_i \) of \( A(K) \) modulo torsion are all in \( \text{Div}_v(X) \). In fact we can modify the multiples \( dA_i \) by vertical divisors to get good divisors \( A_i \) working fiber by fiber. To make \( dA_i \) good for the fiber \( X_v \), since \( A_i \) is in
Div\(_d(X)\) we merely need to solve the equations

\[
\left( d \mathcal{A}_i + \sum_{a > 1} b_a X_{va} \right) \cdot X_{vj} = 0 \quad 1 < j \leq h_v
\]

By Cramer’s rule this system has an integral solution for each bad \( v \) because \( d_v \) divides \( d \). We find that

\[
(\Omega; F; \{\mathcal{A}'_i\}; \{X_{va}\}) = (\Omega; F; \{d \mathcal{A}_i\}; \{X_{va}\}) = d^{2r}(\Omega; F; \{\mathcal{A}_i\}; \{X_{va}\}).
\]

Since the \( \mathcal{A}'_i \) were all chosen good, the intersection matrix \((\Omega; F; \{\mathcal{A}'_i\}; \{X_{va}\})\) is of the form in the diagram, where empty regions stand for zero matrices.

\[
\begin{array}{cccccc}
\Omega & F & \{\mathcal{A}'_i\} & \{X_{va}\} & \cdots & \{X_{za}\} \\
\Omega & \Omega^2 & \alpha & \star & \cdots & \star \\
F & \alpha & 0 & \star & \cdots & \star \\
\{\mathcal{A}'_i\} & & & \star & \cdots & \star \\
\{X_{va}\} & & & & \star & \cdots \\
\cdot & & \cdot & \cdot & \cdot & \cdot \\
\{X_{za}\} & & & \star & \cdots & \star \\
\end{array}
\]

By Tate’s criterion, \((\mathcal{A}'_i \cdot \mathcal{A}'_j) = (d \alpha_i, d \alpha_j)/\log q\). This matrix then has absolute value of its determinant

\[
\alpha^2 |(\mathcal{A}'_i, \mathcal{A}'_j)| \cdot \prod_v |X_{va} \cdot X_{vb}| = d^{2r} \alpha^2 \frac{\det(\alpha, \alpha)}{(\log q)^r} \prod_v |X_{va} \cdot X_{vb}|
\]

whence the lemma. \( \blacksquare \)
2. Relation to Néron models. For the moment let $(X, V, f)$ be a local model for $X_K$ over $V$, with $A_V$ the Néron model over $V$ for $A$, the Jacobian of $X_K$. The following theorem [22; §8.1.2] links the structure of the closed fiber $X_v$ of $X$ with that of the fiber $A_v$ of the Néron model at $v \in |V|$.

**Theorem 5.2** (Raynaud): Let $(\bar{X}, \bar{V}, \bar{f})$ be a strictly local model with closed fiber $X_s = d, R_s = \sum p_j X_j$, $1 \leq j \leq n$.

1. There is a non-exact complex of abelian groups

$$0 \to D_0(\bar{X}) \xrightarrow{i} D(\bar{X}) \xrightarrow{\alpha} \mathbb{Z}^n \xrightarrow{\beta} \mathbb{Z} \to 0$$

where the maps are defined as

$$i: X_s \mapsto \sum p_j X_j;$$

$$\alpha: E \mapsto (X_1 \cdot E), (X_2 \cdot E), \ldots, (X_n \cdot E);$$

and

$$\beta: (a_1, \ldots, a_n) \mapsto \sum a_j p_j$$

is "total degree".

$D(\bar{X})$ denotes the group of divisors supported on the closed fiber, and $D_0(\bar{X})$ is the subgroup of principal divisors in $D(\bar{X})$.

2. $\ker \alpha/\text{im } i = \mathbb{Z}/d_v$.

3. If $f$ is cohomologically flat (in dimension zero), $\ker \beta/\text{im } \alpha = A_s/A_s^0$.

By negative semi-definiteness of the intersection matrix of partial fiber components it is clear that $\ker \alpha = \mathbb{Z} \cdot R_s$ and that we can express $	ext{im } \alpha$ by the exact sequence

$$(\text{Im}) \quad 0 \to \mathbb{Z} \cdot R_s \to \bigoplus X_j \to \text{im } \alpha \to 0.$$

We restate (3) of this theorem in a form more convenient for later use.

