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WEIGHTED INTERSECTION NUMBERS ON HILBERT MODULAR SURFACES

Yue Lin Lawrence Tong*

§1. Introduction

In [5] Hirzebruch and Zagier computed the intersection numbers among a series of curves T_m, $m = 1, 2, \dots$ in the non-compact Hilbert modular surface with quotient singularities $X = H^2/SL_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of $Q(\sqrt{p})$ and $p \equiv 1 \pmod 4$ is a prime. We abbreviate their result in the form $T_m \cdot T_n = \Sigma_{z \in T_m \cap T_n} H_p(z)$ where $T_m \cap T_n$ is the set theoretical intersection and $H_p(z)$ counts the multiplicity which may be a rational number if the intersection occurs at a singular point. The $H_p(z)$ are given by number theoretical functions and assuming mn is not a square the intersections are all transversal.

In [10], [5] the authors also constructed a series of cusp forms $\omega_m^{(k+2)} \in S_{k+2}(SL_2(\mathcal{O}))$ for any even $k+2 \geq 2$ (the index $k+2$ is necessitated by conventions in §2). By compactifying X at the cusps and resolving the corresponding singularities there is a surface $\tilde{X}$, and the cusp forms of weight 2 give rise to a series of forms (and cohomology classes) of type (1, 1) $j(\omega_m^{(2)})$ on $\tilde{X}$ with compact support in X. From numerous evidences it is conjectured that, on $\tilde{X}$, $j(\omega_m^{(2)})$ is the Poincaré dual of T_m^{co}, essentially the "finite part" of the completion of T_m in $\tilde{X}$. A possible proof is sketched in [11][1]. This would imply that up to some universal constants

$$\int_{\tilde{X}} j(\omega_m^{(2)}) \wedge j(\omega_n^{(2)}) = T_m^{co} \cdot T_n^{co} = \sum_{z \in T_m \cap T_n} H_p(z) + (\text{intersection at cusps}).$$

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[1] This has been proved by T. Oda: On modular forms associated with indefinite quadratic forms of signature (2, $n-2$), *Math. Ann. 231*, 97–144 (1977).

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Now for $k > 0$ it is proved in [11] that the Petersson product of $\omega_m^{(k+2)}$ and $\omega_n^{(k+2)}$ can be expressed as an analogous sum

$$\sum_{z\in T_m\cap T_n} H_p(z)\frac{\rho^{k+1}-\bar{\rho}^{k+1}}{\rho-\bar{\rho}}+(\text{extra terms}) \tag{1}$$

where ρ, $\bar{\rho}$ are invariants at the intersection points. A natural question then arises whether there is a geometric interpretation of this number as in the case of weight 2 (cf. [11], p. 167 Remark 2).

In the language of distributions or currents, the duality of $j(\omega_m^{(2)})$ and T_m^{co} means that as currents these two give the same cohomology class. Our purpose here is to show that by generalizing the formalism of Shimura [8], one can introduce a vector bundle E on X for each $k+2$, and we construct a canonical section s_m for $E \mid T_m$ so that the corresponding current $\hat{s}_m$ that it defines is the exact analogy of T_m for $k=0$. Namely with respect to a canonical metric $\mu: E\to E^*$ and modulo some constants

$$\int_X \langle s_m, \hat{\mu s_n}\rangle = \sum_{z\in T_m\cap T_n} H_p(z)\frac{\rho^{k+1}-\bar{\rho}^{k+1}}{\rho-\bar{\rho}}$$

Note that X, E and integrals should be taken here in the sense of V-manifolds of Satake and Baily [1] because of the quotient singularities. The proof of the above relation makes use of the formalism developed in [9].

Next the bundle E is extended to a bundle over $\tilde{X}$ and j is generalized to a map $S_{k+2}(SL_2(\mathcal{O}))\to H^{1,1}(X, E)$. The remaining question is to show that over $\tilde{X}$ $j(\omega_m^{(k+2)})$ is in the same cohomology class as $\hat{\tilde{s}}_m^c$ (cf. §5), which would lead to intersection interpretation of the Doi Naganuma Lifting for forms of higher weight. By making use of a formal computation of Zagier this is reduced to a question about the cohomology of $H^{1,1}(X, E)$.

