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COMMUTING VARIETIES OF SEMISIMPLE LIE ALGEBRAS AND ALGEBRAIC GROUPS

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Introduction

Let $\mathfrak{g} = \mathfrak{gl}_n(K)$ be the Lie algebra of all $n \times n$ matrices over the algebraically closed field K and let $\mathcal{C}(\mathfrak{g})$ be the variety of all pairs (x, y) of elements of \mathfrak{g} such that $[x, y] = 0$. We call $\mathcal{C}(\mathfrak{g})$ the *commuting variety* of \mathfrak{g} . Gerstenhaber [6] has shown that $\mathcal{C}(\mathfrak{g})$ is an irreducible algebraic variety. In this paper we shall generalize Gerstenhaber's result to reductive Lie algebras and simply connected semisimple algebraic groups, both over algebraically closed fields of characteristic zero. Our basic result states that every commuting pair of elements in a reductive Lie algebra (resp. simply connected semisimple algebraic group) can be approximated by a pair of elements belonging to a Cartan subalgebra (resp. maximal torus). Precisely, for Lie algebras we prove the following theorem:

THEOREM A: *Let \mathfrak{g} be a reductive Lie algebra over the algebraically closed field K of characteristic zero and let $\mathcal{C}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}$. Let $(x, y) \in \mathcal{C}(\mathfrak{g})$ and let N be a neighbourhood of (x, y) in $\mathcal{C}(\mathfrak{g})$. Then there exists a Cartan subalgebra \mathfrak{h} of \mathfrak{g} such that N meets $\mathfrak{h} \times \mathfrak{h}$.*

The conclusion of Theorem A implies that $\mathcal{C}(\mathfrak{g})$ is an irreducible algebraic variety. We remark that if K is the field \mathbb{C} of complex numbers, then N can be taken to be an arbitrary neighbourhood of (x, y) in the topology of $\mathcal{C}(\mathfrak{g})$ as a complex space.

We also prove similar theorems for semisimple Lie algebras and algebraic groups over the field \mathbb{R} of real numbers or, more generally, over a local field of characteristic zero. In this case, there may be more than one conjugacy class of Cartan subalgebras or Cartan subgroups, so that the analogue of the irreducibility statement concerning $\mathcal{C}(\mathfrak{g})$ does not hold.

Now for a few words about the proofs. We consider the case of Lie algebras. By an inductive argument using the Jordan decomposition of elements of \mathfrak{g} , we can quickly reduce the proof to the case of commuting pairs (x, y) , where x is a nilpotent element of \mathfrak{g} whose centralizer \mathfrak{g}_x does not contain any non-zero semisimple elements. In the recent paper [1] of Carter and Bala on the classification of nilpotent conjugacy classes in semisimple Lie algebras, such elements x are called *distinguished* nilpotents. A key technical result of [1] states that distinguished nilpotent elements are of parabolic type (for definition, see §4). For commuting pairs (x, y) where x is a nilpotent of parabolic type, the argument is more delicate. It uses an idea of Dixmier [5], who shows that such an x is the limit of semisimple elements a such that $\dim \mathfrak{g}_a = \dim \mathfrak{g}_x$.

§1. Preliminaries

Our basic reference for algebraic groups and algebraic geometry is [2]. All algebraic varieties will be taken over an algebraically closed field K of characteristic zero and we shall identify an algebraic variety X with the set $X(K)$ of its K -points. We shall denote the Lie algebra of an algebraic group G, H, U etc., by the corresponding lower case German letter $\mathfrak{g}, \mathfrak{h}, \mathfrak{u}$ etc. If G (resp. \mathfrak{g}) is a group (resp. Lie algebra) and if $x \in G$ (resp. $x \in \mathfrak{g}$), then G_x (resp. \mathfrak{g}_x) denotes the centralizer of x in G (resp. \mathfrak{g}). An affine algebraic group G is *reductive* if G is connected and the unipotent radical of G is a torus. If H is an algebraic group, then H^0 denotes the identity component of H .

§2. Proof of Theorem A

Let \mathfrak{g} be a reductive Lie algebra over K . We let $\mathcal{E}'(\mathfrak{g})$ be the set of all pairs $(s, t) \in \mathfrak{g} \times \mathfrak{g}$ such that there exists a Cartan subalgebra \mathfrak{h} of \mathfrak{g} which contains both s and t . Let $\mathcal{E}(\mathfrak{g})$ denote the closure of $\mathcal{E}'(\mathfrak{g})$ in $\mathcal{C}(\mathfrak{g})$. Clearly $\mathcal{E}(\mathfrak{g})$ is contained in $\mathcal{C}(\mathfrak{g})$. To prove Theorem A we must prove that $\mathcal{E}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})$. The proof is by induction on $\dim \mathfrak{g}$. The proof is clear for $\dim \mathfrak{g} = 0$. We assume $\dim \mathfrak{g} > 0$ and we make the inductive hypothesis that if \mathfrak{k} is a reductive Lie algebra with $\dim \mathfrak{k} < \dim \mathfrak{g}$, then $\mathcal{E}(\mathfrak{k}) = \mathcal{C}(\mathfrak{k})$.

Let \mathfrak{c} denote the centre of \mathfrak{g} and let $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$. Then \mathfrak{s} is a semisimple Lie algebra and \mathfrak{g} is the direct sum of \mathfrak{c} and \mathfrak{s} .

LEMMA 2.1: *Let $(x, y) \in \mathcal{C}(\mathfrak{g})$ and write $x = x_1 + x_2$, $y = y_1 + y_2$, with $x_1, y_1 \in \mathfrak{c}$ and $x_2, y_2 \in \mathfrak{e}$. Then $(x, y) \in \mathcal{Z}(\mathfrak{g})$ if and only if $(x_2, y_2) \in \mathcal{Z}(\mathfrak{e})$.*

The proof of Lemma 2.1 follows immediately from the fact that every Cartan subalgebra \mathfrak{h} of \mathfrak{g} is of the form $\mathfrak{h} = \mathfrak{c} + \mathfrak{a}$, where \mathfrak{a} is a Cartan subalgebra of \mathfrak{e} .

LEMMA 2.2: *Assume that $\mathfrak{c} \neq \{0\}$. Then $\mathcal{C}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})$.*

PROOF: \mathfrak{e} is a semisimple Lie algebra and $\dim \mathfrak{e} < \dim \mathfrak{g}$. Hence by the inductive hypothesis $\mathcal{C}(\mathfrak{e}) = \mathcal{Z}(\mathfrak{e})$. It follows immediately from Lemma 2.1 that $\mathcal{C}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})$.

Lemma 2.2 reduces the proof to the case of semi-simple \mathfrak{g} . For the rest of the proof, we assume \mathfrak{g} to be semisimple.