**Theorem 5.2':** Let $\bar{V}$ be a strictly local scheme with closed point $s$. To a strictly local model $(\bar{X}, \bar{V}, \bar{f})$ is associated an exact sequence of abelian groups

$$(*) \quad 0 \to \text{im } \alpha_s \to \ker \beta_s \to A_s/A_s^0 \to 0$$

which expresses the structure of the (group of $\bar{k}$-rational points on

---

the fiber $A_v$ of the Néron model in terms of the closed fiber $X_v$, provided that the map $\tilde{f}$ is cohomologically flat (in dimension zero).

Now return to the $k$-fibration $(X, V, f)$ for which $A = \text{Jac}(X_K)$ has Néron model $A_V$ over $V$. To study the fiber $A_v$ of $A_V$ at $v$ we can restrict our attention to the associated local model $(X(v), V(v), f)$ and to the associated strictly semi-local model $(X(v) \times_k \kbar, V(v) \times_k \kbar, \tilde{f})$. This semi-local model has deg$(v)$ (disjoint) closed fibers, which are just the geometric fibers $X_v$ of $(\overline{X}, \overline{V}, \overline{f})$ lying above the closed fiber $X_v$. For each (geometric) fiber $X_v$ there is an exact sequence $(\ast \mu)$ and thus we get an associated sequence for the strictly semi-local model

$$(\bigoplus \ast \mu) \quad 0 \to \bigoplus \text{im} \alpha \to \bigoplus \ker \beta \to \bigoplus A_v / A_v^0 \to 0.$$ 

Write the maps $\bigoplus \alpha, \bigoplus \beta$ as $\tilde{\alpha}, \tilde{\beta}$. Since $G(k/k)$ permutes the set of $k$-irreducible components of $X_v \times_k \kbar$ it acts on $\text{im} \tilde{\alpha}, \ker \tilde{\beta}$. Write $\text{im} \alpha, \ker \beta$ for $G(k/k)$-fixed elements.

**Proposition 5.3:** $[A_v : A_v^0] = [\ker \beta : \text{im} \alpha] \cdot \delta_v$ where $\delta_v$ is some integer dividing $\Delta_v$.

**Proof:** Taking Galois cohomology of $(\bigoplus \ast \mu)$ gives an exact sequence

$$0 \to \text{im} \alpha \to \ker \beta \to H^0(G, \bigoplus A_v / A_v^0) \to C \to 0$$

of groups for some subgroup $C$ of $H^1(G, \text{im} \tilde{\alpha})$. The lemma follows by calculations that $H^1(G, \text{im} \tilde{\alpha}) = \mathbb{Z} / \Delta_v$ and $H^0(G, \bigoplus A_v / A_v^0) = A_v / A_v^0$, both calculations are made from the Hochschild–Serre Spectral Sequence.

Consider a Galois module of a particular type: for $H$ a subgroup of the abelian group $G$ and $M$ an $H$-module, we define a $G$-module $\bigoplus M$, the direct sum extended over $G/H$, by having $G/H$ act merely by permuting components, and having $H$ act on each component by its original action on $M$. Such a $G$-module is induced by the $H$-module $M$. For a $G$-module induced by the $H$-module $M$, Shapiro’s lemma holds: $H^p(G, \bigoplus M) = H^p(H, M)$. In fact in the Hochschild–Serre Spectral Sequence $H^p(G/H, H^q(H, \bigoplus M)) \Rightarrow H^{p+q}(G, \bigoplus M)$ the groups $H^q(H, \bigoplus M) = \bigoplus H^q(H, M)$ are coinduced as $G/H$-modules: $G/H$ merely permutes the components. But then the spectral sequence degenerates to $H^0(G/H, \bigoplus H^q(H, M)) = H^q(G, \bigoplus M)$, and the first group is just $H^q(H, M)$. 