I wish to thank Hirzebruch and Zagier for some discussions and particularly to the latter for posing the question to me.

§2. Shimura's formalism and the map j

$k>0$ will be a fixed even integer from here on. Let M be the representation of $GL(2,\mathbb{R})$ in the vector space V of k fold symmetric tensors on $\mathbb{R}^2$. There exists a unique symmetric bilinear form P on V (cf. [6], [8]) such that if $\binom{u}{v}^k$ denotes the k fold symmetric power of the vector $\binom{u}{v} \in \mathbb{R}^2$, then P in matrix form satisfies

$$(2) \qquad \left(\binom{u}{v}^k\right)^t P \binom{w}{z}^k = (uz - vw)^k \text{ for all } \binom{u}{v}, \binom{w}{z} \in \mathbb{R}^2.$$

This implies that

$$(3) \qquad M(\sigma)^t P M(\sigma) = (\det \sigma)^k P \text{ for all } \sigma \in GL(2, \mathbb{R}).$$

In particular $M(\sigma)$ is an automorphism of the form P for all $\sigma \in SL_2(\mathbb{R})$.

Now $SL_2(\mathcal{O})$ operates on H^2, the 2 fold product of the upper half plane in the standard way, and it operates on $V \otimes V$ by the representation $\sigma \to M(\sigma) \otimes M(\sigma')$. The quotient $(H^2 \times V \otimes V)/SL_2(\mathcal{O})$ has a natural vector bundle structure over the surface X in the sense of [1] (cf. [7]) which will be denoted by E. From the invariance of the symmetric bilinear form $P \otimes P$ under $SL_2(\mathcal{O})$ we get a canonical metric μ on the bundle E. μ may be viewed equivalently as a bundle isomorphism of E to its dual $\mu : E \xrightarrow{\cong} E^*$.

Henceforth we shall consider only the complexification of the vector bundle E, still denoted by E, obtained by complexifying V. μ is naturally extended to a Hermitian metric on the complex bundle E which is the same as extending it to a conjugate linear isomorphism $\mu : E \to E^*$. These bundles are obviously holomorphic of rank $(k+1)^2$.

On X' let $A^{p,q}(X', \mathbb{C})$ denote the smooth forms of type (p, q). The natural metric of X' defines the Hodge $\bar{*}$ operator $A^{p,q}(X', \mathbb{C}) \to A^{2-p,2-q}(X', \mathbb{C})$. This coupled with μ gives rise to a duality operator $\bar{*}_\mu : A^{p,q}(X', E) \to A^{2-p,2-q}(X', E^*)$. Finally the trace or contraction map $E \otimes E^* \to \mathcal{O}_{X'}$ is denoted by $\langle , \rangle$.

By generalizing the Shimura procedure we define a map $j : S_{k+2}(SL_2(\mathcal{O})) \to A^{1,1}(X', E)$ as follows. $(z_1, z_2) \in H^2$ being the standard coordinates, we put for simplicity $z_i^{(k)}$ as the vector $\binom{z_i}{1}^k$ then for $\sigma \in GL(2, \mathbb{R})$, $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$(4) \qquad (\sigma \circ z_i)^{(k)} d(\sigma \circ z_i) = \det(\sigma) J(\sigma, z)^{k+2} (M(\sigma) z_i^{(k)}) dz_i$$

where $J(\sigma, z) = (cz + d)^{-1}$. This implies that for $F \in S_{k+2}(SL_2(\mathcal{O}))$ and $\eta = F(z_1, z_2) z_1^{(k)} \otimes z_2^{(k)} dz_1 \wedge dz_2$, then

$$\eta \cdot (\sigma z_1, \sigma' z_2) = M(\sigma) \otimes M(\sigma') \cdot \eta.$$

η therefore descends to an E valued holomorphic 2 form on X. Now to get forms of type (1, 1) and the map j, we imitate the recipe in [5].