LEMMA 2.3: *Let $(x, y) \in \mathcal{C}(\mathfrak{g})$ and assume that either x or y is not nilpotent. Then $(x, y) \in \mathcal{Z}(\mathfrak{g})$.*

PROOF: Let $x = x_s + x_n$ be the Jordan decomposition of x ; here x_s is semisimple, x_n is nilpotent and $[x_s, x_n] = 0$. Assume that x is not nilpotent, i.e. that $x_s \neq 0$. Let $\mathfrak{k} = \mathfrak{g}_{x_s}$. Then \mathfrak{k} is a reductive Lie algebra, $\dim \mathfrak{k} < \dim \mathfrak{g}$ and $(x, y) \in \mathcal{C}(\mathfrak{k})$. By the inductive hypothesis, $\mathcal{Z}(\mathfrak{k}) = \mathcal{C}(\mathfrak{k})$. But every Cartan subalgebra of \mathfrak{k} is a Cartan subalgebra of \mathfrak{g} . Therefore $\mathcal{Z}(\mathfrak{k}) \subset \mathcal{Z}(\mathfrak{g})$. Consequently $(x, y) \in \mathcal{Z}(\mathfrak{g})$. Since $(x, y) \in \mathcal{Z}(\mathfrak{g})$ if and only if $(y, x) \in \mathcal{Z}(\mathfrak{g})$, we also see that $(x, y) \in \mathcal{Z}(\mathfrak{g})$ if y is not nilpotent.

LEMMA 2.4: *Let $(x, y) \in \mathcal{C}(\mathfrak{g})$ and assume that there exists a non-zero semisimple element $s \in \mathfrak{g}_x$. Then $(x, y) \in \mathcal{Z}(\mathfrak{g})$.*

PROOF: For $t \in K$, let $a_t = ty + (1-t)s$. Then $(x, a_t) \in \mathcal{C}(\mathfrak{g})$ for every $t \in K$. Let D denote the set of $t \in K$ such that a_t is not nilpotent. Then D is an open subset of K and D is non-empty since $0 \in D$. By Lemma 2.3 we have $(x, a_t) \in \mathcal{Z}(\mathfrak{g})$ for every $t \in D$. Since $\mathcal{Z}(\mathfrak{g})$ is a closed subset of $\mathcal{C}(\mathfrak{g})$ and D is dense in K , we see that $(x, a_t) \in \mathcal{Z}(\mathfrak{g})$ for every $t \in K$. Thus $(x, y) = (x, a_1) \in \mathcal{Z}(\mathfrak{g})$. This proves Lemma 2.4.

Following [1], we say that a nilpotent element x of \mathfrak{g} is *distinguished* if the centralizer \mathfrak{g}_x does not contain any non-zero semisimple elements. Lemma 2.4 reduces our proof to the case of commuting pairs (x, y) with x a distinguished nilpotent element. It will be shown in §4, Corollary 4.7, that if (x, y) is such a pair, then

$(x, y) \in \mathcal{E}(\mathfrak{g})$. This completes the proof of Theorem A, modulo the proof of Corollary 4.7. (In the interests of brevity, it is more convenient to give our arguments involving distinguished nilpotent elements in \mathfrak{g} and distinguished unipotent elements in a semisimple algebraic group in the same §. For this reason, we have postponed our proof of the result mentioned above until §4.)

As a corollary to Theorem A, we have:

COROLLARY 2.5: *Let \mathfrak{g} be a reductive Lie algebra. Then $\mathcal{C}(\mathfrak{g})$ is an irreducible algebraic variety.*

PROOF: Let G be the adjoint group of G and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . Define a morphism

$$\eta: G \times \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{g} \times \mathfrak{g} \text{ by } \eta(a, x, y) = (a(x), a(y)).$$

Since any two Cartan subalgebras of \mathfrak{g} are conjugate under G , we see that the image of η is $\mathcal{E}'(\mathfrak{g})$; consequently $\mathcal{E}'(\mathfrak{g})$ is irreducible. It follows immediately that $\mathcal{C}(\mathfrak{g}) = \mathcal{E}(\mathfrak{g})$ is irreducible.

REMARK 2.6: It is easy to give examples of solvable Lie algebras \mathfrak{g} such that $\mathcal{C}(\mathfrak{g})$ is not an irreducible variety.

§3. Commuting varieties of reductive groups

THEOREM B: *Let G be a reductive algebraic group and let $\mathcal{C}(G) = \{(x, y) \in G \times G \mid xy = yx\}$ be the commuting variety of G . Let $(x, y) \in \mathcal{C}(G)$ and assume that there exists z belonging to $Z(G)$, the centre of G , such that $zy \in G_x^0$. Let N be a neighbourhood of (x, y) in $\mathcal{C}(G)$. Then there exists a maximal torus T of G such that N meets $T \times T$.*

The proof of Theorem B will be given in a series of lemmas. We let $\mathcal{E}'(G)$ be the set of all $(s, t) \in G \times G$ such that there exists a maximal torus T of G which contains both s and t . Let $\mathcal{E}(G)$ denote closure of $\mathcal{E}'(G)$ in $\mathcal{C}(G)$. To prove Theorem B, we must show that if $(x, y) \in \mathcal{C}(G)$ and if there exists $z \in Z(G)$ such that $zy \in G_x^0$, then $(x, y) \in \mathcal{E}(G)$. The proof will be by induction on $\dim G$. It is clear for $\dim G = 0$. We assume $\dim G > 0$ and that Theorem B holds for reductive groups H with $\dim H < \dim G$.

LEMMA 3.1: *Let $(x, y) \in \mathcal{C}(G)$ and let $z, w \in Z(G)$. Then $(x, y) \in \mathcal{E}(G)$ if and only if $(zx, wy) \in \mathcal{E}(G)$.*

PROOF: Define a morphism $\tau: G \times G \rightarrow G \times G$ by $\tau(a, b) = (za, wb)$; τ is an automorphism of the algebraic variety $G \times G$. If T is a maximal torus of G , then $Z(G) \subset T$ and hence $\tau(T \times T) = T \times T$. It follows immediately that $\tau(\mathcal{Z}(G)) = \mathcal{Z}(G)$. This proves Lemma 3.1.

LEMMA 3.2: *Let $(x, y) \in \mathcal{C}(G)$ with $y \in G_x^0$. Let $x = x_s x_u$ and $y = y_s y_u$ be the Jordan decompositions of x and y . Assume that either $x_s \notin Z(G)$ or $y_s \notin Z(G)$. Then $(x, y) \in \mathcal{Z}(G)$.*

PROOF: Assume first that $x_s \notin Z(G)$. Let $H = G_{x_s}^0$. Then H is a reductive group and $\dim H < \dim G$. By standard properties of the Jordan decomposition we have $G_x \subset G_{x_s}$; hence $y \in H$. Clearly $x_s \in H$. Since we are in characteristic zero, $x_u \in H$. Therefore $x = x_s x_u \in H$. Consequently $(x, y) \in \mathcal{C}(H)$ and $y \in H_x^0 = G_x^0$. By the inductive hypothesis $(x, y) \in \mathcal{Z}(H)$. Since every maximal torus of H is a maximal torus of G , we see that $\mathcal{Z}(H) \subset \mathcal{Z}(G)$ and hence that $(x, y) \in \mathcal{Z}(G)$.