All the Galois modules we deal with are induced. Write $G'$ for $G(k(v))$. $\oplus A_{\mu}/A_{\mu}^\circ$ is induced by the $G'$-module $A_{\mu}/A_{\mu}^\circ$; $\text{im} \bar{\alpha}$ is induced by the $G'$-module $\text{im} \alpha_{\mu}$; and after rewriting the exact sequence of $G'$-modules (Im) above as

$$(\text{Im}) \quad 0 \to Z R_{\mu} \xrightarrow{\phi_{\mu}} \bigoplus_a Z X_a = \bigoplus_a \mathcal{Z}_a \to \text{im} \alpha_{\mu} \to 0,$$

we recognize the $G'$-module $\bigoplus \mathcal{Z}_a$ as a direct sum of modules induced by $G_a$-modules for various subgroups $G_a$ of $G'$. We now compute:

(1) \quad $H^0(G, \oplus A_{\mu}/A_{\mu}^\circ) = A_{\mu}/A_{\mu}^\circ$.

By Shapiro's lemma, $H^0(G, \oplus A_{\mu}/A_{\mu}^\circ)$ is $H^0(G', A_{\mu}/A_{\mu}^\circ)$. By Lang's Theorem, this is $A_{\mu}/A_{\mu}^\circ(k(v))$, isomorphic by Lemma 5.6 to the group of components $A_{\mu}/A_{\mu}^\circ$.

(2) \quad $H^1(G, \text{im} \bar{\alpha}) = Z/\Delta_v$.

First, $H^1(G, \text{im} \bar{\alpha}) = H^1(G', \text{im} \alpha_{\mu})$. Take cohomology of the exact sequence (Im). $H^1(G', \bigoplus \mathcal{Z}_a) = \bigoplus H^1(G_a, Z X_a) = 0$. Similarly, $H^2(G', Z R_{\mu}) = H^2(G', Z)$ is known to be $Q/Z$, and $H^2(G', \bigoplus \mathcal{Z}_a) = \bigoplus_a Q/Z$. The map $H^2(\varphi_{\mu})$: $Q/Z \to \bigoplus_a Q/Z$ is given by $\varphi_{\mu}: n \mapsto (n \cdot m_1 q_1, \ldots, n \cdot m_h q_h)$ since $G_a$ has index $q_a$ in $G'$ and $\varphi_{\mu}$ takes $R_{\mu}$ to $(m_1 X_1, \ldots, m_h X_h)$. $H^1(G', \text{im} \alpha_{\mu})$, the kernel, is $Z/\gcd(m_a q_a) = Z/\Delta_v$ by definition. \hfill \blacksquare

In the next lemma, the explicit form of the map $\bar{\alpha}$ provides a numerical value for $[\ker \beta : \text{im} \alpha]$.

**Lemma 5.4:** $[\ker \beta : \text{im} \alpha]$ and $|X_{va} \cdot X_{eb}|$ satisfy

$$[\ker \beta : \text{im} \alpha] \cdot \frac{\prod_a q_a \deg(v)}{\deg(v)} = \frac{\Delta_v}{(m_v)^2} |X_{va} \cdot X_{eb}|.$$

**Proof:** Write part of Raynaud’s complex as

$$(\#_{\mu}) \quad 0 \to \text{im} \alpha_{\mu} \hookrightarrow Z^n \xrightarrow{(1/d_{\mu}) \beta_{\mu}} Z \to 0$$

where the last map, $(a_1, \ldots, a_n) \mapsto \sum a_m j$, is surjective by definition of $d_{\mu}$. Summing as usual over the geometric fibers of the semi-local
model and using $(\text{Im})$ to express $\text{im} \: \alpha_\mu$, we get the complex of $G(\bar{k}/k)$-modules

$$0 \to \bigoplus \mathbb{Z} X_{\mu a} \bigotimes \mathbb{Z} R_\mu \to \mathbb{Z}^n \to \mathbb{Z} \to 0$$

where the non-indexed sums are over the fibers $X_\mu$ of the semi-local model. Galois fixed-part yields the complex (5.1) below. To extract information on $[\ker \beta : \text{im} \: \alpha]$ notice that $D' = \bigoplus_{a>1} \mathbb{Z} X_{va}$, the free $\mathbb{Z}$-module generated by all the $k(v)$-components of $X_v$ except the first one, has index $m_{v1}$ in $\text{im} \: \alpha$.

$$0 \longrightarrow \text{im} \: \alpha = \bigoplus \mathbb{Z} X_{va} \bigotimes \mathbb{Z} R_v \xrightarrow{\alpha} \mathbb{Z}^h \xrightarrow{d_v} \mathbb{Z} \longrightarrow ?$$

$$D' = \bigoplus_{a>1} \mathbb{Z} X_{va} \xrightarrow{\delta} \mathbb{Z}^h \xrightarrow{\pi_1} \mathbb{Z} \longrightarrow ?$$