Let ϵ be a fundamental unit for $Q(\sqrt{p})\epsilon > 0 > \epsilon'$, and put

$$(jF)(z_1, z_2) = F(\epsilon z_1, \epsilon' \bar{z}_2) z_1^{(k)} \otimes \bar{z}_2^{(k)} dz_1 \wedge d\bar{z}_2. \tag{5}$$

Note that $\epsilon'\bar{z}_2$ is in the upper half plane and the above expression is invariant under $SL_2(\mathcal{O})$. It is also possible to have jF symmetric in z_1, z_2 as in [5] by symmetrizing the right hand side. From (2) and (3) it follows that

$$\langle z_1^{(k)} \otimes \bar{z}_2^{(k)}, \mu(z_1^{(k)} \otimes \bar{z}_2^{(k)})\rangle = (\text{const.}) y_1^k y_2^k.$$

Therefore given $F, G \in S_{k+2}(SL_2(\mathcal{O})$.

$$\langle jF, *_\mu jG\rangle = (\text{const.}) F(\epsilon z_1, \epsilon' \bar{z}_2)\overline{G(\epsilon z_1, \epsilon' \bar{z}_2)} \times y_1^k y_2^k dx_1 dy_1 dx_2 dy_2 \tag{6}$$

($z_i = x_i + \sqrt{-1}\, y_i$) so that the pairing may be related to the Petersson product.

§3. A canonocal section of $E \mid T_m$

For our purpose the definition of the curves T_m is most conveniently described in the form in [11] §6. Let

$$\mathcal{A} = \{A \in M_2(\mathcal{O}) \mid A^* = A'\}$$

where $A^* = (\det A)A^{-1}$ and A' has for entries the conjugates of A. The group $G = SL_2(\mathcal{O})/\{\pm 1\}$ acts on $\mathcal{A}$ by $B \cdot A = B^*AB'$. Each $A \in \mathcal{A}$ with $\det A > 0$ defines a curve in H^2 via its graph $\{(z, Az) \mid z \in H\}$ and T_m consists of the images in $H^2/SL_2(\mathcal{O})$ of these graphs for all A with $\det A = m$. Let A_i $i = 1, \ldots, r$ denote the representatives of the $SL_2(\mathcal{O})$ equivalence classes of A with $\det A = m$. and G_i the isotropy groups

$$G_i = \{B \in G \mid B^*A_iB' = \pm A_i\}. \tag{7}$$

Then $T_m = \bigcup_{i=1}^r H/G_i$ where the ith component is embedded by $z \to (z, A_i z)$ and we denote it by T_m^i.

In this setting we define a canonical section of the bundle E over the curve T_m as follows. The bilinear pairing P of §2 may be considered as a linear map $P \in \operatorname{Hom}(V, V^*)$ we have the composition

$(PM(A_i))^{-1} = M(A_i)^{-1}P^{-1} \in \mathrm{Hom}(V^*, V) \simeq V \otimes V$, which is a fiber of the bundle E. We now assign the constant real matrix $M(A_i)^{-1}P^{-1}$ over the graph $\{(z, A_i z) \mid z \in H\}$ and we want to show that this descends to a section of $E \mid T^i_m$. It amounts to showing the invariance of $M(A_i)^{-1}P^{-1}$ under the isotropy group.

LEMMA: *$M(A_i)^{-1}P^{-1}$ considered as a tensor in $V \otimes V$ is invariant under G_i.*

PROOF: Let $B \in G_i$, then the action $M(B) \otimes M(B')(M(A_i)^{-1}P^{-1})$ is given in matrix products as $M(B)M(A_i)^{-1}P^{-1}M(B')^t$. Using the symmetry of P and the relation (3) we get

$$P^{-1}M(B')^t = M(B')^{-1}P^{-1}.$$

And for $B \in SL_2(\mathcal{O})B^{-1} = B^*$ so that the total product above is $M(B)M(A_i)^{-1}M(B')^*P^{-1}$. Finally by (7) this is just $M(A_i)^{-1}P^{-1}$ which proves the lemma.