Assume now that $x_s \in Z(G)$ and $y_s \notin Z(G)$. By Lemma 3.1 we may reduce to the case in which x is unipotent. Hence we have $(y, x) \in \mathcal{C}(G)$ and, since x is unipotent, $x \in G_y^0$. By the argument given above, $(y, x) \in \mathcal{Z}(G)$. But clearly if $(y, x) \in \mathcal{Z}(G)$, then $(x, y) \in \mathcal{Z}(G)$. This proves Lemma 3.2.

It follows from Lemmas 3.1 and 3.2 that we need only consider pairs $(x, y) \in \mathcal{C}(G)$ with x and y both unipotent. In this case both x and y belong to the derived group of G , which is semisimple. Thus we can, and shall, assume that G is semisimple.

LEMMA 3.3: *Let $(x, y) \in \mathcal{C}(G)$ with x and y both unipotent and assume that G_x contains a non-trivial torus A . Then $(x, y) \in \mathcal{Z}(G)$.*

PROOF: Let $Y = \{g \in G_x^0 \mid g_s \notin Z(G)\}$. Then Y is an open subset of G_x^0 . (Recall that $Z(G)$ is finite since G is semisimple.) Moreover Y is non-empty since if $a \in A$ and $a \notin Z(G)$, then $a \in Y$. Thus Y is dense in G_x^0 . If $g \in Y$, then it follows from Lemma 3.2 that $(x, g) \in \mathcal{Z}(G)$. Therefore $(x, b) \in \mathcal{Z}(G)$ for every $b \in G_x^0$. In particular $(x, y) \in \mathcal{Z}(G)$. This proves Lemma 3.3.

We say that a unipotent element u of the semisimple group G is *distinguished* if the connected centralizer G_u^0 is a unipotent subgroup of G . We see from Lemma 3.3 that, in order to prove Theorem B, it suffices to prove that if $(x, y) \in \mathcal{C}(G)$ with (i) x and y both unipotent and (ii) x a distinguished unipotent, then $(x, y) \in \mathcal{Z}(G)$. But according to Corollary 4.14, if the commuting pair (x, y) satisfies (i) and (ii), then $(x, y) \in \mathcal{Z}(G)$. Thus the proof of Theorem B is not complete, modulo the proof of Corollary 4.14.

REMARK 3.6: The following example shows that it is not necessarily the case that $\mathcal{Z}(G) = \mathcal{C}(G)$. Let G be the special orthogonal group $SO_3(K)$, let $x = \text{diag}(1, -1, -1)$ and $y = \text{diag}(-1, -1, 1)$. Then $(x, y) \in \mathcal{C}(G)$ but it is not difficult to show that $(x, y) \notin \mathcal{Z}(G)$. However, such examples do not exist if G is simply connected, as is shown by the following theorem:

THEOREM C: *Let G be a simply connected semisimple algebraic group, let $(x, y) \in \mathcal{C}(G)$ and let N be a neighbourhood of (x, y) in $\mathcal{C}(G)$. Then there exists a maximal torus T of G such that N meets $T \times T$. Consequently $\mathcal{C}(G)$ is irreducible.*

The basic property of simply connected semisimple groups which we use in the proof is that the centralizer of a semisimple element in such a group is connected. For a proof of this, see [11, pp. E-31–E-37].

PROOF OF THEOREM C: We must prove that $\mathcal{C}(G) = \mathcal{Z}(G)$. Let $(x, y) \in \mathcal{C}(G)$. Since G is simply connected, we see that $H = G_{y_s}$ is connected, hence reductive. Clearly $(x, y) \in \mathcal{C}(H)$, $y_s \in Z(H)$ and $y_u = y_s^{-1}y \in H_x^0$. Thus, by Theorem B, $(x, y) \in \mathcal{Z}(H)$. Since $\mathcal{Z}(H) \subset \mathcal{Z}(G)$, we see that $(x, y) \in \mathcal{Z}(G)$. The proof that $\mathcal{C}(G)$ is irreducible is similar to the proof of Corollary 2.5 and will be omitted.

§4. Nilpotent and unipotent elements of parabolic type

Let G be a semisimple algebraic group, let P be a parabolic subgroup of G and let U denote the unipotent radical of P . It is shown in [10] that there exists $x \in \mathfrak{u}$ (resp. $u \in U$) such that $c_P(x)$ (resp. $C_P(u)$), the P -conjugacy class of x (resp. u) is a dense open subset of \mathfrak{u} (resp. U). This motivates the following definition:

DEFINITION 4.1: *Let G be a semisimple algebraic group and let x (resp. u) be a nilpotent (resp. unipotent) element of \mathfrak{g} (resp. G). Then x (resp. u) is of parabolic type if there exists a parabolic subgroup P of G with unipotent radical U such that $c_P(x)$ (resp. $C_P(u)$) is a dense open subset of \mathfrak{u} (resp. U).*

REMARK 4.2: (i) For \mathfrak{g} (resp. G) of type A_n , all nilpotent (resp. unipotent) elements are of parabolic type. If \mathfrak{g} (resp. G) is not of type A_n , then there exist nilpotent (resp. unipotent) elements which are not of parabolic type.

(ii) Let G, P, U be as above and let $x \in \mathfrak{u}$ (resp. $u \in U$) be such that $c_P(x)$ (resp. $C_P(u)$) is a dense open subset of \mathfrak{u} (resp. U). Then it is shown in [10] that $\mathfrak{p}_x = \mathfrak{g}_x$ (resp. that $P_x^0 = G_x^0$).

(iii) Let x be a nilpotent of \mathfrak{g} and let S be the set of semisimple elements s of \mathfrak{g} such that $\dim \mathfrak{g}_s = \dim \mathfrak{g}_x$. Then it is shown in [5] that x is of parabolic type if and only if x belongs to the closure of S .

Now let G, P, U be as above, $G \neq P$, and let M be a Levi subgroup of P ; thus P is the semi-direct product of M and U . Let $A = Z(M)^0$. Then A is a torus and $R = AU$ is the solvable radical of P ; the Lie algebra $\mathfrak{r} = \mathfrak{a} + \mathfrak{u}$ is the radical of \mathfrak{p} . In particular R and \mathfrak{r} are stable under the action of P . Let $\mathfrak{a}' = \{a \in \mathfrak{a} \mid \mathfrak{g}_a = \mathfrak{m}\}$; \mathfrak{a}' is a dense open subset of \mathfrak{a} . The following result is proved in [5].