(5.1)

The map $\alpha$, induced by $\tilde{\alpha}$ on $(\text{im} \: \tilde{\alpha})^G$, has matrix

$$\begin{pmatrix} 1 \\
\frac{1}{q_a \cdot \deg(v)} X_{va} \cdot X_{vb} \end{pmatrix} \quad 1 \leq a, b \leq h_v$$

and $\beta$, induced by $\tilde{\beta}$, is the map $(e_1, \ldots, e_h) \mapsto \Sigma p_a q_a e_a$. The map $\gamma$ is just a change of coordinates converting $\beta/d_v$ to projection on the first coordinate, and fixing all other coordinates. Commutativity of the diagram forces $\delta$ to have $h \times (h-1)$ matrix

$$\begin{pmatrix} 0 & \cdots & 0 \\
\frac{1}{q_a \cdot \deg(v)} X_{va} X_{vb} \end{pmatrix} \quad 1 \leq a, b \leq h_v$$

By linear algebra,

$$[\ker \pi_1 : \text{im} \: D'] = \left| \frac{1}{q_a \cdot \deg(v)} X_{va} \cdot X_{vb} \right| \quad 1 \leq a, b \leq h_v$$

(5.2)

Chasing around the commutative square, the index is also

$$[\ker \pi_1 : \text{im} \: D'] = m_{v1} [\ker \beta : \text{im} \: \alpha] \frac{m_{v1} q_1}{\Delta_v}$$

(5.3)

since the vector $(0, b_2, \ldots, b_h) \in \ker \pi_1$ is in the image of the map $\gamma \Leftrightarrow$
there is some $b_1$ with $0 = \Sigma m_a q_a b_a$, $1 \leq a \leq h_v$. This happens if and only if $m_1 q_1$ divides the sum $\Sigma m_a q_a b_a$, $1 < a \in h_v$. Combine (5.2) and (5.3) to prove the lemma. ■

Combining Lemma 5.4 with Propositions 5.1 and 5.3 we can give a more arithmetic interpretation to the determinant of the matrix.

PROPOSITION 5.5: The intersection determinant satisfies

$$(\Omega; F; \{\mathcal{A}_i\}; \{\mathcal{X}_{ab}\}) = \alpha^e \frac{\det(\alpha_i, \alpha_j)}{(\log q)^v} \prod_v |X_{oa} \cdot X_{ob}|$$

where for some integer $e_v$ dividing $\Delta_v^2$,

$$\frac{e_v}{(m_{e_v})^2} |X_{oa} \cdot X_{ob}|_{v < a, b} = [A_v : A_{e_v}] \prod_a q_a \frac{\deg v}{\deg v} 1 \leq a \leq h_v$$

LEMMA 5.6: Let $A$ be an algebraic group defined over $k$ which splits over $\bar{k}$ as

$$A \times_k \bar{k} = A_1 \times \cdots \times A_r$$

a product of transitively conjugate algebraic groups, defined over $k$, by necessity. Then $[A(k)] = [A_i(k_i)]$ for any choice of $i$.

PROOF: $A(k_r) = A_1(k_r) \times \cdots \times A_r(k_r)$; $A(k)$ is the fixed set under (arithmetic) Frobenius. Yet if $(P_1, \ldots, P_r)$ is a point of $A_1(k_r) \times \cdots \times A_r(k_r)$, $\varphi(P_1, \ldots, P_r) = (P_1^{(q)}, P_1^{(q)}, \ldots, P_r^{(q)})$ where $P^{(q)}$ denotes the point whose coordinates are $q$th-powers of those of $P$, and the numbering of the $A_i$ has been arranged so that $\varphi$ takes $A_1$ into $A_2$, etc. ■

6. The main Theorem

1. Deduction from the stronger result. After all this preparation we will prove the equivalence of the arithmetic and geometric conjectures. We actually prove that the equivalence amounts to an equality between orders of groups.