We denote by s^i_m the section of $E \mid_{T^i_m}$ defined above and s_m the totality of these in $E \mid_{T_m}$. Note that applying the operator μ on s^i_m we get a section in $E^* \mid_{T^i_m}$ which in matrices has the formula

$$\mu(s^i_m) = PM(A_i)^{-1}. \tag{8}$$

§4. Intersections with coefficients in a bundle

We describe briefly how to compute the intersection numbers of currents supported on submanifolds with coefficients in a vector bundle. Let X^n be an n-dim. complex manifold Y^r and Y'^s complex submanifolds of codimensions s and r respectively so that $r + s = n$. Let $E \to X$ be a holomorphic vector bundle with sections $s_Y \in \Gamma(Y, E \mid_Y)$, $s_{Y'} \in \Gamma(Y', E^* \mid_{Y'})$. These sections define currents $\hat{s}_Y, \hat{s}_{Y'}$ where e.g. for any form with compact support $\phi \in A^{r,r}_c(X, E^*)$

$$\hat{s}_Y[\phi] = \int_Y \langle s_Y, \phi \mid_Y \rangle \in \mathbb{C}. \tag{9}$$

$\hat{s}_Y$ represents a cohomology class in $H^{s,s}(X, E)$ which assuming Serre duality is uniquely characterized by the equation (9) for all $\phi \in H^{r,r}_c(X, E^*)$. $\hat{s}_Y$ is also the image of s_Y by the so called Gysin map associated to the embedding $\iota_Y : Y \to X$. By taking smooth representatives the cohomology classes $\hat{s}_Y$ and $\hat{s}_{Y'}$ have an "intersection number" defined by $\int_X \langle \hat{s}_Y, \hat{s}_{Y'} \rangle$ assuming the integrand has compact support. Our purpose is to describe a method of computing this number geometrically in terms of the data at points of the intersection $Z = Y \cap Y'$. We describe the method following [9] which uses Čech cohomology, and this may be related to Dolbeault cohomology by the standard isomorphism. In [9] X is compact, but it is readily extended to cohomology classes with compact support on open X.

We express (9) by the commutative diagram

$$
\begin{tikzcd}
\hat{s}_{Y'} \in H^Y(X, E^* \otimes \Omega^Y_X) \arrow[r, "\langle \hat{s}_Y, \cdot \rangle"] \arrow[d, "\iota^*_Y"] & H^n_c(X, \Omega^n_X) \arrow[dr, "\int_X"] & \\
& & \mathbf{C} \\
H^Y(Y, E^*|_Y \otimes \Omega^Y_Y) \arrow[r, "\langle s_Y, \cdot \rangle"] & H^Y_c(X, \Omega^Y_Y) \arrow[ur, "\int_Y"'] &
\end{tikzcd} \tag{10}
$$

Now suppose Z consists of isolated points, then there is a compatible sequence of maps by the Grothendieck duality theory,

$$
\begin{tikzcd}
\hat{s}_{Y'} \in H^r(X, E^* \otimes \Omega^r_X) \arrow[r, "\iota^*"] & H^r(Y, E^*|_Y \otimes \Omega^r_Y) \arrow[r, "\langle s_Y, \cdot \rangle"] & H^r_c(Y, \Omega^r_Y) \arrow[r, "\int_Y"] & \mathbf{C} \\
\tilde{S}_{Y'} \in \mathrm{Ext}^r(X; \mathcal{O}_{Y'}, E^* \otimes \Omega^r_X) \arrow[u] \arrow[r, "\iota^\#"] & \mathrm{Ext}^r(Y; \mathcal{O}_Z, E^*|_Y \otimes \Omega^r_Y) \arrow[u] \arrow[r, "\langle s_Y, \cdot \rangle"] & \mathrm{Ext}^r_c(Y; \mathcal{O}_Z, \Omega^r_Y) \arrow[u] \arrow[r, "\simeq"] & H^0_c(Y, \underline{\mathrm{Ext}}^r(\mathcal{O}_Z, \Omega^r_Y)) \arrow[u, "\mathrm{Res}"']
\end{tikzcd} \tag{11}
$$

and our problem is to find a geometric description of the maps and objects in the bottom row. For submanifolds with codimension greater than one and for higher dimensional Z there is an analogous sequence which requires the elaborate machinery developed in [9]. The Holomorphic Lefschetz formula also follows from such a sequence where the coefficients are slightly more subtle.