4.3: Let $r = a + v$ with $a \in \mathfrak{a}'$ and $v \in \mathfrak{u}$. Then r is P -conjugate to a . In particular r is a semisimple element of \mathfrak{g} , $\mathfrak{p}_r = \mathfrak{g}_r$ and $\dim \mathfrak{p}_r = \dim \mathfrak{m}$.

Now let $m = \dim \mathfrak{m}$, and let $\mathfrak{r}' = \{r \in \mathfrak{r} \mid \dim \mathfrak{p}_r = m\}$. Then \mathfrak{r}' is a P -stable dense open subset of \mathfrak{r} and $\mathfrak{a}' + \mathfrak{u} = \{a + v \mid a \in \mathfrak{a}', v \in \mathfrak{u}\}$ is contained in \mathfrak{r}' . Let $x \in \mathfrak{u}$ be such that the P -conjugacy class of x is a dense open subset of \mathfrak{u} . Then $x \in \mathfrak{r}'$.

LEMMA 4.4: Let $\mathcal{R} = \{(r, t) \in \mathfrak{r}' \times \mathfrak{p} \mid t \in \mathfrak{p}_r\}$ and let $\pi: \mathcal{R} \rightarrow \mathfrak{r}'$ denote the restriction to \mathcal{R} of the projection $\mathfrak{r}' \times \mathfrak{p} \rightarrow \mathfrak{r}'$. Then π is an open mapping.

PROOF: Let $c \in \mathfrak{r}'$ and let $\mathfrak{d} = \mathfrak{p}_c$. Let $\text{Gr}_m(\mathfrak{p})$ denote the Grassmann variety of m -dimensional vector subspaces of \mathfrak{p} ; $\text{Gr}_m(\mathfrak{p})$ is a projective algebraic variety. Let \mathfrak{f} be a vector subspace of \mathfrak{p} such that \mathfrak{p} is the direct sum of \mathfrak{d} and \mathfrak{f} . Let \mathcal{S} be the subset of $\text{Gr}_m(\mathfrak{p})$ consisting of all m -dimensional subspaces \mathfrak{b} such that $\mathfrak{b} \cap \mathfrak{f} = \{0\}$; \mathcal{S} is an open subset of $\text{Gr}_m(\mathfrak{p})$. For every $T \in \text{Hom}_K(\mathfrak{d}, \mathfrak{f})$, let $\alpha(T)$ be the vector subspace $\{d + T(d) \mid d \in \mathfrak{d}\}$ of \mathfrak{p} . Then $\alpha(T) \in \mathcal{S}$ and $\alpha: \text{Hom}_K(\mathfrak{d}, \mathfrak{f}) \rightarrow \mathcal{S}$ is an isomorphism of algebraic varieties; \mathcal{S} is a “big Schubert cell” on $\text{Gr}_m(\mathfrak{p})$ and, if one represents elements of $\text{Hom}_K(\mathfrak{d}, \mathfrak{f})$ by matrices, then α^{-1} gives “Schubert coordinates” on \mathcal{S} .

It is easy to see that the map $r \mapsto \mathfrak{p}_r$ of \mathfrak{r}' into $\text{Gr}_m(\mathfrak{p})$ is a morphism of algebraic varieties. Let $\mathfrak{r}'' = \{r \in \mathfrak{r}' \mid \mathfrak{p}_r \in \mathcal{S}\}$; \mathfrak{r}'' is an open neighbourhood of c in \mathfrak{r}' . We define a morphism $\tau: \mathfrak{r}'' \times \mathfrak{d} \rightarrow \mathcal{R}$ as follows: let $r \in \mathfrak{r}''$; then \mathfrak{p}_r is a point of \mathcal{S} ; let $T_r = \alpha^{-1}(\mathfrak{p}_r)$; then $T_r \in \text{Hom}_K(\mathfrak{d}, \mathfrak{f})$ and $\mathfrak{p}_r = \{d + T_r(d) \mid d \in \mathfrak{d}\}$; we define $\tau(r, d) = (r, d + T_r(d))$. It is a straightforward matter to check that τ defines an isomorphism (of algebraic varieties) of $\mathfrak{r}'' \times \mathfrak{d}$ onto $\pi^{-1}(\mathfrak{r}'')$.

Using the morphism τ , it is now easy to check that if $d \in \mathfrak{d} = \mathfrak{p}_c$, then π maps every neighbourhood of (c, d) in \mathcal{R} onto a neighbourhood of c in \mathfrak{r}' . This proves Lemma 4.4.

In Proposition 4.5 and Corollary 4.7 below, we use the notation of §2. We assume the inductive hypothesis made at the beginning of §2 and Lemmas 2.1–2.4.

PROPOSITION 4.5: *Let \mathfrak{g} be semisimple and let $(x, y) \in \mathcal{C}(\mathfrak{g})$, with x a nilpotent element of parabolic type. Then $(x, y) \in \mathcal{Z}(\mathfrak{g})$.*

PROOF: Let G be the adjoint group of \mathfrak{g} . Choose a parabolic subgroup P of G , with unipotent radical U , such that the P -conjugacy class of x is dense in \mathfrak{u} . Let the notation be as above. Let N be an open neighbourhood of (x, y) in $\mathcal{C}(\mathfrak{g})$ and let $N' = N \cap \mathcal{R}$. By Lemma 4.4, $\pi(N')$ is an open subset of \mathfrak{r}' . In particular $\pi(N')$ meets $\mathfrak{a}' + \mathfrak{u}$. Hence, by 4.3, $\pi(N')$ contains a semisimple element s . Thus there exists $t \in \mathfrak{p}$, such that $(s, t) \in N'$. By Lemma 2.4, $(s, t) \in \mathcal{Z}(\mathfrak{g})$. We have shown that every neighbourhood of (x, y) meets $\mathcal{Z}(\mathfrak{g})$. Since $\mathcal{Z}(\mathfrak{g})$ is closed, $(x, y) \in \mathcal{Z}(\mathfrak{g})$.

The following result is proved in [1, Prop. 4.3]:

4.6: In a semisimple Lie algebra, every distinguished nilpotent element is of parabolic type.

COROLLARY 4.7: *Let $(x, y) \in \mathcal{C}(\mathfrak{g})$, with x a distinguished nilpotent element. Then $(x, y) \in \mathcal{Z}(\mathfrak{g})$.*

We now wish to prove the analogues of Proposition 4.5 and Corollary 4.7 for a semisimple algebraic group G . We have to be a bit more careful here because of the possibility of non-connected centralizers of elements of G . Let P, U, M, A and R be as defined earlier in this §. Since the centralizer of a torus in a reductive group is connected, we see that $Z_G(A) = M$. Let $A' = \{a \in A \mid G_a = M\}$. It follows from the argument given in [2, Prop. 8.18] that A' is a non-empty open subset of A . The following result is proved in [7, Lemma 1.3]:

4.8: Let $r = av$ with $a \in A'$ and $v \in U$. Then r is P -conjugate to a . In particular r is semisimple, $\dim G_r = m$, G_r is connected and $G_r = P_r$.