THEOREM 6.1: Let $k$ be a finite field, $K$ a function field in one variable over $k$ with complete nonsingular (geometrically irreducible)
model V. Let $X$ be a smooth surface defined over $k$ and equipped with a proper, flat, cohomologically flat (in dimension zero) $k$-rational map $f: X \to V$. If the generic fiber $X_K$ of $f$ (as curve defined over $K$) is nonsingular and geometrically irreducible, and contains a $K$-rational cycle of degree 1, then the conjectures $AT(X, k)$ and $BSD(A, K)$ are equivalent, where $A$ denotes the $K$-abelian variety $Jac(X_K)$.

Before stating the stronger version, we need a definition. We consider a geometric property of $S$ an algebraic $F$-scheme for an arbitrary field $F$. The property is said to hold locally for the $F$-étale topology at $s$, a closed point of $S$, if there is an $s$-punctured scheme $S(s)$ étale over $S$ for which the property holds. Of course, a reasonable property holds locally for the $F$-étale topology exactly when it holds for the ordinary henselization.

**LEMMA 6.2:** Let $(X, V, f)$ be a $k$-fibration; $f$ is of degree $\delta$ locally for the $k$-étale topology at $v$ exactly when $d_v \Delta_v = \delta$.

**PROOF:** Consider the strictly local model $(X(\mu), V(\mu), f(\mu))$ at a geometric point $\mu$ above $v$. If the closed fiber $X_v = \Sigma p_a X_a$, the geometric fiber (of the strictly local model) is $X_v = \Sigma p_a \Sigma X_{aj}$. Raynaud has shown [22, §7.1.1] the existence of a horizontal divisor $D_{aj}$ on $X(\mu)$ intersecting the closed fiber $X_v$ only on the component $X_{aj}$ and with $(D_{aj} \cdot X_{aj}) = p_a$. Consequently the strictly local model has degree divisible by $d_v = \gcd(p_a)$. On the other hand, a horizontal divisor $D$ on $X(\mu)$ is base-extended from a divisor defined locally for the $k$-étale topology if and only if it is $G(k/k)$-invariant. By the proof of Proposition 3.1, in $G$-invariant form Raynaud's result shows the existence at $v$ of locally $k$-étale divisors $D_v$ of degrees $p_a q_{va}$, whence there is a locally $k$-étale divisor at $v$ of degree $d_v \Delta_v$. By Proposition 3.1, we see that $X$ is locally of degree exactly $d_v \Delta_v$. □

**THEOREM 6.3:** Let $k, K, X,$ and $V$ be as in Theorem 6.1. Assume that the fibration $(X, V, f)$ is of degree $\alpha$ and that locally everywhere for the $k$-étale topology it is of degree 1. Then the equivalence, above,
of the Artin–Tate and Birch–Swinnerton-Dyer conjectures holds exactly when the equality

$$\alpha^2 [\text{Br}(X, k)] = [\text{III}(A, K)]$$

holds between the orders of the groups.

To deduce Theorem 6.1 from this result, just notice that the Zariski closure on $X$ of the given $K$-rational cycle on $X_K$ of degree 1 is a $k$-rational divisor of degree 1 on $X$, so $\alpha = 1$ and $(X, V, f)$ is locally everywhere of degree 1. By Theorem 6.3 we need only check the equality $[\text{Br}(X, k)] = [\text{III}(A, K)]$ which follows by Artin’s Theorem 2.3.

2. The Proof. Collected here are five results easily combined to prove Theorem 6.3. Of these, one depends purely on the structure of the degenerate fibers, two are global statements about the surface, and two relate to the structure of the Néron–Severi group. Proof of the two as-yet-unproven results follow that of the theorem.

**Proposition 6.4:** Let $S$ denote any finite set of closed points of $V$ containing all $v$ such that $X_v$ is degenerate or, for the given invariant differential form $\omega$ on $A_K$, the reduced form $\omega_v$ is not non-zero non-polar. Write $B$ for the $K|k$ trace of $A$. Then

$$P_2(X, q^{-s}) = (1 - q^{1-s})^2 P_1(B, q^{-s}) P_1(B, q^{1-s}) \frac{L(A, s)}{|\mu| \cdot d} \prod_{v \in S} \frac{f_{A(K_v)} |\omega_v| |\mu_v| \cdot \Pi_a (1 - N_v q_v^{a(1-s)})}{P_1(X_v, N_v)} \cdot (1 - N_v)^{-s}.$$

**Proposition 4.6:** The divisors $\Omega$; $F$ any complete vertical fiber; $\{S_i\}$; and $\{X_{va} \text{ for } a > 1\}$ generate a free submodule of $\text{NS}(X)$ modulo torsion of finite index

$$\frac{[A(K)]_{tor}}{[B(k)]} \cdot \frac{d_v m_v^{\alpha_1}}{[\text{NS}(X)]_{tor}}$$

and rank $\rho(X) = 2 + r + \sum (h_v - 1)$ inside that group.