For the application at hand, we need $n = 2$, $r = s = 1$ and it is easy to write down explicitly this sequence as follows: consider the complex of length one (the dual Koszul complex)

$$\check{K}^0(Y') \xrightarrow{\partial_{Y'}} \check{K}^1(Y')$$

where $\check{K}^0(Y') = \mathcal{O}_X$ the structure sheaf, $\check{K}^1(Y') = L_{Y'}$ the line bundle of the divisor Y' and $\partial_{Y'}$ is given in each open set by multiplication by the defining function of Y'. Suppose $\mathcal{U}$ is a Stein covering of X and consider $\check{C}^*(\mathcal{U}, \check{K}^*)$ the bicomplex of Čech cochains with coefficients in $\check{K}^*$. The cohomology of the associated total complex is $\operatorname{Ext}^*(X; \mathcal{O}_{Y'}, \mathcal{O}_X)$. Now if we tensor $\check{K}^*$ with the coefficients $\Omega^1_X \otimes E^*$ then we are computing $\operatorname{Ext}^*(X; \mathcal{O}_{Y'}, \Omega^1_X \otimes E^*)$. In this group there is a class $\tilde{s}_{Y'}$ which goes to $\hat{s}_{Y'}$ by the first vertical map in (11) and which we describe in detail. Being of total degree 1, $\tilde{s}_{Y'}$ has two components $(\tilde{s}_{Y'})^{0,1} + (\tilde{s}_{Y'})^{1,0}$ with the superscripts indicating the bidegree. In an open set $U_\alpha \in \mathcal{U}$.

$$(\tilde{s}_{Y'})^{0,1}_\alpha = (\bar{s}_{Y'})_\alpha \otimes df_\alpha \otimes e_\alpha$$

where $(\bar{s}_{Y'})_\alpha$ is an extension of $s_{Y'} \in \Gamma(Y' \cap U_\alpha, E^* \mid_{Y'})$ to a section over U_α, f_α is the defining function for Y' in U_α and e_α is a generator of $L_{Y' \mid U_\alpha}$. In any overlap $U_\alpha \cap U_\beta \neq \phi$ it is easily checked that the restrictions of $(\tilde{s}_{Y'})^{0,1}_\alpha$ and $(\tilde{s}_{Y'})^{0,1}_\beta$ to $Y' \cap U_\alpha \cap U_\beta$ agree (the transition functions of df and e being inverse to each other on Y'). This implies that the Čech coboundary $(\delta(\tilde{s}^{0,1}_{Y'}))_{\alpha\beta}$ vanishes on Y' and so must be equal to some $\partial_{Y'}(\eta)$. This term η gives $(\tilde{s}^{1,0}_{Y'})_{\alpha\beta}$ and we have constructed $\tilde{s}_{Y'}$. (Although we do not use it here, $\tilde{s}_{Y'} \to \tilde{s}^{1,0}_{Y'} = \hat{s}_{Y'}$ is the first vertical map in (11)).

Next we construct $\iota^\#_Y$. By restriction we have a chain map $\check{K}^*(Y') \to \check{K}^*(Y') \mid_Y$, and we need to find a chain map $\check{K}^*(Y') \mid_Y \to \check{K}^*(Z)$. It amounts to finding u in the commutative diagram

$$\begin{tikzcd}
L_{Y'} \arrow[r, "i^*"] \arrow[d, "\partial_{Y'}"] & L_{Y'}|_Y \arrow[r, "u"] \arrow[d, "\partial_{Y'|Y}"] & L_Z \arrow[d, "\partial_Z"] \\
\mathcal{O}_X \arrow[r, "i^*"] & \mathcal{O}_Y \arrow[r, "1"] & \mathcal{O}_Y
\end{tikzcd}$$

Assuming the covering $\mathcal{U}$ sufficiently fine so that at most one point of Z is contained in any open set, then if g_α is a holomorphic function on $Y \cap U_\alpha$ defining the point $z \in Z$, $u_\alpha = \dfrac{g_\alpha}{f_{\alpha \mid Y}}$. We conclude that the component of type $(0, 1)$ in $\iota^\#_Y \tilde{s}_{Y'}$ evaluated on $U_\alpha \cap Y$ is $((\bar{s}_{Y'})_\alpha \otimes df_\alpha) \mid_Y \otimes \dfrac{g_\alpha}{f_{\alpha \mid Y}} \epsilon_\alpha$ where ϵ_α is a basis of $L_{Z \mid U_\alpha \cap Y}$. Contracting with the section s_Y we get $\langle (s_Y)_\alpha, (\bar{s}_{Y'})_\alpha \rangle df_{\alpha \mid Y} \otimes \dfrac{g_\alpha}{f_{\alpha \mid Y}} \epsilon_\alpha$. The third horizontal arrow in bottom of (11) asserts that this $(0, 1)$ component is all we need for the residue map. To an element $h_\alpha \epsilon_\alpha \in \Gamma(U \cap Y, \Omega^1_Y \otimes L_Z)$ the residue map assigns $\mathrm{Res}_z\left\{\frac{1}{2\pi i}\frac{h_\alpha}{g_\alpha}\right\}$ the usual Cauchy residue of $\frac{1}{2\pi i}\frac{h_\alpha}{g_\alpha}$ at z. For our intersection number we get therefore