Now let $R' = \{r \in R \mid \dim P_r = m\}$. Then R' is a P -stable dense open

subset of R and R' contains $A'U$. If $u \in U$ is such that $C_P(U)$ is dense in U , then $u \in R'$.

LEMMA 4.9: *Let $\mathcal{R} = \{(r, g) \in R' \times P \mid g \in P_r\}$. Let $\pi: \mathcal{R} \rightarrow R'$ denote the restriction to \mathcal{R} of the projection $R' \times P \rightarrow R'$. Let $(r, g) \in \mathcal{R}$ be such that $g \in P_r^0$. Then π maps every neighbourhood of (r, g) in \mathcal{R} onto a neighbourhood of r in R' .*

Lemma 4.9 is a special case of the following result:

PROPOSITION 4.10: *Let the algebraic group H act morphically on the irreducible normal algebraic variety X and assume that all orbits of H on X have the same dimension. Let $\mathcal{V} = \{(x, h) \in X \times H \mid h \cdot x = x\}$ and let $\pi: \mathcal{V} \rightarrow X$ denote the restriction to \mathcal{V} of the projection $X \times H \rightarrow X$. Let $(y, g) \in \mathcal{V}$ be such that $g \in H_y^0$. Then π maps every neighbourhood of (y, g) in \mathcal{V} onto a neighbourhood of y in X .*

REMARK 4.11: In Proposition 4.10 it is not necessarily the case that π is an open mapping.

We shall postpone the proof of Proposition 4.10 for the moment.

In Proposition 4.12 and Corollary 4.14 below, we use some of the notations of §3. We assume the inductive hypothesis of §3 and Lemmas 3.1–3.3.

PROPOSITION 4.12: *Let G be a semisimple algebraic group and let $(u, v) \in \mathcal{C}(G)$ with u a unipotent element of parabolic type and $v \in G_u^0$. Then $(u, v) \in \mathcal{Z}(G)$.*

PROOF: We continue with the notation introduced earlier. We may assume that $u \in U$ and that $C_P(u)$ is a dense open subset of U . By Remark 4.2. (ii), $G_u^0 = P_u^0$. Thus $(u, v) \in \mathcal{R}$. Let N be a neighbourhood of (u, v) in $\mathcal{C}(G)$ and let $N' = N \cap \mathcal{R}$. By Lemma 4.9, $\pi(N')$ is a neighbourhood of u in R' . Hence $\pi(N')$ meets $A'U$. But by 4.8, if $r \in A'U$, then r is semisimple and G_r is connected. Thus N' contains a pair (r, g) with r semisimple and $g \in G_r^0 = P_r^0$. By Lemma 3.4, $(r, g) \in \mathcal{Z}(G)$. We have shown that every neighbourhood of (u, v) meets $\mathcal{Z}(G)$. Thus $(u, v) \in \mathcal{Z}(G)$. This proves 4.12.

The following result is proved in [1, Prop. 4.3]:

4.13: Every distinguished unipotent element in a semisimple al-

gebraic group is of parabolic type.

As an immediate consequence of 4.12 and 4.13 we have:

COROLLARY 4.14: *Let $(u, v) \in \mathcal{C}(G)$ with u and v both unipotent and u a distinguished unipotent. Then $(u, v) \in \mathcal{E}(G)$.*

It remains to prove Proposition 4.10.

PROOF OF PROPOSITION 4.10: By an argument given in [9, p. 64], there exists a non-empty, smooth H -stable open subset Y of X such that $\pi^{-1}(Y)$ is a smooth subvariety of $H \times Y$ and the restriction of π to $\pi^{-1}(Y)$ is a submersion. Since $\pi^{-1}(Y)$ is smooth, the irreducible components of $\pi^{-1}(Y)$ are the same as the connected components. Let A' be the irreducible component of $\pi^{-1}(Y)$ which contains $Y \times \{e\}$ and let A denote the Zariski closure of A' in V . Clearly A is an irreducible variety and $X \times \{e\} \subset A$. Let $p: A \rightarrow X$ denote the restriction of π . We note that $\dim A = q + \dim X$, where q is the common dimension of the stabilizers H_x , $x \in X$. Let $x \in X$. Then $p^{-1}(x) \subset \pi^{-1}(x) = \{x\} \times H_x$. Since $(x, e) \in A$, we see from the standard theorem on the dimension of fibres of a morphism [2, p. 38] that (i) $p^{-1}(x) \supset \{x\} \times H_x^0$ and that (ii) if $(x, h) \in p^{-1}(x)$, then $\{x\} \times hH_x^0 \subset p^{-1}(x)$. In particular we see that each irreducible component of each fibre $p^{-1}(x)$, $x \in X$, has dimension q . By a theorem of Chevalley [2, p. 81], the map $p: A \rightarrow X$ is an open map.

Now let $(y, g) \in \mathcal{V}$ with $g \in H_y^0$ and let N be a neighbourhood of (y, g) in \mathcal{V} . Then $(y, g) \in A$ by (i) above and $N' = N \cap A$ is a neighbourhood of (y, g) in A . Since p is an open map, $p(N')$ is a neighbourhood of y in X . This proves Proposition 4.10.

§5. Preliminaries on varieties over local fields

We recall that a local field is a (commutative) non-discrete locally compact topological field. We denote by k a local field of characteristic zero and we let K be an algebraically closed extension field of k . It is known that, to within isomorphism, k is either the field \mathbf{R} of real numbers, the field \mathbf{C} of complex numbers, or a finite extension field of a p -adic field \mathbf{Q}_p , for some rational prime p .

If V is a finite-dimensional vector space over k , then we always consider V as a topological space with the topology determined by the topology of k . Thus if (e_1, \dots, e_n) is a basis of V , then the map $(a_1, \dots, a_n) \mapsto \sum_{i=1}^n a_i e_i$ is a homeomorphism of k^n onto V . Subsets of

V are given the induced topology. In particular, if \mathfrak{g} is a finite-dimensional Lie algebra over k , then the commuting variety $\mathcal{C}(\mathfrak{g})$ is given the induced topology as a subset of $\mathfrak{g} \times \mathfrak{g}$. More generally, if X is an algebraic variety defined over k , then the set $X(k)$ of k -rational points of X is given the topology determined by the topology of k , i.e. the topology of $X(k)$ as an analytic space over k . If G is an algebraic group defined over k , then $G(k)$ is a locally compact topological group.