**Proposition 3.3:** The $\zeta$-function of the closed fiber $X_v$ is

$$Z(X_v, T) = \frac{P_1(X_v, T)}{(1 - T) \cdot \Pi(1 - (N_v T)^{a_v})}$$
with the product taken for $1 \leq a \leq h_v$. Writing $A_v^0$ for the connected component of identity of $\text{Pic}_{X_v}$ the precise value holds:

$$P_1(X_v, N_v^{-1}) = [A_v^0(k(v))]/(N_v)^d.$$

**PROPOSITION 5.5:** The intersection determinant satisfies

$$(\Omega; F; \{\mathcal{A}_i\}; \{X_{v,a}\}) = \alpha^2 \frac{|\det(a_i, \alpha_i)|}{(\log q)^r} \prod_v |X_{v,a} \cdot X_{v,b}|$$

where for some integer $\epsilon_v$ diving $\Delta_v^2$,

$$\frac{\epsilon_v}{(m_v^a)} |X_{v,a} \cdot X_{v,b}|_{1 < a, b} = [A_v : A_v^0] \frac{\prod_a q_a \deg v}{\deg v} 1 \leq a \leq h_v$$

**PROPOSITION 6.5:** Write $\alpha(X) = \chi(X, \mathcal{O}_X) - 1 + \dim \text{Pic}_X$, and write $b$ for $\dim B$. Then

$$|\mu|^d q^{b+\deg \omega} = q^{\alpha(X)}.$$

**PROOF OF THEOREM 6.3:** Proposition 6.4 gives an explicit formula for $P_2(X, q^{-s})$. To prove the theorem, we merely need to let $s$ approach 1, combine terms, and interpret the various factors. As $s \to 1$, BSD$(A, K)$ tells us that

$$s - 1 \log q - \frac{1 - q^{-1-s}}{1 - q^{-1-s}}$$

Because $B$ is an abelian variety of dimension $b$ defined over $k$, we can interpret $P_1(B, T)$ – cf. the proof of Proposition 3.3 – as the characteristic polynomial of arithmetic Frobenius operating on $T_v(B(k)) \otimes_{\mathbb{Q}_v} \mathbb{Q}_v$. This interpretation yields the functional equation

$$P_1(B, T) = q^{bT^2} P_1(B, (qT)^{-1}).$$
Substituting $T = 1$, we find that

\begin{equation}
(6.1.3) \quad P_1(B, q^{-1})P_1(B, 1) = P_1(B, 1)^2/q^b = [B(k)]^2/q^b.
\end{equation}

Combining equations (6.1) and (6.1.1)-(6.1.3), and the “rank” statement of Proposition 4.6, we find that as $s \to 1$,

\begin{equation}
(6.2) \quad P_2(X, q^{-s}) \sim (1 - q^{1-s})^{\rho^{(X)}[B(k)]^2/\mu^d q^b} \cdot \prod_{v \in S} \frac{\int_{A(K_v)} |\omega|_{\mu^d v} \cdot \prod q_a \deg(v)}{P_1(X_v, N_v^{-1}) \deg(v)}
\end{equation}

In the final product factor, the factors $P_1(X_v, N_v^{-1})$ almost cancel with the integrals. Precisely, for $\omega'$ (possibly different from $\omega$) a differential form with nonzero nonpolar reduction at $v$,

\begin{equation}
\int_{A(K_v)} |\omega|_{\mu^d v} = \left| \frac{\omega}{\omega'} \right|_{A(K_v)} \int_{A(K_v)} |\omega'|_{\mu^d v}
\end{equation}

\begin{equation}
= \left| \frac{\omega}{\omega'} \right|_{A(K_v)} [A(K_v) : A^0(K_v)] [A^0(K_v) : A^1(K_v)] \int_{A^1(K_v)} |\omega'|_{\mu^d v}
\end{equation}