$$\mathrm{Res}_z\left\{\frac{1}{2\pi i}\frac{\langle (s_Y)_\alpha, \overline{(s_{Y'})_\alpha}\rangle df_{\alpha|Y}}{f_{\alpha|Y}}\right\} \tag{12}$$

Combining with (10) and (11) we have derived the following result.

PROPOSITION: $\displaystyle\int_X \langle \hat{s}_Y, \hat{s}_{Y'}\rangle = \sum_{z \in Z = Y \cap Y'} \mathrm{Res}_z\left\{\frac{1}{2\pi i}\frac{\langle (s_Y)_\alpha, (\bar{s}_{Y'})_\alpha\rangle df_{\alpha|Y}}{f_{\alpha|Y}}\right\}$

REMARKS: 1. The Proposition is applicable to intersection with higher multiplicities which will arise when one tries to deal with the cusps.

2. When Z consists of isolated points an alternative method for computing such intersection numbers is to use harmonic theory and de Rham's formulas representing these numbers in terms of Green's kernel of the Laplacian.

We now apply this formula to the bundle and sections constructed earlier. Let $Y = T^i_m$, $Y' = T^j_n$, $s^i_m = M(A_i)^{-1}P^{-1}$, $\mu(s^j_n) = PM(B_j)^{-1}$ by (8).

We will assume that mn is not a square so that the intersections are transversal and $\mathrm{Res}_z\left\{\frac{1}{2\pi i}\frac{df_{\alpha|Y}}{f_{\alpha|Y}}\right\} = 1$, and the multiplicity of the intersection is just described by the functions $H_p(z)$ in §1. Note that at a quotient singularity according to the definition of integrals on V manifolds [1], $\int \langle \hat{s}_Y, \hat{s}_{Y'}\rangle$ is defined by lifting to a local uniformizing neighborhood integrating and then dividing by the order of isotropy group. We can take open sets in $H \times H$ as local uniformizing *nbhd* where the curves and sections have the same local equations so that the weight factor $\langle (s_Y)_\alpha, (\bar{s}_{Y'})_\alpha\rangle$ is unchanged by this procedure. For weight 2 ($k = 0$) case this way of counting multiplicity is the same as that employed in [5] where the numbers $H_p(z)$ are obtained. It only remains to make explicit the weight factor.

$$\langle M(A_i)^{-1}P^{-1}, PM(B_j)^{-1}\rangle \tag{13}$$

To compute this pairing we have by (3)

$$PM(B_j)^{-1} = \frac{1}{(\det B_j)^k} M(B_j)^t P \text{ and } \det B_j = n$$

(13) is therefore given by

$$\frac{1}{n^k}\operatorname{Trace}\{M(A_i)^{-1}P^{-1}(M(B_j)^tP)^t\} = \frac{1}{n^k}\operatorname{Trace}\{M(A_i^{-1}B_j)\} \tag{14}$$

Now suppose ρ and $\bar\rho$ are the characteristic roots of the matrix $A_i^{-1}B_j$, then the characteristic roots of $M(A_i^{-1}B_j)$ are the numbers $\rho^{k-\iota}\bar\rho^{\iota}$, $\iota = 0, \ldots, k$ so that

$$\operatorname{Trace} M(A_i^{-1}B_j) = \frac{\rho^{k+1} - \bar\rho^{k+1}}{\rho - \bar\rho}. \tag{15}$$