If X is an algebraic variety defined over k , we shall need to consider the Zariski topology on X , the Zariski k -topology on X and the topology on $X(k)$ as an analytic space over k . In order to avoid confusion, in §6 and §7 all topological terms which refer to the Zariski topology will be given the prefix *Zariski*. Thus an open set (resp. k -open set) in the Zariski topology is *Zariski-open* (resp. *Zariski- k -open*).

§6. Reductive Lie algebras over local fields

In this section we wish to prove the analogue of Theorem A for a reductive Lie algebra \mathfrak{g} over the local field k . The reader should bear in mind, however, that there are two differences with the situation of Theorem A:

(i) The topology on the commuting variety $\mathcal{C}(\mathfrak{g})$ is stronger than the Zariski- k -topology on $\mathcal{C}(\mathfrak{g})$; and

(ii) It is not necessarily the case that all Cartan subalgebras of \mathfrak{g} are conjugate. (However, it is known that there are only a finite number of conjugacy classes of Cartan subalgebras of \mathfrak{g} [3, 8].)

THEOREM D: *Let \mathfrak{g} be a reductive Lie algebra over the local field k of characteristic zero and let $\mathcal{C}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}$. Let $(x, y) \in \mathcal{C}(\mathfrak{g})$ and let N be a neighbourhood of (x, y) in $\mathcal{C}(\mathfrak{g})$. Then there exists a Cartan subalgebra \mathfrak{h} of \mathfrak{g} such that N meets $\mathfrak{h} \times \mathfrak{h}$.*

The proof of Theorem D will be given in a series of lemmas. The lines of the proof are the same as those of Theorem A. One just needs to check that all of the constructions made in that proof can be carried out over the field k . We let $\mathcal{E}'(\mathfrak{g})$ be the set of all $(x, y) \in \mathcal{C}(\mathfrak{g})$ such that there exists a Cartan subalgebra of \mathfrak{g} which contains both x and y . Let $\mathcal{E}(\mathfrak{g})$ be the closure of $\mathcal{E}'(\mathfrak{g})$ in $\mathcal{C}(\mathfrak{g})$. We must prove that $\mathcal{E}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})$. The proof is by induction on $\dim \mathfrak{g}$. We assume that $\dim \mathfrak{g} > 0$ and that Theorem D holds for reductive Lie algebras of dimension less than $\dim \mathfrak{g}$.

LEMMA 6.1: Let c denote the centre of \mathfrak{g} and let $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$. Let $(x, y) \in \mathcal{C}(\mathfrak{g})$ and write $x = x_1 + x_2$, $y = y_1 + y_2$ with $x_1, x_2 \in c$ and $y_1, y_2 \in \mathfrak{d}$. Then $(x, y) \in \mathcal{C}(\mathfrak{g})$ if and only if $(x_2, y_2) \in \mathcal{C}(\mathfrak{d})$.

LEMMA 6.2: Assume $c \neq \{0\}$. Then $\mathcal{C}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})$.

Lemmas 6.1 and 6.2 reduce the proof to the case of semisimple \mathfrak{g} . For the rest of the proof we assume that \mathfrak{g} is semisimple.

LEMMA 6.3: Let $(x, y) \in \mathcal{C}(\mathfrak{g})$ and assume that either x or y is not nilpotent. Then $(x, y) \in \mathcal{C}(\mathfrak{g})$.

LEMMA 6.4: Let $(x, y) \in \mathcal{C}(\mathfrak{g})$ and assume that there exists a non-zero semisimple element in \mathfrak{g}_x . Then $(x, y) \in \mathcal{C}(\mathfrak{g})$.

The proofs of Lemma 6.1–6.4 are the same as those of Lemmas 2.1–2.4.

Lemmas 6.1–6.4 reduce the proof of Theorem D to the case of pairs $(x, y) \in \mathcal{C}(\mathfrak{g})$ where x is a distinguished nilpotent element of \mathfrak{g} . To show that the arguments of §2 can be carried out over the field k we need the Jacobson-Morosov Theorem.

DEFINITION 6.5: Let \mathfrak{k} be a semisimple Lie algebra over a field F of characteristic zero. Then a triple of elements (x, h, y) in \mathfrak{k} , distinct from $(0, 0, 0)$, is an $\mathfrak{sl}_2(F)$ -triple if they satisfy the following commutation rules:

$$[h, x] = 2x, [h, y] = -2y \quad \text{and} \quad [x, y] = -h.$$

6.6. (Jacobson-Morosov Theorem): Let \mathfrak{k} be a semisimple Lie algebra over F and let x be a non-zero nilpotent element of \mathfrak{k} . Then there exists an $\mathfrak{sl}_2(F)$ -triple (x, h, y) containing x .

For more details on \mathfrak{sl}_2 -triples, see [4].

We now return to the proof of Theorem D. Let x be a non-zero nilpotent element of \mathfrak{g} and let (x, h, y) be an $\mathfrak{sl}_2(k)$ -triple in \mathfrak{g} . It follows easily from the representation theory of $\mathfrak{sl}_2(k)$ that all the eigenvalues of $ad_{\mathfrak{g}}h$ are integers. We say that x is an *even nilpotent* if all the eigenvalues of $ad_{\mathfrak{g}}h$ are even integers. (This is independent of the choice of $\mathfrak{sl}_2(k)$ -triple with first element x .) The following result is a key technical result in the paper [1] by Carter and Bala on the classification of nilpotent conjugacy classes in semisimple Lie algebras:

6.7: Every distinguished nilpotent element of a semisimple Lie algebra is an even nilpotent.

For the proof of 6.7, see [1, Thm. 4.27]. The proof is complicated and involves classification. It would be of interest to have a more elementary proof.

In order to be able to use the results of §4 it will be convenient to change our point of view slightly. Let $\mathfrak{g} = \mathfrak{g} \otimes_k K$. Then \mathfrak{g} is a semisimple Lie algebra over K with k -structure $\mathfrak{g} = \mathfrak{g}(k)$. If V is a vector subspace defined over k of the K -vector space \mathfrak{g} , we set $V(k) = V \cap \mathfrak{g}(k) = V \cap \mathfrak{g}$.

Now let x be a distinguished nilpotent element of \mathfrak{g} and let (x, h, y) be an $\mathfrak{sl}_2(k)$ -triple in \mathfrak{g} . We consider (x, h, y) as an $\mathfrak{sl}_2(K)$ -triple in \mathfrak{g} . For each (even) integer j , let \mathfrak{g}_j be the j -eigenspace of $ad_{\mathfrak{g}} h$. Then \mathfrak{g} is the direct sum of the \mathfrak{g}_j 's. Let $\mathfrak{p} = \sum_{j \geq 0} \mathfrak{g}_j$, let $\mathfrak{m} = \mathfrak{g}_0$ and let $\mathfrak{u} = \sum_{j > 0} \mathfrak{g}_j$. Let G be the adjoint group of the semisimple Lie algebra \mathfrak{g} . Then there exists a parabolic subgroup P of G with unipotent radical U and a Levi subgroup M of P such that \mathfrak{p} (resp. \mathfrak{u} , \mathfrak{m}) is the Lie algebra of P (resp. U , M); P , M and U are all defined over k . It is shown in [1, Prop. 4.3] that $c_P(x)$, the P -conjugacy class of x is a dense Zariski-open subset of \mathfrak{u} . (Warning: If $P(k)$ denotes the group of k -rational points of P , then the $P(k)$ -orbit of x is not necessarily dense in $\mathfrak{u}(k)$.)