where $A^0(K_v)$, $A^1(K_v)$ are the subgroups of $A(K_v)$ reducing to nonsingular points, or the origin, on $A_v$. Under the uniformization of an abelian variety via local parameters at the origin $A^1(K_v)$ corresponds to the space $m^d_v$; also $A^0(K_v)/A^1(K_v)$ isomorphically reduces to $A^0(k(v))$, as does $A(K_v)/A^0(K_v)$ to the group $A_d/A^d_v$ of connected components of $A_v$. By Proposition 3.3,

\begin{equation}
(6.2.1) \quad \int_{A(K_v)} |\omega|_{\mu^d v} P_1(X_v, N_v^{-1}) = \left| \frac{\omega}{\omega'} \right|_{A_v} [A_v : A^0_v].
\end{equation}

Using (6.2.1) we can rewrite the last two factors of (6.2) as

\begin{equation}
(6.2.2) \quad \frac{1}{q^d \omega} \frac{\det(\alpha, \alpha)}{(\log q)^2} \prod_{v \in S} [A_v : A^0_v] \frac{\prod q_a \deg(v)}{\deg(v)}
\end{equation}

By Proposition 5.5, this is – except for the $q$-power factor –

\begin{equation}
(6.2.3) \quad \frac{1}{\alpha^2} \Omega; \{\mathcal{A}_i\}; \{X_{\text{red}}\} \prod_{v \in S} \frac{\epsilon_v}{(m_v^e)^2}.
\end{equation}
Substitute (6.2.2) and (6.2.3) into (6.2) and use Proposition 6.5 to find

\[
P_2(X, q^{-s}) \sim (1 - q^{1-s})^{\rho(X)} \frac{\#(A, K)[B(k)]^2}{q^{a(X)}[A(K)_{\text{tor}}]^2} \cdot \frac{(\Omega; F; \{\mathcal{A}_i\}; \{X_{\text{tor}}\})}{\alpha^2} \prod_{v \in S} \frac{\epsilon_v}{(m_v)^2}.
\]

By linear algebra, \((\Omega; F; \{\mathcal{A}_i\}; \{X_{\text{tor}}\})\) is the square of the index in Proposition 4.6. Using this to eliminate from (6.3) assorted terms, we come up with yet another expression

\[
P_2(X, q^{-s}) \sim (1 - q^{1-s})^{\rho(X)} \frac{\#(A, K)[D_i \cdot D_j]}{q^{a(X)}[NS(X)_{\text{tor}}]^2} \prod d_v^2 \epsilon_v.
\]

Comparing (6.4) with the statement of the Artin–Tate conjecture, we have reduced the equivalence of conjectures to the equality

\[\#(A, K) \cdot \prod d_v^2 \epsilon_v = \alpha^2 [\text{Br}(X, k)].\]

The “locally everywhere of degree 1” assumption guarantees, via Lemma 6.2, that for all \(v\), \(d_v \Delta_v = 1\). Since \(\epsilon_v\) divides \(\Delta_v^2\), the theorem is proved.

3. The leftover proofs

**Proposition 6.4**: The equality below holds:

\[
P_2(X, q^{-s}) = (1 - q^{1-s})^2 P_1(B, q^{-s}) P_1(B, q^{1-s}) \frac{L^S(A, s)}{\mu^d} \prod_{v \in S} \frac{f_{A(K_v)} [\omega_v] \mu_v^d \cdot \Pi_{\alpha} (1 - N_{v}^{q_v(1-s)})}{P_{1}(X_v, \mathbb{Q}^{-s}) \cdot (1 - N_{v}^{1-s})}.
\]

**Proof**: We start by writing

\[\zeta(X, s) = \prod \zeta(X_v, s) \quad v \in |V|\]