Note that clearly $\frac{1}{n^k}\operatorname{Trace}(M(A_i^{-1}B_j)) = \frac{1}{m^k}\operatorname{Trace}(M(B_j^{-1}A_i))$ and in terms of the characteristic roots ρ, $\bar\rho$ above the intersection is also $\frac{1}{m^k}\frac{(\frac{1}{\bar\rho})^{k+1} - (\frac{1}{\rho})^{k+1}}{(\frac{1}{\bar\rho}) - (\frac{1}{\rho})}$. Combining these results with the above Proposition and the results of Hirzebruch and Zagier stated in the Introduction we have the formula

$$\int_{X'} \langle \hat s_m, \hat\mu s_n\rangle = \frac{1}{n^k}\sum_{z\in T_m\cap T_n}\frac{\rho^{k+1} - \bar\rho^{k+1}}{\rho - \bar\rho} H_p(z). \tag{16}$$

§5. Extension of the bundle over $\tilde X$ and relation of s_m to modular forms

We recall Hirzebruch's procedure of resolving the cusp singularities described in [4, §2]. To avoid confusion we change his notations slightly and the singular cusp is transformed to the point ∞ in $\overline{H^2/G(N, U)}$ ($N = M$, and $U = V$ in notations of [4]). We want to extend $E \to H^2/G(N, U)$ to the nonsingular model. Now there is a biholomorphic map defined by logarithms,

$$Y - \bigcup_{j\in Z} S_j \xrightarrow{\Phi} \mathbb{C}^2/N,$$

and consider the vector bundle $E_1 = \mathbb{C}^2 \times V \otimes V/_N$ where N consists of the matrices $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)$ and act on $V \otimes V$ by the representation in §2. E_1 is a locally constant vector bundle and its Chern classes vanish. On the other hand $\mathbb{C}^2/N$ is biholomorphic to the Stein manifold $\mathbb{C}^* \times \mathbb{C}^*$ over which all vector bundles split into line bundles and are then determined by their Chern classes (cf. [2]). It follows that E_1 is trivial and we can take the trivial $V \otimes V$ bundle on Y to extend E_1. Next the infinite cyclic group U acts compatibly on Y^+ and $H^2/_N$ and the bundle $Y^+ \times V \otimes V/_U \to Y^+/_U$ is the extension of $E \to H^2/G(N, U)$. In this way we get a bundle still denoted by E over X which extends the bundle $E \to X$ from §2. Note that since U acts with trivial isotropy group on the curves S_k of the resolution, the restriction $E \to S_k$ is trivial by the above description.

Following [3], [5], consider now the decomposition

$$H^{1,1}(\tilde{X}, E) = H_!^{1,1}(\tilde{X}, E) \oplus H_{\inf}^{1,1}(\tilde{X}, E)$$

where $H_!^{1,1}(\tilde{X}, E)$ is the image of $H_c^{1,1}(X, E)$ induced by the inclusion $X \to \tilde{X}$, and $H_{\inf}^{1,1}(\tilde{X}, E)$ is the orthogonal complement depending on the cusp resolutions. Let $\bar{s}_m$ be the closure of s_m in $E \to \tilde{X}$, $\hat{\bar{s}}_m$ the current it defines and $\hat{\bar{s}}_m^c$ its component in $H_!^{1,1}(\tilde{X}, E)$. By analogy with weight 2 case one expects that $j(\omega_m^{k+2}) = \hat{\bar{s}}_m^c$ in $H_!^{1,1}(\tilde{X}, E)$. By duality and the above decomposition, it suffices to show that

$$\int_{\tilde{X}} \langle j(\omega_m^{k+2}), \bar{*}_\mu \phi \rangle = \int_{\tilde{X}} \langle \hat{\bar{s}}_m^c, \bar{*}_\mu \phi \rangle \quad \forall \phi \in H_!^{1,1}(\tilde{X}, E). \tag{17}$$

Concerning this equation there is the following.

PROPOSITION: *The equation* (17) *holds for all* $\phi = j(F)$ *where* $F \in S_{k+2}(SL_2(\mathcal{O}))$.