Let \mathfrak{a} denote the centre of \mathfrak{m} and let $\mathfrak{r} = \mathfrak{a} + \mathfrak{u}$; \mathfrak{r} is the radical of \mathfrak{p} . As in §4, we define $\mathfrak{r}' = \{r \in \mathfrak{r} \mid \dim \mathfrak{p}_r = \dim \mathfrak{m}\}$ and $\mathfrak{a}' = \{a \in \mathfrak{a} \mid \mathfrak{g}_a = \mathfrak{m}\}$. Then \mathfrak{r}' and \mathfrak{a}' are Zariski- k -open subsets of \mathfrak{r} and \mathfrak{a} respectively.

LEMMA 6.8: *Let $\mathfrak{r}'(k) = \mathfrak{r}' \cap \mathfrak{r}(k)$ and $\mathfrak{a}'(k) = \mathfrak{a}' \cap \mathfrak{a}(k)$. Then*

- (i) $\mathfrak{a}'(k) + \mathfrak{u}(k) \subset \mathfrak{r}'(k)$;
- (ii) every element of $\mathfrak{a}'(k) + \mathfrak{u}(k)$ is semisimple; and
- (iii) $\mathfrak{a}'(k) + \mathfrak{u}(k)$ is a dense open subset of $\mathfrak{r}(k)$.

PROOF: Let $a \in \mathfrak{a}'(k)$ and $v \in \mathfrak{u}(k)$. By 4.3, $a + v$ is P -conjugate to a . This proves (i) and (ii). The proof of (iii) is elementary.

LEMMA 6.9: *Let $\mathcal{R} = \{(r, z) \in \mathfrak{r}'(k) \times \mathfrak{p}(k) \mid z \in \mathfrak{p}_r(k)\}$. Let $\pi: \mathcal{R} \rightarrow \mathfrak{r}'(k)$ denote the restriction to \mathcal{R} of the projection $\mathfrak{r}'(k) \times \mathfrak{p}(k) \rightarrow \mathfrak{r}(k)$. Then π is an open mapping.*

The proof of Lemma 5.9 is almost exactly the same as the proof of Lemma 4.4, except that we work in the category of analytic spaces

over k instead of algebraic varieties over K . We omit details of the proof.

LEMMA 6.10: *Let $(x, z) \in \mathcal{C}(\mathfrak{g})$, with x a distinguished nilpotent of \mathfrak{g} . Then $(x, z) \in \mathcal{E}(\mathfrak{g})$.*

The proof of 6.10 is essentially the same as that of Proposition 4.5. Lemma 6.10 completes the proof of Theorem D.

§7. Reductive groups over local fields

Let G be a reductive linear algebraic group defined over the local field k of characteristic zero. A subgroup A of $G(k)$ is a *Cartan subgroup* of $G(k)$ if there exists a Cartan subalgebra \mathfrak{h} of the k -Lie algebra $\mathfrak{g}(k)$ such that $A = \{g \in G(k) \mid (\text{Ad}_g)(x) = x \text{ for every } x \in \mathfrak{h}\}$; equivalently A is a Cartan subgroup of $G(k)$ if there exists a maximal torus T of G , defined over k , such that $A = T(k)$. It follows from the second definition that a Cartan subgroup of $G(k)$ is abelian. It is known that $G(k)$ has only a finite number of conjugacy classes of Cartan subgroups (This follows from the corresponding result for Cartan subalgebras of $\mathfrak{g}(k)$.)

THEOREM E: *Let G be a reductive linear algebraic group defined over a local field k of characteristic zero and let $\mathcal{C}(G(k)) = \{(x, y) \in G(k) \times G(k) \mid xy = yx\}$ be the commuting variety of $G(k)$. Let $(x, y) \in \mathcal{C}(G(k))$ be such that there exists $z \in Z(G)(k)$ such that $zy \in (G^{\mathfrak{h}})(k)$ and let N be a neighbourhood of (x, y) in $\mathcal{C}(G(k))$. Then there exists a Cartan subgroup A of $G(k)$ such that $A \times A$ meets N .*

The proof of Theorem E will occupy most of the rest of this section. Roughly, the proof goes as follows: By arguments similar to those in the proof of Theorem B, we reduce to the case of a commuting pair (x, y) , where x and y are distinguished unipotent elements of G . To treat the case of a pair of distinguished unipotent elements, we use Theorem D and the exponential and log maps.

It follows from [2, Prop. 1.10] that we may assume that G is a k -subgroup of $GL_n(K)$; in order to simplify our exposition we shall make this assumption. Thus $G(k) = G \cap GL_n(k)$ and $\mathfrak{g}(k) = \mathfrak{g} \cap \mathfrak{gl}_n(k)$.

We let $\mathcal{E}'(G(k))$ be the set of all $(x, y) \in \mathcal{C}(G(k))$ such that there exists a Cartan subgroup A of $G(k)$ which contains both x and y . We let $\mathcal{E}(G(k))$ denote the closure of $\mathcal{E}'(G(k))$ in $\mathcal{C}(G(k))$. We must prove

that $\mathcal{E}(G(k)) = \mathcal{C}(G(k))$. The proof is by induction on $\dim G$. We assume $\dim G > 0$ and that the theorem holds for reductive k -groups G' with $\dim G' < \dim G$.

LEMMA 7.1: *Let $(z, w) \in Z(G)(k)$ and let $(x, y) \in \mathcal{C}(G(k))$. Then $(x, y) \in E(G(k))$ if and only if $(zx, wy) \in E(G(k))$.*

PROOF: We note that $Z(G)(k)$ is contained in the kernel of the adjoint representation of G , hence in every Cartan subgroup of $G(k)$. The rest of the proof follows in exactly the same manner as that of Lemma 3.1.

LEMMA 7.2: *Let $(x, y) \in \mathcal{C}(G(k))$ with $y \in (G_x^0)(k)$. Let $x = x_s x_u$ and $y = y_s y_u$ be the Jordan decompositions of x and y . Assume that either $x_s \notin Z(G)(k)$ or $y_s \notin Z(G)(k)$. Then $(x, y) \in \mathcal{E}(G(k))$.*

The proof of Lemma 7.2 is essentially the same as that of Lemma 3.2. We omit details.