By the formalism of the Weil Conjectures, since \(X\) is smooth we can
(using Poincaré Duality) write this as

\[
\frac{P_1(X, q^{-s})P_1(X, q^{1-s})}{(1 - q^{-s})(1 - q^{1-s})} = \prod_{\nu \in S} \frac{P_1(X_\nu, N_{\nu^{-1}}^{-1})}{(1 - N_{\nu^{-1}}^{-1}) \cdot \prod (1 - N_{\nu^{-1}}^{q_{\nu}(1-s)})}
\]

where the explicit form of the right side of the equation comes from Proposition 3.3. Grouping factors on the right, we can write

\[
\zeta(V, s) = \prod (1 - N_{\nu}^{-s})^{-1}
\]

and similarly for \(\zeta(V, s - 1)\). Since \(V\) is also smooth, Poincaré Duality allows us to rewrite (6.5) as

\[
\text{LHS} = \frac{P_1(V, q^{-s})}{(1 - q^{-s})(1 - q^{1-s})} \cdot \frac{P_1(V, q^{1-s})}{(1 - q^{1-s})(1 - q^{2-s})} \cdot \prod_{\nu \in S} P_1(X_\nu, N_{\nu}^{-s}) \cdot \prod_{\nu \in S} P_1(X_\nu, N_{\nu}^{-s}) \cdot \prod_{\nu \in S} (1 - N_{\nu}^{-s})^{-1}
\]

Solving (6.6) for \(P_2(X, q^{-s})\) and noticing that \(P_1(X, T)/P_1(V, T) = P_1(B, T)\) by Proposition 4.4, we reach

\[
P_2(X, q^{-s}) = (1 - q^{1-s})^2 P_1(B, q^{-s}) P_1(B, q^{1-s}) \frac{1}{\prod_{\nu \in S} P_1(X_\nu, N_{\nu}^{-s})} \cdot \prod_{\nu \in S} \frac{\Pi_{a} (1 - N_{\nu}^{q_{\nu}(1-s)})}{P_1(X_\nu, N_{\nu}^{-s})(1 - N_{\nu}^{1-s})}
\]

In both (6.6) and (6.7) we have split the product-factor into "good" and "bad" parts. By definition,

\[
L^X(A, s) = \frac{|\mu|^d}{\prod_{\nu \in S} P_1(X, N_{\nu}^{-s}) \Pi_{\nu \in S} f_{A(K_\nu)} |\omega_{\nu}||\mu_{\nu}|^d}
\]

since, as in the proof of Proposition 3.3, \(P_1(X_\nu, T)\) can be interpreted as the characteristic polynomial of arithmetic Frobenius acting on \(A_\nu\) whenever \(X_\nu\) is nondegenerate. Substituting (4) into (3) proves the proposition. 

**PROPOSITION 6.5:** \(|\mu|^d q^{b + \deg \omega} = q^\sigma(X)\).

**PROOF:** Use the Leray Spectral Sequence for \(f_* \mathcal{O}_X\). Recall that for étale sheaves lifted from quasi-coherent (Zariski) sheaves, \(H^{\ell}_i = H^{\ell}_{\mathbb{Zar}}\).
This shows that $H^0(V, R^2f_*\mathcal{O}_X)$ is zero since each stalk of $R^2f_*\mathcal{O}_X$ is zero; also $H^2(V, \mathcal{O}_V)$ and $H^3(V, \mathcal{O}_V)$ are both zero. The spectral sequence then gives the exact sequences

$$H^0(V, \mathcal{O}_V) \Rightarrow H^0(X, \mathcal{O}_X)$$
$$0 \to H^1(V, \mathcal{O}_V) \to H^1(X, \mathcal{O}_X) \to H^0(V, R^1f_*\mathcal{O}_X) \to 0$$
$$0 \to H^2(X, \mathcal{O}_X) \Rightarrow H^1(V, R^1f_*\mathcal{O}_X) \to 0$$

Combine these to get $\chi(X, \mathcal{O}_X) = \chi(V, \mathcal{O}_V) - \chi(V, R^1f_*\mathcal{O}_X)$. By Lemma 2.4, $\chi(V, R^1f_*\mathcal{O}_X) = -\deg \omega + d \cdot \chi(V, \mathcal{O}_V)$; these two equalities, together with the fact that $V$ has genus $g$, give

$$\chi(X, \mathcal{O}_X) = (1 - d)(1 - g) + \deg \omega.$$ 

Proposition 4.4 shows that $g + b = \dim \text{Pic}^0_X$. It is well known that for the particular choice made for $\mu$, $|\mu| = q^{g-1}$. Therefore

$$|\mu|^d q^b q^{-\deg \omega} = q^{d(g-1)+b+\deg \omega}$$

which exponent is just $\chi(X, \mathcal{O}_X) - 1 + \dim \text{Pic}^0_X$. □

REFERENCES