The proof is just a translation of a formal argument due to Zagier. The left hand side of (17) by (6) is

$$(\text{const.}) \int \omega_m(\epsilon z_1, \epsilon' \bar{z}_2) F(\epsilon \bar{z}_1, \epsilon' z_2) y_1^k y_2^k dx_1 dy_2 dx_2 dy_2$$

By the proof of Theorem 6 of [11], this is just the sum

$$(\text{const.}) \sum_{i=1}^{r} \int_{H/G_i} F(\epsilon \bar{z}, \epsilon' A_i z) J(A_i, z)^{k+2} y^k dx dy \tag{18}$$

Next the right hand side of (17) is $\int_{T_m} \langle s_m, (\bar{*}_\mu j(F)) \big|_{T_m} \rangle$ since jF is orthogonal to the curves of singularity resolutions. (cf. [7] p. 165). To make explicit this integrand we have $(\bar{*}_\mu j(F)) \big|_{T_m}$ is a sum of

$F(\epsilon\bar{z}, \epsilon' A_i z)$ multiplied by the vector valued form

$$\mu\{\bar{z}^{(k)} \otimes (A_i z)^{(k)} dz \wedge d(\overline{A_i z}) = \det(A_i) J(A_i z)^{k+2} \mu\{\bar{z}^{(k)} \otimes M(A_i)(z)^{(k)} dz \wedge d\bar{z}\}$$

by (4). Finally $\langle s_m^i, \mu(\bar{z}^{(k)} \otimes M(A_i)(z)^{(k)})\rangle$ in terms of products of matrices is

$$\begin{aligned} &\mathrm{Trace}\,\{M(A_i)^{-1} P^{-1} (P\bar{z}^{(k)} (z^{(k)t} M(A_i)^t) P^t)^t\} \\ &= \mathrm{Trace}\,\{\bar{z}^{(k)} (z^{(k)})^t P\} = (\mathrm{const.}) y^k. \end{aligned}$$

This proves (17) up to some universal constants for $\phi = j(F)$. To complete the proof of (17) it is necessary to know the precise structure of $H_!^{1,1}(\tilde{X}, E)$. In the case of the trivial representation $E = \mathbb{C}$ (weight 2) it is known that this group consists of the images of the cusp forms and the first Chern class of X. Since the first Chern class is orthogonal to $j(\omega_m^2)$ but not to T_m^c, this explains the notation T_m^{co} needed to have the Poincaré dual of $j(\omega_m^2)$ in §1. In the case of higher weight it is not known if there are classes in $H_!^{1,1}(\tilde{X}, E)$ other than the cusp forms. (cf. [3]).

REFERENCES

[1] W. L. BAILY, JR: The decomposition theorem for V-manifolds. *Amer. Jour. Math.* *78* (1956) 862–888.

[2] P. A. GRIFFITHS: Function theory of finite order on algebraic varieties I(A). *Jour. Diff. Geometry, 6,* 285–306 (1972).

[3] G. HARDER: On the cohomology of discrete arithmetically defined groups. *Proc. Int. Colloq. on Discrete subgroups of Lie groups,* 129–160, Bombay (1973).

[4] F. HIRZEBRUCH: Hilbert modular surfaces. *L'Ens. Math, 19,* 183–281 (1973).

[5] F. HIRZEBRUCH and D. ZAGIER: Intersection numbers of curves on Hilbert Modular surfaces and modular forms of Nebentypus, *Inventiones math. 36,* 57–113 (1976).

[6] S. LANG: Introduction to Modular forms. Grundlehren der mathematischen Wissenschaften 222, Springer-Verlag 1976.

[7] Y. MATSUSHIMA and S. MURAKAMI: On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds. *Annals of Math. 78,* No. 2, 365–416 (1963).

[8] G. SHIMURA: Sur les intégrales attachées aux formes automorphes, *Journal of Math. Soc. of Japan, 11,* No. 4, 291–311 (1959).

[9] D. TOLEDO and Y. L. L. TONG: Duality and intersection theory in complex manifolds I. *Math. Ann. 237,* 41–77 (1978).

[10] D. ZAGIER: Modular forms aseociated to real quadratic fields. *Inventiones math. 30,* 1–46 (1975).

[11] D. ZAGIER: Modular forms whose Fourier coefficients involve Zeta functions of quadratic fields, collected papers in International Summer School on Modular Functions Bonn 1976. Springer-Verlag Lecture Notes in Mathematics 627.

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