By Lemmas 7.1 and 7.2, we see that it suffices to consider pairs $(x, y) \in \mathcal{C}(G(k))$ such that x and y are both unipotent. If x and y are both unipotent, then they are contained in the derived group of G , which is a semisimple k -group. Hence we may assume that G is a semisimple k -group. For the rest of the proof of Theorem E, we shall make this assumption.

LEMMA 7.3: *Let $(x, y) \in \mathcal{C}(G(k))$ with x and y unipotent and assume that G_x contains a non-trivial torus. Then $(x, y) \in \mathcal{E}(G(k))$.*

PROOF: Let $D = G_x^0$; D is a connected k -subgroup of G and $y \in D(k)$. Since D contains a non-trivial torus, D is not a unipotent group. Let $D' = \{d \in D \mid d_s \notin Z(G)\}$; since $Z(G)$ is finite, D' is a non-empty Zariski- k -open subset of D . It follows easily that $D'(k)$ is a dense open subset of $D(k)$. Thus there exists a sequence (d_n) of elements of $D'(k)$ such that $d_n \rightarrow y$. By Lemma 7.2, $(x, d_n) \in \mathcal{E}(G(k))$ for every n . Thus $(x, y) \in \mathcal{E}(G(k))$. This proves Lemma 7.3.

We have now reduced the proof of Theorem E to the consideration of pairs $(x, y) \in \mathcal{C}(G(k))$ such that x and y are both distinguished unipotent elements of G . Let (x, y) be such a pair. Let $\mathcal{N}(\mathfrak{g}(k))$ be the set of all nilpotent elements in the Lie algebra $\mathfrak{g}(k)$ and let $\mathcal{U}(G(k))$ be the set of all unipotent elements in $G(k)$. Let $\mu : \mathcal{N}(\mathfrak{g}(k)) \rightarrow \mathcal{U}(G(k))$ be the map given by the usual exponential power series; $\mu(x) = \sum_{j=0}^{\infty} (x^j / j!)$. Since the elements of $\mathcal{N}(\mathfrak{g}(k))$ are nilpotent, μ is poly-

nomial map. It is well-known that μ is a homeomorphism of $\mathcal{N}(\mathfrak{g}(k))$ onto $\mathcal{U}(G(k))$. Let $\lambda : \mathcal{U}(G(k)) \rightarrow \mathcal{N}(\mathfrak{g}(k))$ be the inverse homeomorphism; λ is also a polynomial map, given by the log series. Let $a = \lambda(x)$ and $b = \lambda(y)$. Then a and b are distinguished nilpotent elements of $\mathfrak{g}(k)$ and $[a, b] = 0$.

We let (a, h, c) be an $\mathfrak{sl}_2(k)$ triple in $\mathfrak{g}(k)$. For each integer j , let $\mathfrak{g}(k)_j$ denote the j -eigenspace of $ad_{\mathfrak{g}(k)}h$. Let $\mathfrak{v} = \sum_{j>0} \mathfrak{g}(k)_j$. Let \mathfrak{a} be the three-dimensional subalgebra of $\mathfrak{g}(k)$ spanned by the triple (a, h, c) ; since a is a distinguished nilpotent element of \mathfrak{g} , it follows that the centralizer of a in \mathfrak{g} is $\{0\}$. (See [1, Cor. 2.15].) Since $[a, b] = 0$, this implies $b \in \mathfrak{v}$. Clearly we have $a \in \mathfrak{v}$.

Corresponding to the inclusion homeomorphism of $\mathfrak{a} \cong \mathfrak{sl}_2(k)$ into $\mathfrak{g}(k) \subset \mathfrak{gl}_n(k)$, there is a compatible homomorphism of groups $\rho : SL_2(k) \rightarrow G(k)$. (See [4, p. 73].) To simplify our notation, let $\rho_t = \rho(\text{diag}(t, t^{-1}))$ for $t \in k - \{0\}$. It is not difficult to see that if $z \in \mathfrak{g}(k)_j$, then $(Ad_{\mathfrak{g}(k)}\rho_t)(z) = t^jz$. Hence if $v \in \mathfrak{v}$, we have $\lim_{t \rightarrow 0} (Ad_{\mathfrak{g}(k)}\rho_t)(v) = 0$.

Now there exists an open neighbourhood J of 0 in $\mathfrak{g}(k)$ such that the exponential map \exp (given by the usual power series) converges in J and defines a homeomorphism of J onto an open neighbourhood L of e in $G(k)$. Since $a, b \in \mathfrak{v}$, we see that after conjugating by $Ad_{\mathfrak{g}(k)}\rho_t$, for an appropriate t , we may assume that $a, b \in J$.

Thus we see that a and b are commuting elements of $\mathfrak{g}(k)$ which belong to J . It follows from Theorem D that there exists a sequence of pairs (a_n, b_n) in $J \times J$ satisfying the following conditions: (i) $(a_n, b_n) \rightarrow (a, b)$; and (ii) for each n there exists a Cartan subalgebra \mathfrak{h}_n of $\mathfrak{g}(k)$ such that $a_n, b_n \in \mathfrak{h}_n$. Since a_n and b_n belong to the Cartan subalgebra \mathfrak{h}_n , it is clear that $\exp(a_n)$ and $\exp(b_n)$ belong to the Cartan subgroup

$$A_n = \{g \in G(k) \mid (Ad_{\mathfrak{g}(k)}g)(x) = x \text{ for every } x \in \mathfrak{h}_n\}.$$

In particular, $(\exp(a_n), \exp(b_n)) \in \mathcal{E}(G(k))$. Now since $(a_n, b_n) \rightarrow (a, b)$, we see that $(\exp(a_n), \exp(b_n)) \rightarrow (\exp(a), \exp(b)) = (x, y)$. Thus $(x, y) \in \mathcal{E}(G(k))$. This completes the proof of Theorem E.

THEOREM F: *Let G be a simply connected semisimple algebraic group defined over the local field k of characteristic zero. Let $(x, y) \in \mathcal{E}(G(k))$ and let N be a neighbourhood of (x, y) in $\mathcal{E}(G(k))$. Then there exists a Cartan subgroup A of $G(k)$ such that N meets $A \times A$.*

The proof of Theorem F is essentially the same as that of Theorem C.

REMARK 7.4: Let $k = \mathbf{R}$ and $K = \mathbf{C}$ and let G be a semisimple k -group. Then the hypothesis of Theorem F requires only that the complex Lie group G be simply connected. This does not necessarily imply that the real Lie group $G(\mathbf{R})$ is simply connected. For example, let $G = SL_n(\mathbf{C})$ and $G(\mathbf{R}) = SL_n(\mathbf{R})$; then G is simply connected, but $G(\mathbf{R})$ is not simply connected.

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